A converse of Cauchy's fundamental sequence theorem

Simon Stevin, *Wis- en Natuurkundig Tijdschrift*
35e Jaargang, Aflevering III (Februari 1962)

A CONVERSE OF CAUCHY'S FUNDAMENTAL SEQUENCE THEOREM

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Let $\{\delta_{mn}\}$ denote a double sequence of non-negative real numbers ($m, n = 1, 2, \ldots$). We shall call it a *strong double sequence* if it has the property that every sequence of real numbers $a_1, a_2, \ldots$ satisfying

$$(1) \qquad |a_m - a_n| \leqslant \delta_{mn} \qquad (m, n = 1, 2, \ldots)$$

is automatically convergent. So Cauchy's theorem on fundamental sequences states that a sufficient condition for $\{\delta_{mn}\}$ to be strong, is that

$$(2) \qquad \lim_{m,n\to\infty} \delta_{mn} = 0$$

(i.e. for every $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that $m > N(\varepsilon)$, $n > N(\varepsilon)$ imply $|\delta_{mn}| < \varepsilon$).

Condition (2) is not necessary, however. It is, for example, not satisfied by $\delta_{mn} = mn/(m^2 + n^2)$ (as $\delta_{mm} = \frac{1}{2}$ for all m), but nevertheless this $\{\delta_{mn}\}$ is strong. For, with this $\{\delta_{mn}\}$ we infer from (1), by making $n \to \infty$ and keeping m fixed, that $\lim a_n$ exists and equals a_m. (So in this case it even follows from (1) that the sequence $\{a_n\}$ is constant.)

We shall show in theorem 1 that (2) is a necessary condition if some extra assumptions are added, viz. symmetry and triangle inequality.

In this connection we remark that, for every double sequence $\{\delta_{mn}\}$ (with $\delta_{mn} \geqslant 0$), we can construct a double sequence $\{d_{mn}\}$ that satisfies the conditions of theorem 1 and that is equivalent to $\{\delta_{mn}\}$ in the following sense: if $\{a_n\}$ satisfies (1) for all m and n, then it satisfies $|a_m - a_n| \leqslant d_{mn}$ for all m and n, and vice versa. For d_{mn} we can take the "shortest distance between m and n", to be obtained as follows. Put $e(k, l) = \min(\delta_{kl}, \delta_{lk})$, and

$$d_{mn} = \inf \{e(k_0, k_1) + e(k_1, k_2) + \ldots + e(k_{N-1}, k_N)\},$$

the infimum being taken over all possible finite chains of positive integers $m = k_0, k_1, \dots, k_{N-1}, k_N = n$, without any restriction on N other than $N \geqslant 0$. It is not difficult to prove that $\{d_{mn}\}$ has the required properties, but this is unimportant for our present purpose.

Theorem 1. Let $\{d_{mn}\}$ satisfy

$$(3) \qquad d_{mn} \geqslant 0, \quad d_{mn} = d_{nm}, \qquad d_{mn} \leqslant d_{mk} + d_{kn}$$

for all $m, n, k, = 1, 2, \dots,$ and assume that it is a strong double sequence. Then we have

$$(4) \qquad \lim_{m,n\to\infty} d_{mn} = 0.$$

Proof. Assuming that (4) does not hold, we shall show that $\{d_{mn}\}$ is not strong.

First we show (assuming that (3) is true and (4) is false) that there exist infinite sets of natural numbers, S and T, and that there exists a positive number p such that for all $m \in S$, $n \in T$ we have $d_{mn} > p$.

We shall apply Ramsey's theorem [1] of which a special case [2] reads: Let N be the set of all positive integers, and let C be the collection of all sets of two elements from N. Let C be divided into two classes. Then there exists an infinite subset V of N, such that all sets of two elements taken from V belong to one and the same class.

As (4) is false, we can find a number $p > 0$ and a sequence $m_1 < n_1 < m_2 < n_2 < \dots$ such that [3]

$$d(m_i, n_i) > 2p \qquad (i = 1, 2, \dots).$$

C consists of all pairs $\{i, j\}$ with $i \neq j$ ($\{i, j\}$ and $\{j, i\}$ are considered to be the same pair). We put $\{i, j\}$ into class A if $d(m_i, m_j) > p$, and otherwise in class B. Let V be an infinite set of i's such that all pairs taken from V belong to one and the same class.

[1] F.P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930) 264-286.

[2] The general case gives the same result for subsets of k elements from N, and division into r classes (k and r arbitrary positive integers).

[3] Occasionally we write $d(m, n)$ instead of d_{mn}, just for typographycal reasons.

If all pairs from V belong to A, then we divide V into two infinite subsets V_1 and V_2. Let S be the set of all m_i with $i \in V_1$, and T the set of all m_i with $i \in V_2$. Obviously $d_{mn} > p$ if $m \in S$, $n \in T$.

If all pairs from V belong to B then we take for S the set of all m_i with $i \in V$, and for T the set of all n_i with $i \in V$. If $m \in S$, $n \in T$ we have $m = m_i$, $n = n_j$ with $i \in V$, $j \in V$, whence

$$d_{mn} = d(m_i, n_j) \geqslant d(m_j, n_j) - d(m_i, m_j) \geqslant 2p - p,$$

so again $d_{mn} > p$.

Now that the existence of S, T and p (with $p > 0$, S, T both infinite, $d_{mn} > p$ if $m \in S$, $n \in T$) has been established, we construct a sequence $a_1, a_2, \ldots$ by taking

$$a_k = \inf_{m \in S} \; d_{km} \qquad (k = 1, 2, \ldots).$$

For all k, l we have

$$a_k + d_{kl} = \inf_{m \in S} (d_{km} + d_{kl}) \geqslant \inf_{m \in S} d_{lm} = a_l,$$

and similarly $a_l + d_{kl} \geqslant a_l$, whence $|a_k - a_l| \leqslant d_{kl}$. Nevertheless the sequence $a_1, a_2, \ldots$ diverges, since $a_k = 0$ if $k \in S$, $a_k \geqslant p$ if $k \in T$. So $\{d_{kl}\}$ is not strong, and our proof is complete.

We give a separate formulation of the result of the first part of the proof of theorem 1, slightly modified with respect to both form and contents:

Theorem 2. Let R be a metric space with infinitely many elements. Then exactly one of the following two statements holds:

a) R contains two infinite subsets S and T with positive distance.

b) R is countable, and if it is arranged into a sequence it becomes a fundamental sequence.

Proof. It only remains to look into the case that R is not countable. Then we can choose a countable subset R_1. If (a) does not hold for R then it does not hold for R_1, whence (b) holds for R_1 (as the countable case has already been dealt with). So R_1 can be considered as a fundamental sequence. This implies that R contains at most one point having distance zero to R_1. Consequently there is a number $p > 0$ such that more than countably many points of R have distance $> p$ to R_1. So this non-countable set has positive distance to the countable set R_1.