Operator substitution

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Abstract

Substitution of an operator into an operator-valued map is defined and studied. A Bezout-type theorem is used to derive a number of results. The tensor map is used to formulate solvability conditions for linear matrix equations. Some applications to system theory are given.

1 Introduction

In this paper, we describe properties of operator substitution into an operator-valued analytic function. The general definitions and many properties are valid for operators defined on Banach spaces, but the most complete results can be obtained for finite-dimensional operators. The substitution operation described here is a generalization of the more familiar operation of substituting an operator A into a scalar function $f(z)$. This latter operation is described rather extensively in functional analysis (see e.g. [Dun 58, Ch. VII]) and it is attributed to M. Riess and N. Dunford. On the other hand, the substitution of an operator into an operator-valued function is only considered rarely. An example is [Gan 60, Ch. IV, §3], where the special case of substitution of matrices into a matrix-valued polynomial is mentioned.

In section 2, we will derive a number of basic results. In particular a Bezout-type theorem and, what we will call the partial substitution rule, will play an crucial role in the remainder of the paper. A number of convenient properties of substitution into scalar functions, like the product rule, are not valid in the general case, unless some commutativity assumption is made. The consequences of such an assumption are discussed in section 3. In section 4, the tensor map corresponding to a function F and an operator A is introduced. This map has nice algebraic properties. It can be used for deriving conditions for the solvability of linear operator equations. This will be the subject of section 4 and 5. Finally, in section 6, we will give some examples of application of the results in system theory.

2 Fundamental concepts

If S and T are Banach spaces, we denote by $\mathcal{L}_{ST}$ the space of linear bounded maps $S \to T$. If $\Omega \subseteq \mathbf{C}$ is a nonempty open set, we denote the space of analytic functions $\Omega \to \mathcal{L}_{ST}$ by $\mathbf{A}(\Omega \to \mathcal{L}_{ST})$. If $F \in \mathbf{A}(\Omega \to \mathcal{L}_{ST})$ and $G \in \mathbf{A}(\Omega \to \mathcal{L}_{TU})$, then $(GF) \in \mathbf{A}(\Omega \to \mathcal{L}_{SU})$ is defined by $(GF)(z) := G(z)F(z)$ for $z \in \Omega$. In particular, we write $(GC)(z)$ for $G(z)C$, where C is a constant function, i.e., $C \in \mathcal{L}_{ST}$. The expression $(BF)(z)$ for constant B is defined similarly.

In this section, we assume that S, T are Banach spaces, Ω is an open set in $\mathbf{C}$, $F \in \mathbf{A}(\Omega \to \mathcal{L}_{ST})$ and $A \in \mathcal{L}_{SS}$ is such that $\sigma(A) \subseteq \Omega$ (where $\sigma(A)$ denotes the spectrum of A).

We define the **right substitute** of A into $F(z)$ as the map

$$F(A) := \oint_{\Gamma} F(z)(zI - A)^{-1} dz, \tag{1}$$

where $\oint$ is an abbreviation of $\frac{1}{2\pi i} \int$ and Γ is a contour enclosing $\sigma(A)$ and contained in Ω. It is a consequence of Cauchy's theorem that this integral is independent of the particular choice of Γ.

REMARK 2.1 Similarly, we may define the **left substitute** of A into $F(z)$ as

$$\oint_{\Gamma} (zI - A)^{-1} F(z) dz.$$

Results about this concept will be similar. We will concentrate on the right substitute. □

If $F(z)$ is **scalar**, i.e. of the form $F(z) = f(z)I$, then $F(A) = f(A)$. Hence right substitution is a generalization of the familiar concept of substitution of a map into a scalar analytic function. Some of the properties of the scalar case remain valid in this more general situation. E.g., it is obvious that $(F+G)(A) = F(A) + G(A), F(\alpha I) = F(\alpha)$ and $(BF)(A) = BF(A)$, where $G \in \mathbf{A}(\Omega \to \mathcal{L}_{ST})$ and $B \in \mathcal{L}_{TU}$ for some Banach space U and $\alpha \in \mathbf{C}$. In particular, we have $(\lambda F)(A) = \lambda F(A)$, for $\lambda \in \mathbf{C}$. Also, if $F_n(z) \to F(z)$ $(n \to \infty)$ holds uniformly on Ω, then $F_n(A) \to F(A)$ $(n \to \infty)$. Similarly, if $H(z,\zeta)$ is analytic on $\Omega \times \Omega$ with values in $\mathcal{L}_{ST}$, it follows easily from (1) that $H(z, A)$ is analytic in Ω.

EXAMPLE 2.2 We mention a number of special cases of right substitute:

1. If F is polynomial, say, $F(z) = F_0 + F_1 z + \ldots + F_n z^n$, then $F(A) = F_0 + F_1 A + \ldots + F_n A^n$. This is an easy consequence of the linearity and the scalar case.

2. If F is given by a power series, say, $F(z) = \sum_{n=0}^{\infty} F_n z^n$ and $\sigma(A)$ is contained in the domain of convergence, then $F(A) = \sum_{n=0}^{\infty} F_n A^n$. This is a consequence of the limit property.

3. Let $F(z) := \int_a^b f(t,z)H(t)dt$, where H is a continuous $\mathcal{L}_{ST}$-valued function and f is a continuous complex function, which is analytic w.r.t. z, for $z \in \Omega$. Then $F(A) = \int_a^b H(t)f(t,A)dt$.

□

Some properties for the scalar case are no longer valid. For instance, the product rule $(fg)(A) = f(A)g(A)$ and the spectral-mapping theorem do not carry over. As to the latter property, one might for instance be tempted to expect that $F(A)$ is invertible if $F(z)$ is invertible for every $z \in \sigma(A)$. The following example shows that this is not true:

EXAMPLE 2.3 Let $S := T := \mathbf{C}^2$ and F and A given by the matrix representations:

$$F(z) := \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} = I + zN, \; A := \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \text{ where } N := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $F(z)$ is invertible for all $z \in \mathbf{C}$. On the other hand, $F(A) = I + NA$ is singular. This example also shows that the product rule does not hold.

In fact, $F(z)F(-z) = I$, but $F(A)F(-A) = I$ can not be true, because otherwise, $F(A)$ would be invertible. □

The main obstacle here is the noncommutativity of maps. In the next section, we will derive a number of results based on certain commutativity assumptions. In this section, we develop some basic results.

