Solution to Problem 80-1: A determinant and an identity

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A Matrix Eigenvalue Problem

Problem 79-2, by G. Efroyman, A. Steger and S. Steinberg (University of New Mexico).

Let $M_n$ denote the $n \times n$ matrix whose $(j, k)$ entry $M_n(j, k)$ is given by

$$
\omega^{(i-1)(k-1)/\sqrt{n}}, \quad 1 \leq j, k \leq n,
$$

where $\omega = e^{2\pi i/n}$. Determine all the eigenvalues of $M_n$. The matrix $M_n$ arises in some work on finite Fourier transforms.


A Determinant and an Identity

Problem 80-1, by A. V. Boyd (University of the Witwatersrand, Johannesburg, South Africa).

(a) Prove that

$$
\det |A_{rs}| = (-1)^{r+s}(2^{2n} - 2)B_{2n}/(2n)!
$$

where $r, s = 1, 2, \ldots, n$,

$$
A_{rs} = \begin{cases}
1/(2r - 2s + 3)! , & s \leq r + 1, \\
0, & s > r + 1,
\end{cases}
$$

and $B_n$ is the Bernoulli number defined by

$$
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}.
$$

(b) Prove that if $n$ is odd,

$$
t^n = \sum_{m=0}^{(n-1)/2} \frac{1 - 2^{2m-1}}{n - 2m + 1} \binom{n}{2m} h^{2m-1} B_{2m} ((t + h)^{n+1-2m} - (t - h)^{n+1-2m}).
$$

Solution by O. G. Ruehr (Michigan Technological University).

From the given generating function for the Bernoulli numbers, the well-known expansion, $x \csch x = \sum_{n=0}^{\infty} q_n x^{2n}$, $q_n = (2 - 2^{2n}) B_{2n}/(2n)!$, is easily obtained. Part (a) is now an immediate consequence of Wronski’s determinant for the reciprocal of a power series, i.e.,

$$
\left[ \sum_{n=0}^{\infty} q_n t^n \right]^{-1} = \sum_{n=0}^{\infty} c_n t^n \Rightarrow c_n = (\frac{-1}{q_n+1}) q_0^{n+1} \cdots q_n \begin{vmatrix}
q_1 & q_0 & 0 & \cdots & 0 \\
q_2 & q_1 & q_0 & \cdots & 0 \\
& \cdots & \cdots & \cdots & \cdots \\
q_n & q_{n-1} & q_{n-2} & \cdots & q_1
\end{vmatrix}.
$$

Part (b) results from the elementary identity

$$
[(xh \csch xh)/2h][e^{x(t+h)} - e^{x(t-h)}] = xe^{xt}.
$$
upon multiplying the series corresponding to the bracketed quantities and employing
the formula
\[
\sum_{m=0}^{\infty} A_m \sum_{n=0}^{\infty} B_n = \sum_{m=0}^{\lceil n/2 \rceil} A_mB_{n+1-2m}.
\]
The stated result (b) is correct for nonnegative integers \( n \), provided that \( (n-1)/2 \) is
replaced by \( \lceil n/2 \rceil \) as the upper limit of summation.

Remark. This solver recently rediscovered the following generalization of Wronski's
determinant which had been quoted by Muir [1] without proof. Let
\[
\left( \sum_{n=0}^{\infty} q_n t^n \right)^{m} = \sum_{n=0}^{\infty} c_n(m) t^n.
\]
Then, \( c_n(m) = ((-1)^{n}/q_0^{n+m}) \det |A_{rs}(m)| \), where
\[
A_{rs} = \begin{cases} [(r-s+1)m + (s-1)]q_{r-s+1}/r, & s \leq r + 1, \\ 0, & s > r + 1. \end{cases}
\]
The proof [2], depends upon elementary properties of lower Hessenberg matrices and
uses the J. C. P. Miller formula [3].

REFERENCES

Solution by O. P. LOSSERS (Technische Hogeschool Eindhoven, the Netherlands).

a) Let
\[
D_n = \det |A_{rs}|, \quad D_0 = 1.
\]
Then by expanding the determinant by the last column and iterating this procedure, we
find
\[
D_n = \sum_{i=0}^{n} \frac{(-1)^{i-1}}{(2i+1)!} D_{n-i}
\]
i.e.,
\[
\sum_{i=0}^{n} \frac{(-1)^{i-1}}{(2i+1)!} D_{n-i} = 0.
\]
Since one easily checks the assertion for \( n = 1, 2 \) it is sufficient to see whether the
asserted value of \( D_n \) also satisfies (1), i.e., whether for \( n > 0 \) we have
\[
\sum_{i=0}^{n} \frac{(2n+1)(2(n-i) - 2)B_{2(n-i)}}{2i + 1} = 0.
\]
For the well-known Bernoulli polynomials, we have
\[
B_{2n+1}(t) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n+1-k} t^k
\]
\[
= \frac{1}{2} \binom{2n+1}{n} t^n + \sum_{i=0}^{n} \binom{2n+1}{2i+1} B_{2(n-i)} t^{2i+1}.
\]
Now substitute \( t = \frac{1}{2} \) and \( t = 1 \) in (3) and use \( B_{2n+1}(\frac{1}{2}) = B_{2n+1}(1) = 0 \) \((n > 0)\).

Then (2) follows by subtracting the two equations.

b) The right-hand side of (b) has the form

\[
\sum_{j=0}^{n} a_j h^j t^n - j.
\]

Clearly \( a_j = 0 \) if \( j \) is odd. For \( j \) even, we find

\[
a_j = \frac{1}{n} \left( \frac{n+1}{2j+1} \right) \sum_{m=0}^{j} (1 - 2^{m-1})(2j+1)2m B_{2m},
\]

which is 0 if \( j > 0 \) according to (2). Substituting \( h = 0 \), we find for the right-hand side

\[
\lim_{h \to 0} \frac{1 - 2^{-1} (t+h)^{n+1} - (t-h)^{n+1}}{n+1} = t^n.
\]

This proves (b).

*Also solved by C. Givens (Michigan Technological University), A. A. Jagers (Technische Hogeschool Twente, