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(A,B)-invariant and stabilizability subspaces,
a frequency domain description
with applications

by

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The Netherlands

(A,B)-INVARIANT AND STABILIZABILITY SUBSPACES,
A FREQUENCY DOMAIN DESCRIPTION WITH APPLICATIONS

by M.L.J. Hautus

ABSTRACT. In a paper of E. Emre and the author a polynomial characterization for (A,B)-invariant subspaces is given. The characterization is used to give a frequency domain criterion for the solvability of the disturbance decoupling problem. In this paper a more elementary and simpler treatment is given. Furthermore, stabilizability subspaces are introduced, are given a frequency domain characterization and are used to solve a variety of problems.

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1. Introduction

Using the geometric approach a variety of design problems in control theory have been solved by W.M. Wonham and A.S. Morse. The most important concepts used in the geometric approach are (A,B) -invariant subspaces and controllability subspaces. In [4], a method is described for translating state space concepts to a polynomial and rational matrix formulation. The method given was based on P. Fuhrmann's realization theory, and it could be applied to systems described in terms of matrix fractions, or more generally, by Rosenbrock's system matrix.

In this paper, we restrict ourselves to systems given in state space form and we give a more elementary and direct method for translating geometric concepts into more algebraic terms. A fundamental tool in this translation is the concept of (ξ, ω) representation, which is in essence equivalent to $X_{[sI-A, B]}$ in the terminology of [4].

In addition, we introduce a new type of subspace, called stabilizability subspace, which is very convenient if in a design problem stability considerations play a role. Intuitively, a stabilizability subspace is an (A,B) -invariant subspace such that the system restricted to this space is stabilizable. For a number of design problems, stabilizability subspaces are more convenient than the closely related and more restricted controllability subspaces. Stabilizability subspaces are also given a frequency domain characterization.

The theory thus developed is applied to give solutions in state space and frequency domain formulation of the disturbance decoupling problem, the disturbance decoupling with stability, output stabilization with respect to disturbances and also with respect to arbitrary initial values and furthermore, strong output stabilization with weak internal stability.

Some of these problems have been treated before in the literature, usually only in state space terms, and sometimes in a more complicated way. The output stabilization problem with respect to disturbances and the problems concerning weak and strong stability, are believed to be new.

The frequency domain characterizations usually have the form of a rational matrix equation with some additional conditions like stability and properness.

Algorithms for constructing solutions of such equations are discussed amply in the literature (see e.g. [1],[5],[7].)

2. Weakly invariant and stabilizability subspaces

Consider the time invariant linear system :

$$(2.1) \quad \dot{x} = Ax + Bu, \quad z = Dx,$$

where $x(t) \in X := \mathbb{R}^n$, $u(t) \in U := \mathbb{R}^m$, $z(t) \in Z := \mathbb{R}^r$,
and $A : X \rightarrow X$, $B : U \rightarrow X$, $D : X \rightarrow Z$ are linear maps

(2.2) DEFINITION. A subspace $V \subseteq X$ is called weakly invariant (w.r.t. Σ) if for every $x_0 \in V$ there exists $u \in \Omega$ such that $\xi_u(t, x_0)$ for all $t \geq 0$.

Here Ω denotes the set of piecewise continuous functions $[0, \infty) \rightarrow U$ and $\xi_u(t, x_0)$ denotes the solution x of (2.1) satisfying $x(0) = x_0$.

(2.3) THEOREM. Given Σ , the following statements are equivalent:

- (i) V is weakly invariant,
- (ii) V is (A, B) -invariant, i.e. $AV \subseteq V + BU$,
- (iii) There exists a linear $F : X \rightarrow U$ such that V is $(A+BF)$ -invariant,

$$(2.4) \quad \text{i.e. } (A+BF)V \subseteq V,$$

- (iv) For every $x \in V$ there exist strictly proper rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in V$ (for every s) and