LEMMA 2.4 *If $P \in \mathcal{L}_{TT}$ has its spectrum in Ω, $G \in \mathbf{A}(\Omega \to \mathcal{L}_{TU})$ and $C \in \mathcal{L}_{ST}$ satisfies $PC = CA$, then $(GC)(A) = G(P)C$. In particular, we have $(FA)(A) = F(A)A$.*

PROOF: Because $(zI - P)^{-1}C = C(zI - A)^{-1}$ for $z \notin \sigma(A) \cup \sigma(P)$, the result follows immediately from the definition. □

PROPERTY 2.5 **(Bezout)**

1. *If $F(z) = M(z)(zI - A) + R$ for some $M \in \mathbf{A}(\Omega \to \mathcal{L}_{ST})$ and $R \in \mathcal{L}_{ST}$, then $F(A) = R$.*

2. *There exists a function $W \in \mathbf{A}(\Omega \to \mathcal{L}_{ST})$ such that $F(z) = F(A) + W(z)(zI - A)$.*

PROOF: The first property follows directly from the definition. As a consequence of this part, we observe that if $F(z) = M(z)z$, we have $F(z) = M(z)(zI - A) + (MA)(z)$ and hence $F(A) = M(A)A$, where we have also used Lemma 2.4.

For the proof of the second statement, we define $H(z, \zeta) := (F(z) - F(\zeta))/(z - \zeta)$. It is easily seen that H is analytic in $\Omega \times \Omega$. Therefore $W(z) := H(z, A)$ is in $\mathbf{A}(\Omega \to \mathcal{L}_{ST})$. Substitution of $\zeta = A$ into the relation $F(z) - F(\zeta) = H(z, \zeta)(z - \zeta)$ yields $F(z) - F(A) = W(z)(zI - A)$, in view of the remarks in the previous paragraph (with z replaced by ζ). □

As a result, we have the **partial-substitution rule**:

COROLLARY 2.6 *If $G \in \mathbf{A}(\Omega \to \mathcal{L}_{TU})$, then $(GF)(A) = (GF(A))(A)$.*

PROOF: Because of Property 2.5,2 we have $(GF)(z)-(GF(A))(z) = G(z)(F(z)-F(A)) = G(z)W(z)(zI - A)$ for some $W \in \mathbf{A}(\Omega \to \mathcal{L}_{ST})$. Because of Property 2.5,1, it follows that $(GF)(A) - (GF(A))(A) = 0$. □

The main result of this section is the following:

THEOREM 2.7 *Let U be a Banach space and $G \in \mathbf{A}(\Omega \to \mathcal{L}_{TU})$. Furthermore, let $P \in \mathcal{L}_{TT}$ be such that $\sigma(P) \subseteq \Omega$. Consider the following statements:*

1. *$F(z)A = PF(z)$ for all $z \in \Omega$,*
2. *$F(A)A = PF(A)$,*
3. *$(GF)(A) = G(P)F(A)$.*

Then we have 1⇒2⇒3.

PROOF:
1 ⇒ 2: We have $(PF)(z) = (FA)(z)$ and hence $PF(A) = (PF)(A) = (FA)(A) = F(A)A$.
2 ⇒ 3: $(GF)(A) = (GF(A))(A) = G(P)F(A)$, where we have applied Lemma 2.4 with $C = F(A)$. □

Next, we investigate the invertibility of $F(A)$. We say that F is **left A-invertible**, if there exists an open set Ω_1 containing $\sigma(A)$ and a function $G \in \mathbf{A}(\Omega_1 \to \mathcal{L}_{TS})$ such that $G(z)F(z) = I_S$ on $\Omega_1 \cap \Omega$.

LEMMA 2.8 *If F is left A-invertible then $F(z_0)$ is left invertible for each $z_0 \in \sigma(A)$. Conversely, if $F(z_0)$ is left invertible for each $z_0 \in \sigma(A)$ and (i) $S = T$ or (ii) $\sigma(A)$ is countable, then F is left A-invertible.*

A proof will be given in the appendix. Notice that the countability condition holds if A is compact, in particular if S is finite dimensional.

COROLLARY 2.9 *Let F be left A-invertible. Then $F(A)$ is left invertible iff there exists a map $P \in \mathcal{L}_{TT}$ such that $F(A)A = PF(A)$.*

PROOF: The 'if' part follows from $G(P)F(A) = I_S$, where G is a left inverse of F. If M is a left inverse of $F(A)$, we can take $P := F(A)AM$. □

Finally, we remark that in the finite-dimensional case, the function F may be replaced by a polynomial.

THEOREM 2.10 *If S has dimension n, there exists a polynomial map P of degree $< n$ such that $P(A) = F(A)$.*

For the proof, we need the following lemma:

LEMMA 2.11 *If $f \in \mathbf{A}(\Omega \to \mathbf{C})$ and p is a polynomial of degree n, there exists $q \in \mathbf{A}(\Omega \to \mathbf{C})$ and a polynomial r of degree $< n$ such that $f(z) = q(z)p(z) + r(z)$.*

PROOF: We may assume that p is monic. We use induction with respect to n. If $n = 1$, p is of the form $p(z) = z - a$. Then we have $f(z) = (z - a)q(z) + f(a)$ (compare Property 2.5). If the result is shown for $n - 1$, we write $p = p_1 p_2$, where $\deg p_1 = n - 1$ and $p_2 = z - a$. Then we have $f = p_1 q_1 + r_1$, with $q_1 \in \mathbf{A}(\Omega \to \mathbf{C})$ and $\deg r_1 < n - 1$. Next we substitute $q_1(z) = (z - a)q(z) + q_1(a)$ into this equation. This yields: $f(z) = p(z)q(z) + r(z)$, where $r(z) := q_1(a)p_1(z) + r_1(z)$ is a polynomial of degree $< n$. □

PROOF OF THEOREM 2.10 Write $F(z) = Q(z)p(z) + P(z)$, where $p(z) := \det(zI - A)$, and $P(z)$ is a polynomial map of degree $< n$. Then $F(A) = Q(A)p(A) + P(A) = P(A)$, because of the Cayley-Hamilton theorem. Notice that $(Qp)(A) = Q(A)p(A)$ is a consequence of Theorem 2.7 □

REMARK 2.12 The Cayley-Hamilton Theorem is actually an easy consequence of our results: If A is a map in a finite-dimensional space then, according to Cramer's rule, we have $\operatorname{adj}(A)A = \det(A)I$. Replacing A with $zI - A$, we find that $B(z)(zI - A) = p(z)I$, where $B(z)$ is a polynomial and $p(z)$ is the characteristic polynomial of A. Substituting $z = A$ gives $p(A) = 0$. □

3 The commutative case

In this section we are going to assume the following set up:

- S, T are Banach spaces,
- $\Omega \subseteq \mathbf{C}$ is open,

- $F \in \mathbf{A}(\Omega \to \mathcal{L}_{SS})$,
- $A \in \mathcal{L}_{SS}$ is such that $\sigma(A) \subseteq \Omega$,
- $F(A)A = AF(A)$. This condition holds if $F(z)A = AF(z)$ for all $z \in \Omega$.

