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A CONTINUITY PROPERTY OF A PARAMETRIC PROJECTION AND AN ITERATIVE PROCESS FOR SOLVING LINEAR VARIATIONAL INEQUALITIES

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Abstract

Let the point–to–set mapping $K : R^n \longrightarrow R^{n^*}$ be continuous (R^{n^*} consists of all closed convex subset in R^n) and GS be the set of all symmetric positive definite matrices of size n. Define

$$P^G_{K(x)}(y) = \text{Solution of } \min_{u \in K(x)} \| y - u \|_G$$

Where $\| x \| = x^T G x \quad x \in R^n$. In this paper, it is shown that the operator $F : R^n x R^n x GS \longrightarrow R^n$ defined by

$$F(x,y,G) := P^G_{K(x)}(y)$$

is continuous at each point $(x,y,G) \in R^n x R^n x GS$. Using this result, we can prove the convergence of an iterative process for solving linear variational inequality problems.

Key words : Projection ; Iterative process; Linear variational inequality.

Introdution

Let the point–to–set mapping $K : R^n \longrightarrow R^{n^*}$ be continuous (R^{n^*} consists of all closed convex subset in R^n; the definition of K being continuous will be given in section 2) and GS be the set of all symmetric positive definite matrices with size n. Define

$$P^G_{K(x)}(y) := \text{Solution of} \min_{u \in K(x)} \| y - u \|_G$$

Where $\| x \| = x^T x \quad x \in R^n$. Now we can define an operator $F : R^n x R^n x GS \longrightarrow R^n$

$$F(x,y,G) := P^G_{K(x)}(y) \qquad (1.1)$$

In order to study the variational inequality problem, [Chen.1982] has proved that F(x,y,G) is a continuous mapping of x and y when $G \equiv I$. In this paper, we show that the operator $F : R^n \times R^n \times GS \longrightarrow R^n$ defined by (1.1) is continuous at each point $(x,y,G) \in R^n x R^n \times GS$. Using this result, we can prove the convergence of an iterative process for solving a linear variational inequality problem. The iterative method (algorithm 3.2) designed in this paper is a generalisation of [Men.1977]'s method, which is a very general iterative scheme for solving the linear complementarity problem, which is a special case of the linear variational inequality problem. Within our knowledge there are few methods specially tend to solve the linear variational inequalities, although there are many iterative methods presented for solving the linear complementarity problems. In fact, the method discussed in this paper for solving linear variational inequalities is to reduce a linear variational inequality into a series simple sub–linear variational inequalities which can be easily solved. In section 2 we will prove the continuity of the operator $F(x, y, G)$. The iterative process will be presented in section 3, and the convergence theorem will be proved by using the result obtained in section 2.

2 The continuity of F

Let GS be the set of all symmetric n by n positive definite matrices. For each $G \in GS$, one can define an inner product

$$\langle x , y \rangle_G := x^T G y \quad \forall x, y \in R^n$$

and a norm

$$\| x \|_G := [\langle x,x \rangle_G]^{\frac{1}{2}} \qquad \forall x \in R^n$$

Suppose $C \subset R^n$ is a closed convex subset, and the projection $P_C^G : R^n \longrightarrow C$ is defined by

$$P_C^G(x) := \text{sol} \min_{y \in C} \| x - y \|_G \qquad (2.1)$$

then we have

<u>Lemma 2.1</u> [Kin.1980] The following two statements are equivalent

1) $x = P_C^G(y)$
2) $\langle x-y , u-x \rangle_G \geq 0 \qquad \forall u \in C \quad (2.2)$

<u>Lemma 2.2</u> [Kin.1980] For each $x,y \in R^n$

$$\| P_C^G(x) - P_C^G(y) \|_G \leq \| x - y \|_G \qquad (2.3)$$

If $G = I$, let

$$\langle \; , \; \rangle := \langle \; , \; \rangle_I \quad \| \; \| := \| \; \|_I$$

and

$$P_C(\,) := P_C^I(\,)$$

Let $K : R^n \longrightarrow R^{n^*}$ be continuous : K is both upper semicontinuous and lower semicontinuous. Recall that K is upper semicontinuous[Ber.1963] at $x \in R^n$, if for each open set

N containing f(x), there exists a neighborhood U of x such that $f(U) = \bigcup_{y \in U} f(y) \subset N$, and K is lower semicontinuous [Ber.1963] at $x \in R^n$, if for each open set N meeting f(x), there is a neighborhood U of x such that f(y) meets N for all $y \in U$. If K is upper (or lower) semicontinuous at each point in R^n, we say K is upper (or lower) semicontinuous in R^n. Define

$$f(x,y) := P_{K(x)}(y) \qquad (2.4)$$

the following lemma is proved in [Chan.1982].