$$(2.5) \quad x = (sI-A)\xi(s) - B\omega(s).$$

PROOF. The equivalence of (i), (ii) and (iii) is well known (see [2]). We show that (iii) \Rightarrow (iv) \Rightarrow (i). If (iii) holds and $x \in V$ then $\xi(s) := (sI - A - BF)^{-1}x$, $\omega(s) := F\xi(s)$ satisfy the conditions of (iv). Furthermore, if (iv) holds we denote by $\tilde{\xi}(t)$ and $\tilde{\omega}(t)$ the time domain functions corresponding to $\xi(s)$ and $\omega(s)$. Then (2.5) and $\xi(s) \in V$ imply

$$\dot{\tilde{\xi}}(t) = A\tilde{\xi}(t) + B\tilde{\omega}(t), \quad \tilde{\xi}(0) = x \text{ and } \tilde{\xi}(t) \in V \text{ for all } t \geq 0.$$

Hence (i) holds. □

(2.6) DEFINITION. If $x \in X$, and (2.5) holds for some strictly proper rational functions $\xi(s)$ and $\omega(s)$, we say that (2.5) is a (ξ, ω) -representation of x

According to Theorem 2.3 a weakly invariant subspace V is characterized by the property that every $x \in V$ has a (ξ, ω) -representation satisfying $\xi(s) \in V$.

(2.7) DEFINITION. Let $K \subseteq X$. Then $V_\Sigma(K)$ denotes the space of points for which there exists a (ξ, ω) -representation satisfying $\xi(s) \in K$. In particular $V_\Sigma := V_\Sigma(\ker D)$ is the space of points with a (ξ, ω) -representation satisfying $D\xi(s) = 0$.

(2.8) THEOREM. $V_\Sigma(K)$ is the largest weakly invariant subspace contained in K .

PROOF. Linearity of $V_\Sigma(K)$ is obvious. Regarding $\xi(s)$ and $\omega(s)$ as power series in s^{-1} and equating of the constant terms in (2.5) yields $x = \xi_1 \in K$ for all $x \in V_\Sigma(K)$. Here ξ_1 is defined by the expansion $\xi(s) = \sum_{k=1}^{\infty} \xi_k s^{-k}$. It follows that $V_\Sigma(K) \subseteq K$.

It remains to be shown that $V_\Sigma(K)$ is weakly invariant. For this it suffices to show that in a (ξ, ω) -representation of x satisfying $\xi(s) \in K$, we actually have $\xi(s) \in V_\Sigma(K)$, or equivalently $\xi_k \in V_\Sigma(K)$ for $k = 1, 2, \dots$. Taking the proper part of

$$s^k x = s^k (sI-A) \xi(s) - s^k B\omega(s)$$

yields

$$\xi_k = (sI-A) (s^{k-1} \xi(s))_- - B(s^{k-1} \omega(s))_- ,$$

where $(v(s))_-$ denotes the strictly proper part of the rational vector $v(s)$. The maximality property of $V_\Sigma(K)$ is immediate from the definition. \square

We have $x \in V_\Sigma$ iff there exist strictly proper rational functions $\xi(s)$ and $\omega(s)$ such that

$$(2.9) \quad x = (sI-A)\xi(s) - B\omega(s), \quad D\xi(s) = 0 .$$

It is possible to eliminate $\xi(s)$ from these equations.

We obtain

(2.10) PROPOSITION. $x \in V_{\Sigma}$ iff there exists a strictly proper rational $\omega(s)$ such that

$$(2.11) \quad D(sI-A)^{-1}x = -R(s)\omega(s)$$

where $R(s) := D(sI-A)^{-1}B$ is the transfer function matrix of Σ □

Now we turn our attention to stability and stabilizability aspects. We consider stability from a general point of view in the following sense. Let \mathbb{C}^- be any subset of \mathbb{C} satisfying $\mathbb{C}^- \cap \mathbb{R} \neq \emptyset$. We say that A is a stability matrix (or map) if $\sigma(A) \subseteq \mathbb{C}^-$ and we call a rational function stable if it has no poles outside \mathbb{C}^- .

(2.12) DEFINITION. The matrix pair (A,B) (and the system Σ) is stabilizable if there exists $F : X \rightarrow U$ such that $\sigma(A + BF) \subseteq \mathbb{C}^-$.

Then we have the following criterion for stability. (see [6]).

(2.13) THEOREM. (A,B) is stabilizable iff for every complex $s \notin \mathbb{C}^-$ we have

$$\text{rank} [sI-A, B] = n.$$

(2.14) DEFINITION. $V \subseteq X$ is called a stabilizability subspace if there exists $F : X \rightarrow U$ such that $(A+BF)V \subseteq V$ and $\sigma((A+BF)|_V) \subseteq \mathbb{C}^-$.

Obviously, a stabilizability subspace is weakly invariant. If V is any weakly invariant subspace and $F : X \rightarrow U$ satisfies $(A+BF)V \subseteq V$, then there exists $G : W \rightarrow U$ (where $W = \mathbb{R}^{\ell}$ for some ℓ), such that $\xi_u(t, x_0) \in V$ for all $t \geq 0$ iff $x_0 \in V$ and u is of the form $u = Fx + Gw$ for some $w : [0, \infty) \rightarrow W$. For this, we can take any G such that $BG \subseteq (BU) \cap V$. The set of such pairs (F,G) will be denoted by $\Phi_{\Sigma}(V)$. For every $(F,G) \in \Phi_{\Sigma}(V)$ we may consider the restricted system $\Sigma_{F,G}$:

$$(2.15) \quad \dot{x} = (A+BF)x + BGw$$

with state space V and input value space W .

(2.16) PROPOSITION. Let $V \subseteq X$ be weakly invariant and let $(F,G) \in \Phi_{\Sigma}(V)$. Then V is a stabilizability subspace iff $(A+BF, BG)|_V$ (i.e. the system (2.15)) is stabilizable.

PROOF. The if part is obvious. In order to prove the only if part we notice that if $(A+B\tilde{F})V \subseteq V$, $\sigma(A+B\tilde{F}) \subseteq \mathbb{C}^-$ for some \tilde{F} , we must have $B(F-\tilde{F}) \in V \cap BU = BGW$, hence $B\tilde{F} = BF + BGL$ for some L . If we define $F_1 = F + GL$, it follows that $\sigma(A+BF_1)|_V = \sigma(A+B\tilde{F})|_V \in \mathbb{C}^-$ and $A + BF_1 = (A+BF) + BGL$. Hence $(A+BF, BG)|_V$ is stabilizable \square

(2.17) PROPOSITION. *There exists $(F, G) \in \Phi_\Sigma(V)$ such that $\sigma(A+BF) \subseteq \mathbb{C}^-$ iff Σ is stabilizable and V is a stabilizability subspace.*

PROOF. The necessity of both conditions is obvious. The proof of the sufficiency part is analogous to the proof of [8, Prop 4.1] \square
The following result gives a frequency domain characterization for stabilizability subspaces.

(2.18) THEOREM. *$V \subseteq X$ is a stabilizability subspace iff every $x \in V$ has a (ξ, ω) -representation such that $\xi(s) \in V$ and $\xi(s)$ and $\omega(s)$ are stable.*

PROOF. Let V be a stabilizability subspace and let $F : X \rightarrow U$ be such that $(A+BF)V \subseteq V$ and $\sigma((A+BF)|_V) \subseteq \mathbb{C}^-$. For $x \in V$ we define $\omega := F\xi$ and $\xi(s) := (sI - A - BF)^{-1}x$. It follows that ξ and ω are stable, since V is $(A+BF)$ -invariant and $x \in V$.

Conversely, assume that x has a (ξ, ω) -representation with $\xi(s) \in V$ and ξ, ω stable. It follows from Theorem (2.3) that V is weakly invariant. Let $(F, G) \in \Phi_\Sigma(V)$. For $x \in V$ we can write