In this situation, F is A-invertible iff $F(z_0)$ is invertible for all $z_0 \in \sigma(A)$. Special cases of the theorems of the previous section are the following:

THEOREM 3.1

- *If* $G \in \mathbf{A}(\Omega \to \mathcal{L}_{ST})$ *then* $(GF)(A) = G(A)F(A)$.
- *If* F *is* A*-invertible then* $F(A)$ *is invertible.*

The A-invertibility of F is not necessary for $F(A)$ to be invertible.

EXAMPLE 3.2 Define over $S := \mathbf{C}^2$:

$$F(z) := \begin{bmatrix} z & 0 \\ 0 & z+1 \end{bmatrix} = I + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A := \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$F(A) = I + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

□

We can obtain information about the spectrum of $F(A)$:

COROLLARY 3.3 $\sigma(F(A)) \subseteq \cup_{\mu \in \sigma(A)} \sigma(F(\mu))$.

PROOF: If $\lambda \in \sigma(F(A))$, then $F(A) - \lambda I$ is not invertible. Because of the previous lemma, $F(z) - \lambda I$ is not A-invertible. Hence, there exists $\mu \in \sigma(A)$ for which $F(\mu) - \lambda I$ is not invertible. □

REMARK 3.4 This is a one-sided version of the spectral-mapping theorem. The two-sided version is not valid because the sets are usually not equal, as follows from the previous example. In fact, $\cup_{\mu\in\sigma(A)}\sigma(F(\mu))$ can be quite a lot bigger than $\sigma(F(A))$. For example if $S = \mathbb{C}^n, F(z) := \text{diag}\,(f_1(z),\ldots,f_n(z))$ and $A := \text{diag}\,(a_1,\ldots,a_n)$, then $F(A) = \text{diag}\,(f_1(a_1),\ldots,f_n(a_n))$. Hence $\sigma(F(A)) = \{f_1(a_1),\ldots,f_n(a_n)\}$, whereas $\cup_{\mu\in\sigma(A)}\sigma(F(\mu)) = \{f_j(a_k)|j,k = 1,\ldots,n\}$. □

Finally, we can derive a formula for a composite function:

COROLLARY 3.5 *Let Ω_1 be an open set in $\mathbb{C}$ which contains the closure Λ of $\cup_{\mu\in\sigma(A)}\sigma(F(\mu))$ and let $G \in \mathrm{A}(\Omega_1 \to \mathcal{L}_{ST})$. Then $(G\circ F)(A) = G(F(A))$.*

PROOF: Define $H(z,\zeta) := (zI - F(\zeta))^{-1}$ for $z \in \mathbb{C}, \zeta \in \Omega$ such that $z \notin \sigma(F(\zeta))$. Substituting $\zeta = A$ into the equality $H(z,\zeta)(zI - F(\zeta)) = I$, we find $H(z,A) = (zI - F(A))^{-1}$ for $z \notin \Lambda$. We choose a contour Γ_1 enclosing Λ and contained in Ω_1. Furthermore, Γ_2 is a contour enclosing $\sigma(A)$ and contained in Ω. Then

$$G(F(A)) = \oint_{\Gamma_1} G(z)(zI - F(A))^{-1}dz =$$

$$= \oint_{\Gamma_1} G(z) \oint_{\Gamma_2} H(z,\zeta)(\zeta I - A)^{-1}d\zeta dz =$$

$$\oint_{\Gamma_2}\oint_{\Gamma_1} G(z)(zI - F(\zeta))^{-1}dz(\zeta I - A)^{-1}d\zeta =$$

$$\oint_{\Gamma_2} G(F(\zeta))(\zeta I - A)^{-1}d\zeta = (G\circ F)(A).$$

□

Notice that, because of Remark 3.4, it may happen that $G(F(A))$ is defined whereas $(G\circ F)(A)$ is not.

4 The tensor map

In this section we drop the commutativity assumption. We assume that S, T, U are Banach spaces, $\Omega \subseteq \mathbb{C}$ is open, $F \in \mathrm{A}(\Omega \to \mathcal{L}_{TU})$ and $A \in \mathcal{L}_{SS}$ such that $\sigma(A) \subseteq \Omega$.

We will define and study a map, $F_A : \mathcal{L}_{ST} \to \mathcal{L}_{SU}$. This map can be used to investigate the solvability of a certain class of linear map equations.

DEFINITION 4.1 *The **(right) tensor map** $F_A : \mathcal{L}_{ST} \to \mathcal{L}_{SU}$ is defined by $F_A X := (FX)(A)$.*

Recall that (FX) stands for the function $z \mapsto F(z)X$.

THEOREM 4.2 *We have the following properties:*

1. $(F + G)_A = F_A + G_A$,
2. $I_A = I$, $(BF)_A = BF_A$,
3. $(GF)_A = G_A F_A$,

provided that in each case the map B and the domains of the functions G are such that the algebraic formulas are well defined.

PROOF: We only show property 3. For $X \in \mathcal{L}_{ST}$, we define $Y := (FX)(A) = F_A X$ and we find

$$(GF)_A X = ((GF)X)(A) = (G(FX))(A) = (G(FX)(A))(A)$$
$$= (GY)(A) = G_A Y = G_A F_A X.$$

□

COROLLARY 4.3 *If F is left A-invertible, then F_A is left invertible. If S and T are finite dimensional, the converse implication holds. Similar statements can be made about right invertibility.*

PROOF: Let G be an analytic left inverse of F on some set Ω_1, containing $\sigma(A)$. Then $G(z)F(z) = I$ implies $G_A F_A = I$.