<u>Lemma 2.3</u> Suppose that $K : R^n \longrightarrow R^{n^*}$ is continuous, then $f : R^n x R^n \longrightarrow R^n$ defined by (2.4) is continuous.

Let $F : R^n x R^n x GS \longrightarrow R^n$ Which is defined by

$$F(x,y,G) := P^G_{K(x)}(y) \qquad (2.5)$$

In this section, we are going to prove

<u>Theorem 2.4</u> Let $K : R^n \longrightarrow R^{n^*}$ be continuous.Then the operator $F : R^n x R^n x GS \longrightarrow R^n$ defined by (2.5) is continuous at each point $(x,y,G) \in R^n x R^n x GS$.

<u>Proof</u> Suppose that $\{ (x_k , y_k , G_k) : k \in N \} \subset R^n x R^n x GS$ and

$$x_k \longrightarrow x_0 \ ; \ y_k \longrightarrow y_0 \ ; \ G_k \longrightarrow G_0$$

Denote

$$w := P^{G_0}_{K(x_0)}(y_0) \ ; \ v_k := P^{G_0}_{K(x_k)}(y_k) \ ; \ u_k := P^{G_k}_{K(x_k)}(y_k)$$

and

$$< \, , \, >_k := < \, , \, >_{G_k} \quad ; \quad \| \; \|_k := \| \; \|_{G_k}$$

From lemma 2.3, one knows that

$$\| v_k - w \| \longrightarrow 0 \qquad (k \longrightarrow \infty)$$

Hence it is sufficient to prove

$$\| u_k - v_k \| \longrightarrow 0 \qquad (k \longrightarrow \infty) \qquad (2.6)$$

By (2.2)

$$< u_k - y_k \, , y - u_k >_k \; \geq \; 0 \qquad \forall \, y \in K(x_k) \quad (2.7)$$

and

$$< v_k - y_k \, , y - v_k >_0 \; \geq \; 0 \qquad \forall \, y \in K(x_k) \quad (2.8)$$

Setting $y = v_k$ in (2.7) and $y = u_k$ in (2.8) gives

$$(v_k - u_k)^T G_k (u_k - y_k) \; \geq \; 0$$

and

$$(v_k - u_k)^T G_0 (y_k - v_k) \geq \; 0$$

respectively. Therefore,

$$(v_k - u_k)^T (G_0 + G_k - G_0)(u_k - y_k)$$

$$+ (v_k - u_k)^T G_0 (y_k - v_k) \geq 0$$

i.e.

$$(v_k - u_k)^T (G_k - G_0)(u_k - y_k)$$

$$+ (v_k - u_k)^T G_0 (u_k - y_k)$$

$$+ (v_k - u_k)^T G_0 (y_k - v_k) \qquad \geq \qquad 0$$

then

$$(v_k - u_k)^T (G_k - G_0)(u_k - y_k)$$

$$\geq (v_k - u_k)^T G_0 (v_k - u_k)$$

Let $\lambda_{min}(G_0)$ be the smallest eigenvalue of G_0, then one has

$$(v_k - u_k)^T (G_k - G_0)(u_k - y_k)$$

$$\geq \lambda_{min}(G_0) \| (v_k - u_k) \|^2$$

hence

$$\|(G_k - G_0)\| \; \|(u_k - y_k)\|$$

$$\geq \lambda_{min}(G_0) \| (v_k - u_k) \|$$

Now it is sufficient to prove that $\{ (\| u_k - y_k \|) \}$ is bounded. Since

$$\lambda_{min}^{\frac{1}{2}}(G_k)\,\| u_k - y_k \| \le \|u_k - y_k\|_k$$

$$= \min_{y \in K(x_k)} \| y - y_k \|_k$$

$$\le \lambda_{max}^{\frac{1}{2}}(G_k) \min_{y \in K(x_k)} \| y - y_k \|$$

$$\le \lambda_{max}^{\frac{1}{2}}(G_k) \min_{y \in K(x_k)} (\| y - y_0 \| + \| y_0 - y_k \|)$$