$$(2.19) \quad x = (sI - A - BF)\xi(s) - B(\omega(s) - F\xi(s))$$

with ξ, ω stable and $\xi(s) \in V$. Since V is $(A+BF)$ -invariant it follows that $v(s) := \omega(s) - F\xi(s) \in GW$. We show that $(A+BF, BG)|_V$ is stabilizable, so that the result follows from Proposition (2.16). Suppose that $(A+BF, BG)|_V$ is not stabilizable. By Theorem (2.13) there exists $\lambda \in \mathbb{C} \setminus \mathbb{C}^-$, $\eta \in V^*$ (the dual space of V), $\eta \neq 0$ such that $\eta(A+BF) = \lambda\eta$, $\eta BG = 0$. Multiplying (2.19) from the left by η and using that $v(s) \in GW$ we obtain $\eta x = (s-\lambda)\xi(s)$. Since $\xi(s)$ has only poles in \mathbb{C}^- , this implies $\eta x = 0$. However, since $\eta \neq 0$, $\eta \in V^*$ and since $x \in V$ is arbitrary, this leads to a contradiction. \square

The following corollary can of course also easily be proved directly:

(2.20) COROLLARY. (A,B) is stabilizable iff there exist strictly proper stable matrices $X(s)$ and $U(s)$ such that

$$(2.21) \quad (sI-A) X(s) - BU(s) = I .$$

PROOF. The space X is a stabilizability subspace iff (A,B) is stabilizable. Applying Theorem (2.18) to $V = X$ and $x = e_1, \dots, e_n$ (the columns of I) yields (2.21) □

(2.22) DEFINITION. Given a subspace $K \subseteq X$ (and Σ, \bar{C}) we define

$$V_{\Sigma}^{-}(K)$$

to be the set of points for which there exists a stable (ξ, ω) -representation satisfying $\xi(s) \in K$. In particular

$$V_{\Sigma}^{-} := V_{\Sigma}^{-}(\ker D) .$$

Again, it can easily be shown that $V_{\Sigma}^{-}(K)$ is the largest stabilizability subspace contained in K .

If Σ is detectable (i.e. (A', \bar{D}') is stabilizable w.r.t. \bar{C}), we can give a characterization of V_{Σ}^{-} analogous to Proposition (2.10):

(2.23) PROPOSITION. If Σ is detectable then $x \in V_{\Sigma}^{-}$ iff there exists a strictly proper stable $\omega(s)$ such that $D(sI-A)^{-1}x = -R(s)\omega(s)$.

PROOF. The necessity of this condition is obtained as in Proposition (2.10), i.e. by eliminating $\xi(s)$ from (2.9). To prove sufficiency we define

$$\xi(s) := (sI-A)^{-1}x + (sI-A)^{-1}B\omega(s) .$$

Then $\xi(s)$ is obviously strictly proper and $D\xi(s) = 0$ by assumption. It remains to be shown that $\xi(s)$ is stable. Since Σ is detectable there exist strictly proper stable matrices $X(s)$ and $Y(s)$ such that

$$X(s) (sI-A) - Y(s)D = I .$$

This is the dual of Corollary (2.20). It follows that

$$\xi(s) = X(s) (sI-A)\xi(s) - Y(s)D\xi(s) = X(s) (x+B\omega(s))$$

is stable. □

(2.24) REMARK. The result of (2.23) is no longer valid if the detectability condition is omitted. Consider the example

$$m = r = 1, n = 2 \quad \mathbb{C}^- = \{s \mid \text{Res} < 0\}, D = [1, 0]$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Then $V := \ker D = \{[0, x_2]' \mid x_2 \in \mathbb{R}\}$ is (A, B) -invariant (even A -invariant); We have $(A+BF)V \subseteq V$ for $F = [f_1, f_2]$ iff $f_2 = 0$. But then

$$A + BF = \begin{bmatrix} f_1 & 0 \\ f_1 & 1 \end{bmatrix}$$

and $\sigma((A+BF) \mid V) = \{1\} \notin \mathbb{C}^-$. Hence V is not a stabilizability subspace. However $R(s) = s^{-1}$ and $D(sI-A)^{-1}x = 0$ for every $x \in V$, so that the equation $D(sI-A)^{-1}x = -R(s)\omega(s)$ has the strictly proper stable solution $\omega(s) = 0$

□

The following results on $V_{\Sigma}^-(X)$, the stabilizable subspace of Σ , follow immediately from the foregoing:

(2.25) COROLLARY.