Let S and T be finite dimensional. Because of Lemma 2.8, it suffices to show that $F(\lambda)$ is injective (and hence left invertible) for every $\lambda \in \sigma(A)$. Suppose that there exists $v \neq 0$ satisfying $F(\lambda)v = 0$. We will identify v with the (injective) map $\alpha \mapsto \alpha v : \mathbf{C} \to S$. We also have an eigenvector $w \in S'$ of the adjoint map $A^* : S' \to S'$ corresponding to the eigenvalue λ. Then $wA = \lambda w$ (Here wA denotes the composition of maps. Recall that $w : S \to \mathbf{C}$). If we define $X := vw$, then $X \neq 0$ and we obtain

$$F_A X = (Fvw)(A) = ((Fv)w)(A) = (Fv)(\lambda)w = F(\lambda)vw = 0,$$

where we have used Lemma 2.4, with $C = w, P = \lambda$.

The statements about right invertibility follow by duality. □

We can also obtain a form of the spectral mapping theorem.

THEOREM 4.4 *Suppose that* $F \in \mathbf{A}(\Omega \to \mathcal{L}_{TT})$ *and hence* $F_A : \mathcal{L}_{ST} \to \mathcal{L}_{ST}$. *Then*

$$\sigma(F_A) \subseteq \cup_{\mu \in \sigma(A)} \sigma(F(\mu)).$$

If S and T are finite dimensional, then we have equality.

PROOF: If $\lambda \in \sigma(F_A)$, then $F_A - \lambda I$ is not invertible. By Corollary 4.3, it follows that $F(z) - \lambda I$ is not A-invertible. Consequently, $F(\mu) - \lambda I$ is not invertible for some $\mu \in \sigma(A)$. Hence $\lambda \in \cup_{\mu \in \sigma(A)} \sigma(F(\mu))$.
If S and T are finite dimensional, the converse implication chain can be made, because, if $F(z) - \lambda I$ is A-invertible then $F_A - \lambda I$ is invertible. □

Next we show how the tensor map appears as derivative of a nonlinear operator map. Let $F(z)$ be an $\mathcal{L}_{SS}$-valued function on Ω. Then $F(z)$ defines the map $\mathcal{F} : X \mapsto F(X) : \mathcal{L}_{SS} \to \mathcal{L}_{SS}$, defined for X with spectrum contained in Ω. We are interested in the derivative (or linearization) of this map. Given A with spectrum in Ω, we have for small Y:

$$F(A+Y) - F(A) = \oint_\Gamma F(z)\{(zI - A - Y)^{-1} - (zI - A)^{-1}\}dz =$$

$$= \oint_\Gamma F(z)(zI - A - Y)^{-1} Y (zI - A)^{-1} dz.$$

It follows easily that the required linearization is

$$\mathcal{L} : Y \mapsto \oint_\Gamma F(z)(zI - A)^{-1} Y (zI - A)^{-1} dz.$$

We can also write this as

$$\mathcal{L}(Y) = \oint_\Gamma W(z) Y (zI - A)^{-1} dz = (WY)(A) = W_A Y,$$

where $W(z) := (F(z) - F(A))(zI - A)^{-1}$. (See Property 2.5)
Here we have used that

$$\oint_{\Gamma}(zI - A)^{-1}Y(zI - A)^{-1}dz = 0,$$

as one can easily see by letting Γ be a circle with radius tending to ∞ and using the estimate $\| (zI - A)^{-1} \| \leq M \| z \|^{-1}$ for $\| z \|$ sufficiently large.

We can use this result to investigate the local invertibility of the map $\mathcal{F}$. To this extent, we apply the inverse-function theorem. This theorem states that $\mathcal{F}$ is locally invertible at A iff $\mathcal{L}$ is invertible, i.e., iff W_A is invertible. According to Corollary 4.3, this is the case if W is A-invertible. The latter condition can be written as: $W(\lambda)$ is invertible for $\lambda \in \sigma(A)$. In the finite-dimensional case, this condition is also necessary.

In the special case of substitution of a map into a scalar function $F(z) = f(z)$, this condition can be simplified. In this case, according to the spectral mapping theorem, $W(\lambda)$ is invertible iff $H(\mu, \lambda) \neq 0$ for $\mu \in \sigma(A)$. Here, $H(z, \zeta) := (f(z) - f(\zeta))/(z - \zeta)$. (Recall that $W(z) = H(z, A)$.) As a consequence, we find:

COROLLARY 4.5 *If S is a finite-dimensional space, $\mathcal{F}$ is defined by $\mathcal{F} : X \mapsto f(X) : S \to S$, where $f(z)$ is a scalar analytic function on Ω, and $A \in \mathcal{L}_{SS}$ is such that $\sigma(A) \subseteq \Omega$, then $\mathcal{F}$ is locally invertible at A iff*

- $f(\lambda) \neq f(\mu)$ $(\lambda, \mu \in \sigma(A),\ \lambda \neq \mu)$
- $f'(\lambda) \neq 0$ $(\lambda \in \sigma(A)$.

Notice that these conditions are exactly the conditions for the function $f(z)$ to be locally invertible on $\sigma(A)$, i.e., for the existence of a function $g(z)$, analytic in an neighborhood of $f(\sigma(A))$, such that $g(f(z)) = z$. Hence the local inverse of $\mathcal{F}$ is given by $\mathcal{G} : X \mapsto g(X)$. We find that we have the following:

COROLLARY 4.6 *If $\mathcal{F} : X \mapsto f(X)$ has a C^1 inverse at a certain map A, then there is an inverse of the form $\mathcal{G} : X \mapsto g(X)$, where g is an analytic function on some neighborhood of A.*

Notice that not every function that is analytic in the neighborhood of a certain map has the representation $g(X)$ (take e.g. $\mathcal{G} = X^T$, the transposed map, with respect to a given basis).

5 Operator equations: Universal solvability

In this section, we investigate the equation

$$(FX)(A) = C \tag{2}$$

for $X \in \mathcal{L}_{ST}$, where $C \in \mathcal{L}_{ST}$. Recall that $(FX)(A)$ denotes the result of substitution of A into the function $z \mapsto F(z)X$. We restrict ourselves to the finite-dimensional case, where the most complete results can be obtained. However, a number of the results, in particular the sufficiency parts, can be generalized to the general case.