Now it is sufficient to prove that $\{(\min_{y \in K(x_k)} \| y - y_0 \|)\}$ is bounded. Let

$$\alpha_k := \min_{y \in K(x_k)} \| y - y_0 \|$$

There always exists $z_0 \in K(x_0)$ such that

$$\alpha_0 := \min_{y \in K(x_0)} \| y - y_0 \| = \| z_0 - y_0 \|$$

By the continuity of $K(x)$, there exist $\{z_k\}$ such that

$$z_k \in K(x_k) \quad ; \quad z_k \longrightarrow z_0$$

Thus

$$\alpha_k := \min_{y \in K(x_k)} \| y - y_0 \|$$

$$\leq \| z_k - y_0 \|$$

$$\leq \| z_k - z_0 \| + \| z_0 - y_0 \|$$

i.e.

$$\overline{\lim}\ \alpha_k \leq \| z_0 - y_0 \|$$

hence $\{ \alpha_k \}$ is bounded. This competes the proof. !

3 Iterative process for solving linear variational inequality

Let $f : R^n \longrightarrow R^n$ be a mapping and C be a closed convex subset in R^n, the variational inequality problem can be stated as follows

Problem VIP(f,C) Find $x^* \in C$ such that

$$< x - x^*, f(x^*) > \geq 0 \qquad \forall x \in C \qquad (3.1)$$

If $C = R^n$, then problem VIP(f,C) is equivalent to find a solution of the equation $f(x) = 0$. Given a matrix $A \in R^{nxn}$ and a vector $b \in R^n$. When $f(x) = Ax - b$, we say Problem VIP(f,C) is a linear variational inequality problem and denoted by LVIP(A,b,C).

It is well known [Fang.1982; Kin.1980, etc.] that x^* satisfies (3.1) if and only if

$$x^* = P_C^G (x^* - G^{-1} f(x^*)) \qquad (3.2)$$

where $G \in GS$. Hence, if f is continuous and C is bounded, then by Brouwer's fixed point theorem [Kin.1980, etc], problem VIP(f,C) always has a solution. If f is strictly monotone, then VIP(f,C) has at most one solution and if f is strongly monotone, VIP(fC) has a unique solution[Kin.1980,etc.].

When $f(x) = Ax - b$, f is strictly monotone if and only if f is strongly monotone if and only if

A is positive, i.e.

$$< Ax - Ay , x-y > > 0 \qquad \forall x \neq y \in R^n$$

If A $\in$ GS, then we have an interesting fact

$$x^* = P_C^A (A^{-1}b) \qquad (3.3)$$

i.e. the solution of LVIP(A,b,C) is the projection of the solution of the equation $Ax=b$ under the norm $\| \ \|_A$ on C. Using (3.2) one can check (3.3).

If $C = R_+^n$, then the problem LVIP(A,b,C) is equivalent to the linear complementary problem i.e.

Problem LCP(A,b) Find $x^* \in R_+^n$ such that

$$Ax^* - b \in R_+^n \ ; \ < x^* \ , \ Ax^* - b > = 0 \qquad (3.4)$$

In this section, we are only interested in the numerical method for solving LVIP(A,b,C), especially, in designing an iteration method which suits any kinds of closed convex subset C in R^n. There are many iterative methods for solving LCP(A,b), in [Pang.1982], one can see a compact survey. Motivated by several earlier works, [Man.1977] proposed a fairly general iterative algorithm for solving LCP(A,b), which is

Algorithm 3.1[Man.1977] Let $x_0 \geq 0$

$$y_k = P_{R_+^n} (x_k - \omega D_k(Ax_k - b + E_k(x_{k+1} - x_k)))$$

$$X_{k+1} = \lambda y_k + (1-\lambda)x_k$$

where

$$0 < \lambda < 1 \ ; \qquad \omega > 0$$

$\{D_k\}$ and $\{E_k\}$ are bounded sequences of matrices in R^{nxn}, with each D_k being a positive diagonal matrix satisfying

$$D_k > \alpha I$$

for some $\alpha > 0$.