- (i) $x \in V_{\Sigma}^-(X)$ iff x has a stable (ξ, ω) -representation,
- (ii) Σ is stabilizable iff $V_{\Sigma}^-(X) = X$,
- (iii) There exists $F : X \rightarrow U$ such that $(sI-A-BF)^{-1}x$ is stable for all $x \in V_{\Sigma}^-(X)$.

□

(2.26) REMARK. $W := V_{\Sigma}^-(X)$ is strongly invariant in the sense that for every $x_0 \in W$ and every $u \in \Omega$ we have $\xi_u(t, x_0) \in W$. Equivalently, $AW + Bu \subseteq W$, or $(A+BF)W \subseteq W$ for every linear $F : X \rightarrow U$.

We prove the last statement, which will be used in the proof of (4.4): First we notice that the reachable space $\langle A \mid B \rangle$ is a stabilizability subspace, hence $\langle A \mid B \rangle \subseteq W$, since W is the largest stabilizability subspace. In particular $Bu \subseteq W$. Hence, if F_1 is chosen such that $(A+BF_1)W \subseteq W$ we have $(A+BF)W \subseteq (A+BF_1)W + Bu \subseteq W$.

More generally, one can say that a weakly invariant subspace V is strongly invariant iff $\langle A \mid B \rangle \subseteq V$

□

3. Disturbance Decoupling

In [9], W.M. Wonham and A.S. Morse consider the following problem:

(3.1) *DDP. Given the system*

$$(3.2) \quad \dot{x} = Ax + Bu + Eq, \quad z = Dx,$$

determine $F : X \rightarrow U$ such that, with the feedback control $u = Fx$, the output z does not depend on (the disturbance) q .

Using the notation $\Sigma := (D, A, B)$, we can formulate the necessary and sufficient condition for the solvability of *DDP* given in [9] (see also [8, Theorem 4.2]), as follows

$$(3.3) \quad E Q \subseteq V_{\Sigma},$$

where $Q = \mathbb{R}^p$ is the space in which the disturbance vector q takes its value. Using the characterization of V_{Σ} given in Proposition (2.10) we see that for every $q \in Q$ there exists a strictly proper $\omega(s)$ such that

$$D(sI-A)^{-1}Eq = -R_1(s)\omega(s),$$

where $R_1(s) := D(sI-A)^{-1}B$ is the transfer matrix of Σ .

Introducing in addition the noise to output transfer matrix

$$R_2(s) := D(sI-A)^{-1}E$$

and choosing for q the elements of a basis of Q we find the following result

(3.4) **THEOREM.** *DDP has a solution iff there exists a strictly proper matrix $Q(s)$ satisfying*

$$(3.5) \quad R_1(s)Q(s) = R_2(s).$$

PROOF. The sufficiency of this condition follows by reversing the foregoing argument. □

Thus, it turns out that *DDP* is equivalent to the exact model matching problem. This equivalence has been noted before (see [3], [4]).

An alternative frequency domain condition can be found by applying (2.9) directly. We find:

(3.6) THEOREM. DDP has a solution iff there exist strictly proper rational functions $X(s), U(s)$ such that

$$(sI-A)X(s) - BU(s) = E, \quad DX(s) = 0$$

or, equivalently

$$\begin{bmatrix} sI-A & B \\ D & 0 \end{bmatrix} \begin{bmatrix} X(s) \\ -U(s) \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}.$$

Now we turn to the stable disturbance decoupling problem (see 8, section 5.5)

(3.7) SDDP Given the system (3.2) determine $F: X \rightarrow U$ such that with the feedback $u = Fx$, the output z does not depend on q and in addition $\sigma(A+BF) \subseteq \mathbb{C}^-$.