We will call equation (2) **universally solvable** if it has a solution for every C. If a solution exists for a particular C, the equation is called **individually solvable**. The following general condition for the universal solvability of equation (2) is an immediate consequence of Corollary 4.3:

THEOREM 5.1 *Equation (2) is universally solvable iff F is right A-invertible, i.e., iff $F(\lambda)$ is right invertible for every $\lambda \in \sigma(A)$. Specifically, a solution is given by $X := (GC)(A)$, where G is a right inverse analytic on $\sigma(A)$.* □

EXAMPLE 5.2 **Sylvester's equation** reads $BX - XA = C$, where $A \in \mathcal{L}_{SS}, B \in \mathcal{L}_{TT}$ and $C \in \mathcal{L}_{ST}$ are given maps. This can be seen as $(FX)(A) = C$, where $F(z) := B - zI$. Theorem 5.1 yields a well-known result: This equation is universally solvable iff $F(\lambda)$ is invertible for every $\lambda \in \sigma(A)$, i.e., iff A and B have no common eigenvalue (see [Mac 60, Theorem 46.2]). □

EXAMPLE 5.3 More generally, consider the equation

$$\sum_{j=0}^{k} p_j(B)Xq_j(A) = C,$$

where p_j and q_j are functions, analytic on $\sigma(B)$ and $\sigma(A)$, respectively, and A and B are as in the previous example. Here we find universal solvability in terms of $p(z,\zeta) := \sum_{j=0}^{k} p_j(z)q_j(\zeta)$. In fact the equation is universally solvable iff $p(\lambda,\mu) \neq 0$ for $\lambda \in \sigma(B), \mu \in \sigma(A)$ (see [Mac 60, Theorem 43.8]). □

EXAMPLE 5.4 We get a further generalization if we consider an equation of the form

$$\sum_{j=0}^{k} F_j X q_j(A) = C,$$

where $F_j \in \mathcal{L}_{ST}$ for $j = 0, \ldots, k$. Now we find universal solvability iff the polynomial map $F(z) := \sum_{j=0}^{k} F_j q_j(z)$ is left invertible for $z \in \sigma(A)$. This result was given in [Hau 82]. The special case where $S = T$ of this result was given in [Dat 66] and [Wim 74]. A recent discussion an an algebraic treatment is given in [Wim 92].

□

EXAMPLE 5.5 Let

$$F(z) := \int_0^T e^{t(B-sI)} dt,$$

where B is a map in $S \to S$ and S a finite-dimensional linear space. Then $F(z)$ is an entire function and

$$(FX)(A) = \int_0^T e^{tB} X e^{-tA} dt.$$

Now F is left A-invertible iff

$$\int_0^T e^{(\lambda-\mu)t} dt \neq 0,$$

for $\lambda, \mu \in \sigma(A)$, i.e., iff $\lambda - \mu \neq 2\pi i k/T$ for any nonzero integer k. Hence the equation

$$\int_0^T e^{tB} X e^{-tA} dt = C$$

is universally solvable iff for all nonzero $k \in \mathbb{Z}$, we have $2\pi i k/T \notin \sigma(A) - \sigma(B)$. (Compare [Ros 66, §14.2]). □

6 Operator equations: Individual solvability

For individual solvability, we will give a generalization of Roth's Theorem.

THEOREM 6.1 **(Roth)** *Let S and T be finite-dimensional linear spaces and $A \in \mathcal{L}_{SS}, B \in \mathcal{L}_{TT}, C \in \mathcal{L}_{ST}$ linear maps. Then the equation $BX - XA = C$ has a solution if and only if the maps*

$$\begin{bmatrix} B & C \\ 0 & A \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} \tag{3}$$

in $\mathcal{L}_{T\oplus S,T\oplus S}$ are similar.

(See [Rot 52]) In order to be able to generalize this theorem, we reformulate it. According to [Gan 60, VI, §4 and §5], two maps M and N are similar iff $zI - M$ and $zI - N$ are C$[z]$-equivalent, i.e., iff there exist invertible polynomial maps $P(z)$ and $Q(z)$ such that $P(z)(zI - M) = (zI - N)Q(z)$. Using the theory of section 2, we can give an easy proof of a generalization of this result, which also holds for infinite-dimensional maps. We will say that P and Q in $\mathbf{A}(\Omega \to \mathcal{L}_{SS})$ are **Ω-equivalent** if there exist maps $F, G \in \mathbf{A}(\Omega \to \mathcal{L}_{SS})$ invertible in Ω and satisfying $P(z)F(z) = G(z)Q(z)$.

THEOREM 6.2 *Let $B, A \in \mathcal{L}_{SS}, \Omega \supseteq \sigma(B) \cup \sigma(A)$. Then $zI - B$ and $zI - A$ are Ω-equivalent if and only if B and A are similar.*

PROOF: If B and A are similar, say $B = P^{-1}AP$ then $(zI - B)F(z) = G(z)(zI - A)$, where $F(z) := G(z) := P$. On the other hand, if $(zI - B)F(z) = G(z)(zI - A)$, we substitute $z = A$ into this equation and obtain $F(A)A = BF(A)$. Now the result follows, since by Corollary 2.9, $F(A)$ is invertible. □

Consequently, Roth's theorem can be reformulated as: *The equation $BX - XA = C$ has a solution iff*

$$\begin{bmatrix} zI - B & -C \\ 0 & zI - A \end{bmatrix}, \begin{bmatrix} zI - B & 0 \\ 0 & zI - A \end{bmatrix}$$

are Ω-equivalent.

The extension of Roth's Theorem reads as follows:

THEOREM 6.3 *Let $F \in \mathbf{A}(\Omega \to \mathcal{L}_{ST}), A \in \mathcal{L}_{SS}$ and $C \in \mathcal{L}_{ST}$. Then the following statements are equivalent:*

1. *The equation $(FX)(A) = C$ has a solution $X \in \mathcal{L}_{ST}$.*
2. *The equation*
$$F(z)U(z) + V(z)(zI - A) = C \tag{4}$$
has a solution $(U(z), V(z)) \in \mathbf{A}(\Omega \to \mathcal{L}_{SS}) \times \mathbf{A}(\Omega \to \mathcal{L}_{ST})$.
3. *The maps*
$$\begin{bmatrix} F(z) & -C \\ 0 & zI - A \end{bmatrix}, \begin{bmatrix} F(z) & 0 \\ 0 & zI - A \end{bmatrix}$$
are Ω-equivalent.