Up to our knowledge, there are few methods for solving LVIP(A,b,C), when C is supposed to be an arbitrary closed convex subset. In this section, we are going to fill part of this gap and to generalise algorithm 3.1 for solving LVIP(A,b,C).

Algorithm 3.2 Let $x_0 \in C$

$$y_k = P_C^{D_k}[\, x_k - D_k^{-1}(\, Ax_k - b + E_k(\, x_{k+1} - x_k))\,] \quad (3.5)$$

$$x_{k+1} = \lambda_k y_k + (1-\lambda_k)x_k \quad (3.6)$$

where

$$0 < \lambda \le \lambda_k < 1 \quad (3.7.1)$$

$\{D_k\}$ and $\{E_k\}$ are bounded sequences of matrices in $\mathbb{R}^{n\times n}$, with each $D_k \in$ GS matrix satisfying

$$\beta I \ge D_k \ge \alpha I \quad (3.7.2)$$

for some α and $\beta > 0$.

Remark 1) In algorithm 3.2, we do not use ω, because, in fact, the ω in algorithm 3.1 can be put into D_k.

2) If one chooses D_k to be diagonal and $C = R^n_+$, then algorithm 3.1 and algorithm 3.2 are identical. One can prove this statement using the first part of the following lemma

Lemma 3.3 1) if $D \in$ GS is diagonal, then

$$P^D_{R^n_+}(y) = P_{R^n_+}(y) \qquad \forall\, y \in R^n$$

2) if $D = \alpha I$ ($\alpha > 0$), then

$$P^D_C(y) = P_C(y) \quad . \quad \forall\, y \in R^n, \forall\, C \in R^{n^*}$$

One can directly use the definition to check this lemma.

Theorem 3.1 We posit

1) A is symmetric;

2) (3.7.1) and (3.7.2) hold;

3) there is a positive number γ such that

$$y^T(\lambda_k^{-1} D_k + E_k - \frac{1}{2} A)y \geq \gamma \|y\|^2 \quad \forall k \quad \forall y \in R^n$$

Then each accumulation point of $\{ x_k \}$ generated by algorithm 3.2, is a solution of LVIP(A,b,C).

Proof Let

$$g(x) := \frac{1}{2} x^T Ax - b^T x \qquad (3.8)$$

and $\{ x_k \}$ are generated by algorithm 3.2, then

$$g(x_{k+1}) - g(x_k)$$

$$= \frac{1}{2} (\lambda_k y_k + (1-\lambda_k)x_k)^T A (\lambda_k y_k + (1-\lambda_k)x_k)$$

$$- b^T (\lambda_k y_k + (1-\lambda_k)x_k) - g(x_k)$$

$$= \frac{1}{2} \{ \lambda_k^2 (y_k - x_k)^T A (y_k - x_k) + 2 \lambda_k x_k^T A (y_k - x_k)$$

$$+ x_k^T Ax_k \} - \lambda_k b^T(y_k - x_k) - b^T x_k - g(x_k)$$

$$= \frac{1}{2} \{ \lambda_k^2 (y_k - x_k)^T A (y_k - x_k) + 2 \lambda_k x_k^T A (y_k - x_k) \}$$

$$- \lambda_k b^T(y_k - x_k)$$

$$= \frac{1}{2}(x_{k+1}-x_k)^T A(x_{k+1}-x_k) + x_k^T A(x_{k+1}-x_k)$$

$$- \quad b^T(x_{k+1}-x_k)$$

$$= (x_{k+1}-x_k)^T \frac{1}{2} A(x_{k+1}-x_k) + (x_{k+1}-x_k)^T (Ax_k - b)$$

$$= (x_{k+1}-x_k)^T (D_k(y_k - x_k) + E_k(x_{k+1}-x_k) + Ax_k - b)$$

$$+ (x_{k+1}-x_k)^T(\frac{1}{2} A - \lambda_k^{-1} D_k - E_k)(x_{k+1}-x_k)$$

$$= (x_{k+1}-x_k)^T(\frac{1}{2} A - \lambda_k^{-1} D_k - E_k)(x_{k+1}-x_k)$$

$$+ (x_{k+1}-x_k)^T D_k\{y_k - x_k + D_k^{-1}[Ax_k - b + E_k(x_{k+1}-x_k)]\}$$

Because

$$y_k = P_C^{D_k}[x_k - D_k^{-1}(Ax_k - b + E_k(x_{k+1}-x_k))]$$

and $x_k \in C$, one has,according to lemma 2.1

$$(x_{k+1}-x_k)^T D_k\{y_k - x_k + D_k^{-1}[Ax_k - b + E_k(y_k - x_k)]\}$$

$$= \lambda_k < (y_k - x_k), y_k - x_k + D_k^{-1}[Ax_k - b + E_k(y_k - x_k)] >_{D_k}$$