We can formulate a necessary and sufficient condition for the existence of a solution of SDDP analogous to (3.3)

(3.8) THEOREM. SDDP is solvable iff

- (i) (A, B) is stabilizable
- (ii) $E Q \subseteq V_{\Sigma}^-$

PROOF. Obviously, (i) is necessary for the existence of any F satisfying $\sigma(A+BF) \subseteq \mathbb{C}^-$. Furthermore, (ii) is necessary since a solution F of SDDP satisfies

$$\langle A+BF | E \rangle \subseteq \ker D, \quad \sigma(A+BF) \subseteq \mathbb{C}^-$$

(see [8, section 5.5])

Hence,

$$E Q \subseteq V := \langle A+BF | E \rangle \subseteq V_{\Sigma}^-,$$

since $\langle A+BF | E \rangle$ is a stabilizability subspace

To see the sufficiency of (i) and (ii) we appeal to Proposition (2.17) according to which there exists F such that

$$(A+BF)V_{\Sigma}^- \subseteq V_{\Sigma}^-, \quad \sigma(A+BF) \subseteq \mathbb{C}^-$$

Hence $E Q \subseteq V_{\Sigma}^-$ implies that

$$\langle A+BF | E \rangle \subseteq V_{\Sigma}^- \subseteq \ker D$$

which together with $\sigma(A+BF) \subseteq \mathbb{C}^-$ shows that F is the required feedback.

□

Again we can give a frequency domain criterion for the solvability of *SDDP*, using Definition (2.22).

We find:

(3.9) THEOREM. *SDDP* has a solution iff

(i) There exist strictly proper stable matrices $X(s)$ and $U(s)$ such that

$$(sI-A)X(s) - BU(s) = I$$

(ii) There exist strictly proper stable matrices $\tilde{X}(s), \tilde{U}(s)$ such that

$$(sI-A)\tilde{X}(s) - B\tilde{U}(s) = E, \quad D\tilde{X}(s) = 0$$

If Σ is detectable, a condition similar to Theorem (3.4) can be given.

(3.10) THEOREM. Let Σ be detectable. Then *SDDP* has a solution iff.

(i) (A,B) is stabilizable

(ii) There exists a stable strictly proper solution $Q(s)$ of the equation

$$(3.5)$$

The proof is analogous to the proof of Theorem (3.4) and based on Corollary (2.23)

(3.11) REMARK. If Σ is not detectable then (i) and (ii) of Theorem (3.10) are still necessary for the solvability of *SDDP* but not sufficient. A counter example can be based on the example in Remark (2.24) with $E = [0, 1]'$. \square

4. Output stabilization with respect to disturbance

Disturbance decoupling by state feedback as described in the previous section is generically impossible. Specifically, the condition $DE = 0$ is necessary for the solvability of *DDP*. In this section we will be content with a more modest objective. Here, we try to find a state feedback $F : X \rightarrow U$, such that the noise to output map of the closed loop is stable:

(4.1) *OSDP* Given the system (3.2) determine $F : X \rightarrow U$ such that $D(sI-A-BF)^{-1}E$ is stable

In order to give a condition for the solvability of this problem, we introduce the following subspace

(4.2) DEFINITION. S_{Σ}^{-} denotes the subspace of points $x \in X$ which have a (ξ, ω) -representation with $D\xi(s)$ stable.

This space is connected to previously introduced spaces by the following result.

(4.3) THEOREM. $S_{\Sigma}^{-} = V_{\Sigma} + V_{\Sigma}^{-}(X)$

PROOF. Obviously $V_{\Sigma} \subseteq S_{\Sigma}^{-}$, $V_{\Sigma}^{-}(X) \subseteq S_{\Sigma}^{-}$, so that we only have to prove that $S_{\Sigma}^{-} \subseteq V_{\Sigma} + V_{\Sigma}^{-}(X)$.

Let $x \in S_{\Sigma}^{-}$ and let (ξ, ω) be a representation described in Definition (4.2). Then ξ and ω can be decomposed (uniquely) such that

$$\xi = \xi_1 + \xi_2, \quad \omega = \omega_1 + \omega_2$$

and ξ_1, ω_1 are stable, ξ_2, ω_2 are completely unstable (i.e. have no poles inside \mathbb{C}^{-}).