PROOF: 1⇒2: According to Property 2.5, there exists a function $W \in \mathbf{A}(\Omega \to \mathcal{L}_{ST})$ such that

$$F(z)X = (FX)(z) = (FX)(A) + W(z)(zI - A) = C + W(z)(zI - A).$$

Hence, we can take $U(z) := X, V(z) := -W(z)$.

2⇒1: Right substsitution of $z = A$ into (4) yields

$$(FU(A))(A) = (FU)(A) = C,$$

so that $X := U(A)$ is a solution.

2⇔3: We apply Gustafson's extension of Roth' Theorem to general commutative rings (see [Gus 79]). In this paper, Gustafson proves that the matrix equation $AU + VB = C$ over an arbitrary commutative ring $\mathcal{R}$ has a solution iff the matrices 3 are $\mathcal{R}$-equivalent. We obtained our desired equivalence by applying this result with the ring of analytic functions on Ω and interpreting the maps as matrices. □

7 Examples in system theory

We give a few examples to demonstrate how the concepts of operator substitution can be useful in system theory. We assume the systems to be finite dimensional. Let L and S be finite-dimensional spaces. We are interested in the equation

$$V(z)G(z) + W(z)(zI - A) = F(z), \tag{5}$$

where $G \in \mathbf{A}(\Omega \to \mathcal{L}_{ST}), A \in \mathcal{L}_{SS}, F \in \mathbf{A}(\Omega \to \mathcal{L}_{ST}), \Omega \supseteq \sigma(A)$ and V, W are the sought functions, which we require to be analytic on Ω. The special case of this equation where $G(z) = C$ is constant, or rather its dual, appears when one tries to find a state feedback achieving some prescribed (matrix) denominator for the transfer function. Specifically, if the system is described by the frequency-domain equations $zx = Ax + Bu$, a feedback of the form $u = Mx + v$, where $M = ND^{-1}$ is the transfer function of the compensator, yields $x = (zI - A - BM)^{-1}Bv = D((zI - A)D - BN)^{-1}B$. So we see that $(zI - A)D - BN$ is the denominator of the transfer function. One is interested in finding a polynomials N and D such that this denominator takes a prescribed value. A additional restriction is that ND^{-1} be (possibly strictly) proper. Equation (5) (with $G(z) = C$) is a dual version of this equation.

THEOREM 7.1 *Let $V \in \mathbf{A}(\Omega \to \mathcal{L}_{TS})$. Then there exists $W \in \mathbf{A}(\Omega \to \mathcal{L}_{SS})$ such that (V, W) is a solution of equation 5 if and only if $(VG)(A) = (VG(A))(A) = F(A)$. Consequently, there exists a solution iff the equation $V(z)G(z) + W(z)(zI - A) = F(A)$ has a solution.*

PROOF: If (V, W) is a solution, we can substitute $z = A$ and find $(VG)(A) = F(A)$. Conversely, if $(VG)(A) = F(A)$, we find $(VG)(z) - F(A) = (VG)(z) - (VG)(A) = -W(z)(zI - A)$ for some $W \in \mathbf{A}(\Omega \to \mathcal{L}_{SS})$, according to property 2.5. □

A pair of maps (C, A), where $C \in \mathcal{L}_{ST}, A \in \mathcal{L}_{SS}$ is called **observable** if the map

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

is left invertible or, equivalently,

$$\text{rank}\begin{bmatrix}\lambda I - A\\ C\end{bmatrix} = n,$$

for every $\lambda \in \sigma(A)$, where n denotes the dimension of S. We have the following corollary:

COROLLARY 7.2 *Equation 5 has a solution (V, W) iff the equation $(VG(A))(A) = F(A)$ has a solution V. Furthermore, the follwing statements are equivalent:*

- *equation (5) is universally solvable (i.e., has a solution for every F).*
- $$\text{rank}\begin{bmatrix}\lambda I - A\\ F(\lambda)\end{bmatrix} = n, \tag{6}$$
  for every $\lambda \in \sigma(A)$
- *$(F(A), A)$ is observable.* □

Here are two examples where one encounters the condition (6).

EXAMPLE 7.3 **(Cascade connection)** Consider a series connection Σ_{ser} of two observable systems $\Sigma_i : zx_i = A_i x_i + B_i u_i, y_i = C_i x_i + D_i u_i$, with state-space dimensions n_i for $i = 1, 2$. Assume that $\sigma(A_1) \cap \sigma(A_2) = \emptyset$. Then it is known (see [Hau 75]) that Σ_{ser} is observable iff $\text{rank}\, H(\lambda) = n_2$ for all $\lambda \in \sigma(A_2)$, where

$$H(z) := \begin{bmatrix} zI - A_2\\ C_2 F_1(z)\end{bmatrix},$$

and $F_1(z) := C_1(zI - A_1)^{-1}B_1 + D_1$. □

EXAMPLE 7.4 **(Sampling)** Consider the observed continuous-time system

$$\dot{x} = Ax, y = Cx,$$

where $C \in \mathcal{L}_{ST}, A \in \mathcal{L}_{SS}$. Assume that the the output is sampled with sampling period τ via the sampling mechanism

$$y_k = \int_0^\tau (dR(\theta))y(\theta + k\tau)\ (k = 0, 1, \ldots),$$

where $R(\theta) \in \mathcal{L}_{TU}$ and U is a finite-dimensional linear space. We assume that R is of bounded variation. The sampling operation results in a discrete-time system

$$x_{k+1} = Fx_k, y_k = Hx_k,$$

where $F := e^{\tau A}, H := \int_0^\tau (dR(\theta))Ce^{\theta A}$. In [Hau 72], it is shown that the sampled system is observable iff

$$\operatorname{rank} \begin{bmatrix} \lambda I - A \\ N(\lambda)C \end{bmatrix} = n,$$

for every $\lambda \in \sigma(A)$, where $N(\lambda) := \int_0^\tau e^{\lambda t} dR(t)$.