$$\leq 0$$

Hence

$$g(x_{k+1}) - g(x_k)$$

$$\le (x_{k+1} - x_k)^T(\frac{1}{2} A - \lambda_k^{-1} D_k - E_k)(x_{k+1} - x_k)$$

i.e.

$$g(x_k) - g(x_{k+1})$$

$$\ge (x_{k+1} - x_k)^T(-\frac{1}{2} A + \lambda_k^{-1} D_k + E_k)(x_{k+1} - x_k)$$

$$\ge \gamma \| (x_{k+1} - x_k) \|^2 \qquad (3.10)$$

Suppose that there exists a subsequence $\{ x_{k_i} \}$ in $\{ x_k \}$ such that

$$x_{k_i} \longrightarrow \hat{x}$$

Because $\{ g(x_k) \}$ is monotone decreasing and

$$g(x_{k_i}) \longrightarrow g(\hat{x})$$

one gets

$$g(x_k) \longrightarrow g(\hat{x})$$

From (3.10), we know that

$$\| (x_{k+1} - x_k) \| \longrightarrow 0$$

Consequently,,we have

$$x_{k_i+1} \longrightarrow \hat{x}$$

Since

$$\| y_{k_i} - x_{k_i} \|$$

$$= \lambda_{k_i}^{-1} \| x_{k_i+1} - x_{k_i} \|$$

$$\leq \lambda^{-1} \| x_{k_i+1} - x_{k_i} \|$$

we also have

$$y_{k_i} \longrightarrow \hat{x}$$

Since $\{ D_k \}$ and $\{ E_k \}$ are bounded sequences, without loss generality we can assume that

$$D_{k_i} \longrightarrow D \ ; \ E_{k_i} \longrightarrow E$$

By assumption , D is still in GS. According to (3.5)

$$y_{k_i} = P_C^{D_{k_i}} [x_{k_i} - D_{k_i}^{-1} (Ax_{k_i} - b + E_{k_i} (x_{k_i+1} - x_{k_i}))]$$

Now using theorem 2.4, we get

$$\hat{x} = P_C^D [\hat{x} - D^{-1} (A\hat{x} - b)]$$

By (3.2), one knows that $\hat{x}$ is a solution of LVIP(A,b,C). This completes the proof.

<u>Corollary 3.4</u> We posit the assumptions of theorem 3.4. If A is positive then

$$x_k \longrightarrow x^*$$

where x^* is the unique solution of LVIP(A,b,C).

<u>Proof</u> First, we know that

$$x^* = P_C^A (A^{-1} b)$$

is the unique solution of LVIP(A,b,C). According to theorem 3.4, each accumulation point of $\{x_k\}$ is a solution of LVIP(A,b,C). Therefore it is sufficient to prove that $\{x_k\}$ is bounded.

Since

$$\frac{1}{2}\lambda_{\min}(A) \| x_k \|^2 - \| x_k \| \| b \|$$

$$\leq \frac{1}{2} x_k^T A x_k - x_k^T b$$

$$= g(x_k) \leq g(x_0) \qquad \forall k$$

one knows that $\{ x_k \}$ is bounded. This completes the proof.

The simplest way to use algorithm 3.2 is to put

$$E^k \equiv 0 \;;\; D^k \equiv \alpha I \text{ for some } \alpha > 0$$

in this case, according to lemma 3.3, each step in algorithm 3.2 is to compute

$$y_k = P_C [x_k - \alpha^{-1} (Ax_k - b)]$$

Remark The following two problems are worth to be considered.

1) the convergence of algorithm 3.2 under a weaker condition then A being positive;

2) [Ahn.1981] proved the convergence of algorithm 3.1 for the case that A is not symmetric, the same problem can be posed for algorithm 3.2.

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