Thus we have:

$$x - (sI-A)\xi_1(s) + B\omega_1(s) = (sI-A)\xi_2(s) - B\omega_2(s).$$

The left-hand side is stable the right-hand side is unstable and both functions are proper. Therefore, the strictly proper part of either side equals zero and the static parts are equal:

$$x = \xi_{11} + \xi_{21} ,$$

$$\xi_{11} = (sI-A)\xi_1(s) - B\omega_1(s) ,$$

$$\xi_{21} = (sI-A)\xi_2(s) - B\omega_2(s) .$$

Here ξ_{11} and ξ_{21} are the coefficients of s^{-1} in the power series expansion of $\xi_1(s)$ and $\xi_2(s)$. Furthermore, $D\xi_2 = D\xi - D\xi_1$ is stable, completely unstable and strictly proper, hence zero. Thus

$$\xi_{11} \in V_{\Sigma}^{-}(X) , \xi_{21} \in V_{\Sigma}^{-} . \quad \square$$

The following result is instrumental:

(4.4) THEOREM. *There exists $F : X \rightarrow U$ such that $(A+BF)V_{\Sigma} \subseteq V_{\Sigma}$ and $D(sI-A-BF)^{-1}x$ is stable for all $x \in S_{\Sigma}^{-}$.*

PROOF. For simplicity we use the notation $V := V_{\Sigma}$, $W := V_{\Sigma}^{-}(X)$ $S := S_{\Sigma}^{-}$. We choose $(F_0, G_0) \in \Phi_{\Sigma}(V)$. Then, as is easily seen, V is the unobservable subspace of $(D, A+BF_0)$ (i.e., the $(A+BF_0)$ -invariant space generated by $\ker D$). Furthermore W and hence $S = V + W$, $V \cap W$ are $(A+BF_0)$ -invariant (see Remark (2.26)). We choose a basis q_1, \dots, q_n of X such that q_1, \dots, q_k is a basis of $V \cap W$, q_1, \dots, q_{ℓ} a basis of V (where $\ell \geq k$), $q_1, \dots, q_k, q_{\ell+1}, \dots, q_n$ a basis of W and hence q_1, \dots, q_n a basis of S . Accordingly we split vectors and matrices.

$$A+BF_0 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \quad BG_0 = \begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & D_3 & D_4 \end{bmatrix}$$

Since W is a stabilizability subspace it follows that

$$\begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix} , \quad \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}$$

is stabilizable. Hence (A_{33}, B_3) is stabilizable (see Theorem (2.13)). Therefore, there exists F_3 such that $\sigma(A_{33} + B_3 F_3) \subseteq \mathbb{C}^-$. If we define $\tilde{F} := [0, 0, GF_3, 0]$ we see that $D(sI - A - B(F_0 + \tilde{F}))^{-1}x = D_3(sI - A_{33} - B_3 F_3)^{-1}x_3$ is stable if $x \in S$ i.e., $x_4 = 0$. Since $\text{im } \tilde{F} \subseteq \text{im } G$ it follows that $(A + B(F_0 + \tilde{F}))V \subseteq V$, so that $F := F_0 + \tilde{F}$ satisfies the requirements. \square

Using this result we can give a solution of the OSDP

(4.5) THEOREM. The following statements are equivalent :

(i) OSDP is solvable

(ii) $E Q \subseteq S_{\Sigma}^-$

(iii) There exist strictly proper matrices $X(s)$ and $U(s)$ such that

(4.6) $(sI - A)X(s) - BU(s) = E$, $DX(s)$ is stable

(iv) There exists a strictly proper matrix $Q(s)$ such that

(4.7) $R_1(s)Q(s) - R_2(s)$

is stable.

PROOF. We show that (i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (i).

(i) \Rightarrow (iii): Define $X(s) = (sI - A - BF)^{-1}E$, $U(s) = FX(s)$

(iii) \Rightarrow (iv) Eliminate $X(s)$ from (4.6) and define $Q(s) = -U(s)$.