□

The results found in this section enable us to give a solvability condition for the following "Operator-interpolation problem" (cmp. [BGR 90]):

PROBLEM 7.5 *Given maps $A_i \in \mathcal{L}_{SS}, C_i \in \mathcal{L}_{ST}, M_i \in \mathcal{L}_{SS}$ for $i = 1, \ldots, k$, determine an open set $\Omega \subseteq \mathbb{C}$ such that $\sigma(A_i) \subseteq \Omega$, $(i = 1, \ldots, k)$, and a map $F \in \mathbf{A}(\Omega \to \mathcal{L}_{TS})$ such that $(FC_i)(A_i) = M_i$ $(i = 1, \ldots, k)$.*

THEOREM 7.6 *In the situation of the previous problem, there exists a solution iff $(\bar{C}, \bar{A})$ is observable, where*

$$\bar{C} := [C_1, \ldots, C_k],$$

$$\bar{A} := \operatorname{diag}(A_1, \ldots, A_k).$$

PROOF: The relations $(FC_i)(A_i) = M_i$ $(i = 1, \ldots, k)$ are equivalent to $(F\bar{C})(\bar{A}) = [M_1, \ldots, M_k]$. Hence we can apply Corollary 7.2 □

COROLLARY 7.7 *If (C_i, A_i) is observable for $i = 1, \ldots, k$ and $\sigma(A_i) \cap \sigma(A_j) = \emptyset$ for $i \neq j$, Problem 7.5 has a solution.*

PROOF: The conditions of the Corollary imply that $(\bar{C}, \bar{A})$ is observable. □

Finally, we have

THEOREM 7.8 *If equation (5) has a solution, there is also a solution where V is a polynomial of degree $< n$, where $n = \dim S$.*

PROOF: According to lemma 2.11 we can write $V(z) = Q(z)p(z) + R(z)$, where $p(z) := \det(zI - A)$ and $R(z)$ is a polynomial map of degree $< n$. Then we have $(VG)(z) = (QG)(z)p(z) + (RG)(z)$, and hence $(VG)(A) = (QG)(A)p(A) + (RG)(A) = (RG)(A)$. □

It is a consequence of this result that if a solution of equation (5) can be found, and if $F(z)$ is a monic polynomial with $\deg F(z) \geq n$, we can find a solution (U, V) with $\deg V < n$. It is easily seen that then U must be a monic polynomial with $\deg U \geq n - 1$. Hence VU^{-1} is proper.

For applications of the solvability conditions of sections 4 and 5 to the regulator problem, we refer to [Hau 83].

8 Appendix: Proof of Lemma 2.8

The first statement is obvious. Assume now that $S = T$ and that $F(z)$ is invertible for $z \in \sigma(A)$. Then $G(z) := (F(z))^{-1}$ exists and is analytic in some neighborhood Ω_1 of $\sigma(A)$ (See [Kat 66, Ch. 7, §1]).

The rest of the proof is concerned with the case where $\sigma(A)$ is countable. Let $\lambda_1, \lambda_2, \ldots$ be an enumeration of $\sigma(A)$. The map $F(\lambda_j)$ is left invertible for every j, with left inverse, say G_j. Since $G_j F(\lambda_j) = I$, there exists $\varepsilon_j > 0$ such that $G_j F(z)$ is invertible for $|z - \lambda_j| < \varepsilon_j$. Next we construct a sequence of discs D_k in the following way. $D_1 := \{z \in \mathbb{C} \,||z - \lambda_1| < \varepsilon_1^*\}$, where $\varepsilon_1^* \leq \varepsilon_1$ is chosen positive and such that the boundary of D_1 contains no points of $\sigma(A)$ and $D_1 \subseteq \Omega$. If $D_1, \ldots, D_k$ have been constructed, we choose the first λ_j that is not contained in $V := \cup_{m=1}^{k} D_m$. Then, by construction, we know that $\lambda_j \notin \bar{V}$. Hence we can find a positive $\varepsilon_{k+1}^* \leq \varepsilon_j$ such that the disc $D_{k+1} := \{z \in \mathbb{C} \,||z - \lambda_j| < \varepsilon_{k+1}^*\}$ is disjoint with $\bar{V}$, is contained in Ω and contains no points of $\sigma(A)$ on its boundary. We can continue this way until $\sigma(A)$ is contained in $\Omega_1 := \cup_{m=1}^{k} D_m$. Note that this must happen for a finite k, because of the compactness of $\sigma(A)$. (Otherwise we would get a countable sequence of open discs covering $\sigma(A)$, which we could reduce to a finite covering.) Now we can define the function $H : \Omega_1 \to \mathcal{L}_{TS}$ by $H(z) := G_j$ for z in the disc D_k with center λ_j. Then $H(z)F(z)$ is analytic and invertible on Ω_1. Hence we can take $G := (HF)^{-1}H$.

REMARK 8.1 The result can be generalized. To this extent, we define a set $\Lambda \subseteq \mathbf{C}$ to be **totally disconnected** if for all $\varepsilon > 0$, there exists a finite set of points $z_1, \ldots, z_n$ such that $\Lambda \subseteq \cup_{j=1}^{n} B(z_j, \varepsilon)$ and $B(z_j, \varepsilon) \cap B(z_k, \varepsilon) = \emptyset$ for $j \neq k$. Here $B(a, r) := \{z \in \mathbf{C} | |z - a| < r\}$. Then we have: *If $F(z_0)$ is left invertible for each $z_0 \in \sigma(A)$ and $\sigma(A)$ is totally disconnected, then F is left A-invertible.* □

References

[BGR 90] J.A.BALL, I. GOHBERG, L. RODMAN, *Interpolation of Rational Matrix Functions*, Birkhäuser Verlag, Basel, 1990

[Dat 66] G.S. DATUASHVILI, "On the spectrum of a generalized matrix polynomial" (in Russian), *Bulletin of the Academy of Sciences of the Georgian SSR*, 44, pp. 7-9, 1966

[Dun 58] N. DUNFORD & J.T. SCHWARTZ, *Linear Operators, Part I*, Interscience, New York, 1958

[Gan 60] F.R. GANTMACHER, *The Theory of Matrices I*, Chelsea, New York, 1960

[Gus 79] W.H. GUSTAFSON, "Roth's theorem over commutative rings", *Linear Algebra and Applications*, 23, pp. 245-251,1979

[Hau 72] M.L.J. HAUTUS, "Controllability and observability of sampled systems", *IEEE Trans. on Automat. Control*, AC-17, pp. 528-531, 1972

[Hau 75] M.L.J. HAUTUS, "Input regularity of cascaded systems", *IEEE Trans. on Automat. Control*, AC-20, pp. 120-123, 1975

[Hau 82] M.L.J. HAUTUS, "On the solvability of linear matrix equations" *Memorandum 1982-07, Dept. of Math. Eindhoven University of Technology*, Eindhoven, 1982

[Hau 83] M.L.J. HAUTUS, "On Linear matrix equations and applications to the regulator problem", in *Math. Tools and Models for Control, Systems Analysis and Signal Processing, CNRS*, 3, pp. 399-412, Paris 1983

[Kat 66] T. KATO, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966

[Mac 60] C.C. MACDUFFEE, *The Theory of Matrices*, Acad. Press, New York, 1972

[Ros 66] M. ROSEAU, *Vibrations non linéaires et théorie de la stabilité*, Springer, Berlin, 1966

[Rot 52] W.E. ROTH, "The equations $AX-YB=C$ and $AX-XB=C$ in matrices", *Proc. Amer. Math. Soc.*, 3, pp. 292-396, 1952

[Wim 74] H.K. WIMMER & A.D. ZIEBUR, "Blockmatrizen und lineare Matrizengleichungen", *Math, Nachr.*, 59, pp. 213-219, 1974

[Wim 92] H.K. WIMMER, "Explicit solutions of the matrix equation $\sum A^i X D_i = C$", *SIAM J. Matrix Anal. Appl.*, 13, pp. 1123-1130, 1992

List of COSOR-memoranda - 1992

Number	Month	Author	Title
92-01	January	F.W. Steutel	On the addition of log-convex functions and sequences
92-02	January	P. v.d. Laan	Selection constants for Uniform populations
92-03	February	E.E.M. v. Berkum H.N. Linssen D.A. Overdijk	Data reduction in statistical inference
92-04	February	H.J.C. Huijberts H. Nijmeijer	Strong dynamic input-output decoupling: from linearity to nonlinearity
92-05	March	S.J.L. v. Eijndhoven J.M. Soethoudt	Introduction to a behavioral approach of continuous-time systems
92-06	April	P.J. Zwietering E.H.L. Aarts J. Wessels	The minimal number of layers of a perceptron that sorts
92-07	April	F.P.A. Coolen	Maximum Imprecision Related to Intervals of Measures and Bayesian Inference with Conjugate Imprecise Prior Densities
92-08	May	I.J.B.F. Adan J. Wessels W.H.M. Zijm	A Note on "The effect of varying routing probability in two parallel queues with dynamic routing under a threshold-type scheduling"
92-09	May	I.J.B.F. Adan G.J.J.A.N. v. Houtum J. v.d. Wal	Upper and lower bounds for the waiting time in the symmetric shortest queue system
92-10	May	P. v.d. Laan	Subset Selection: Robustness and Imprecise Selection
92-11	May	R.J.M. Vaessens E.H.L. Aarts J.K. Lenstra	A Local Search Template (Extended Abstract)
92-12	May	F.P.A. Coolen	Elicitation of Expert Knowledge and Assessment of Imprecise Prior Densities for Lifetime Distributions
92-13	May	M.A. Peters A.A. Stoorvogel	Mixed H_2/H_∞ Control in a Stochastic Framework

Number	Month	Author	Title
92-14	June	P.J. Zwietering E.H.L. Aarts J. Wessels	The construction of minimal multi-layered perceptrons: a case study for sorting
92-15	June	P. van der Laan	Experiments: Design, Parametric and Nonparametric Analysis, and Selection
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92-21	June	J.A. Hoogeveen S.L. van de Velde	Polynomial-time algorithms for single-machine bicriteria scheduling
92-22	June	P. van der Laan	The best variety or an almost best one? A comparison of subset selection procedures
92-23	June	T.J.A. Storcken P.H.M. Ruys	Extensions of choice behaviour
92-24	July	L.C.G.J.M. Habets	Characteristic Sets in Commutative Algebra: an overview
92-25	July	P.J. Zwietering E.H.L. Aarts J. Wessels	Exact Classification With Two-Layered Perceptrons
92-26	July	M.W.P. Savelsbergh	Preprocessing and Probing Techniques for Mixed Integer Programming Problems

Number	Month	Author	Title
92-27	July	I.J.B.F. Adan W.A. van de Waarsenburg J. Wessels	Analysing $E_k \| E_r \| c$ Queues
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92-30	August	G.A.P. Kindervater M.W.P. Savelsbergh	Local Search in Physical Distribution Management
92-31	August	M. Makowski M.W.P. Savelsbergh	MP-DIT Mathematical Program data Interchange Tool
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92-33	August	O.J. Boxma J.A.C. Resing	Tandem queues with deterministic service times
92-34	September	J.H.J. Einmahl	A Bahadur-Kiefer theorem beyond the largest observation
92-35	September	F.P.A. Coolen	On non-informativeness in a classical Bayesian inference problem
92-36	September	M.A. Peters	A Mixed H_2/H_∞ Function for a Discrete Time System
92-37	September	I.J.B.F. Adan J. Wessels	Product forms as a solution base for queueing systems
92-38	September	L.C.G.J.M. Habets	A Reachability Test for Systems over Polynomial Rings using Gröbner Bases
92-39	September	G.J. van Houtum I.J.B.F. Adan J. Wessels W.H.M. Zijm	The compensation approach for three or more dimensional random walks

Number	Month	Author	Title
92-40	September	F.P.A. Coolen	Bounds for expected loss in Bayesian decision theory with imprecise prior probabilities
92-41	October	H.J.C. Huijberts H. Nijmeijer A.C. Ruiz	Nonlinear disturbance decoupling and linearization: a partial interpretation of integral feedback
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92-46	November	C.P.M. van Hoesel A.P.M. Wagelmans L.A. Wolsey	Polyhedral characterization of the Economic Lot-sizing problem with Start-up costs
92-47	November	C.P.M. van Hoesel A. Kolen	A linear description of the discrete lot-sizing and scheduling problem
92-48	November	L.C.G.J.M. Habets	A Reliable Stability Test for Exponential Polynomials
92-49	November	E.H.L. Aarts J. Wessels P.J. Zwietering	The Applicability of Neural Nets for Decision Support
92-50	December	F.P.A. Coolen M.J. Newby	Bayesian Reliability Analysis with Imprecise Prior Probabilities
92-51	December	M.J.L. Hautus	Operator substitution