(iv) \Rightarrow (ii) Note that $x \in S_{\Sigma}^-$ iff there exists a strictly proper $\omega(s)$ such that $D(sI - A)^{-1}x + R_1(s)\omega(s)$ is stable. Therefore (4.7) implies $E Q \subseteq S_{\Sigma}^-$

(ii) \Rightarrow (i) According to Theorem (4.4), we choose F such that $D(sI - A - BF)^{-1}x$ is stable for all $x \in EQ$, hence such that $D(sI - A - BF)^{-1}E$ is stable. \square

(4.8) REMARK. Since $V^-(X) \supseteq \langle A B \rangle$, condition (ii) is generically satisfied and hence OSDP is generically solvable. If (ii) is satisfied and we define $E_1 := V_{\Sigma} \cap EQ$ then the feedback in Theorem (4.5) can be chosen such that the noise q satisfying $Eq \in E_1$ is completely suppressed while the remaining noise is stabilized \square

A particular case of the foregoing is the output stabilization problem:

(4.9) OSP Determine $F : X \rightarrow U$ such that $D(sI - A - BF)^{-1}$ is stable

This is a generalization of the problem (corresponding to the case $\mathbb{C}^- = \{s \in \mathbb{C} \mid \text{Res} < 0\}$) of determining a feedback $u = Fx$ such that for arbitrary initial state, the output tends to zero for $t \rightarrow \infty$. (see [8, section 4.4])

Problem (4.9) is a specialization of Problem (4.1) obtained by setting $E = I$. Consequently, we have the following condition for solvability:

(4.10) THEOREM. OSP has a solution iff

$$(4.11) \quad S_{\Sigma}^{-} = X$$

or, equivalently, iff there exist strictly proper matrices $P(s)$ and $Q(s)$ such that

$$(sI-A)P(s) - BQ(s) = I, \quad DP(s) \text{ is stable.} \quad \square$$

(4.11) REMARK. In [8, Theorem 4.4] the following condition is given for the solvability of OSP:

$$(4.12) \quad \langle A|B \rangle + V_{\Sigma} \supseteq X^{+}(A)$$

where $X^{+}(A)$ is the subspace corresponding to the unstable eigenvalues of A .

It is not difficult to see that $V_{\Sigma}^{-}(X) = \langle A|B \rangle + X^{-}(A)$, so that the conditions of Theorems (4.5) and (4.10) can be reformulated in this terminology. Thus,

(4.6) is equivalent to

$$EQ \subseteq V_{\Sigma} + \langle A|B \rangle + X^{-}(A)$$

and (4.11) to

$$V_{\Sigma} + \langle A|B \rangle + X^{-}(A) = X$$

which is easily seen to be equivalent to (4.12) □

Finally, we consider the problem of stabilizing the output strongly with respect to the disturbance, while stabilizing the state weakly. Specifically, we assume that we are given two sets $\mathbb{C}_2 \subseteq \mathbb{C}_1 \subseteq \mathbb{C}$ and we want to find a feedback F such that:

$$(4.13) \quad (sI-A-BF)^{-1} \text{ is 1-stable, } D(sI-A-BF)^{-1}E \text{ is 2-stable.}$$

For this purpose we introduce the space

$$S_{12} := \{x \in X \mid \text{There exist 1-stable strictly proper } \xi(s), \omega(s) \text{ such that } D\xi \text{ is 2-stable and } x = (sI-A)\xi - B\omega\}.$$

Analogously to Theorem (4.5) one can show:

(4.14) THEOREM. The following statements are equivalent:

- (i) There exists F satisfying (4.13),
- (ii) $E Q \subseteq S_{12}$, (A,B) is 1-stabilizable,
- (iii) There exist strictly proper, 1-stable matrices $X(s), U(s)$ such that $(sI-A)X(s) - BU(s) = E$, $DX(s)$ is 2-stable and (A,B) is 1-stabilizable.

(iv) (if Σ is 1-detectable) There exists a strictly proper 1-stable matrix $Q(s)$ such that (4.7) is 2-stable and (A,B) is 1-stabilizable.

The proof is analogous to the proof of Theorem (4.5) and will be omitted.

(In particular, one uses $S_{12} = V_{\Sigma}^{2-}(X) + V_{\Sigma}^{1-}$)

Again, this result can be specialized to strong output and weak state stabilization with respect to arbitrary initial states.

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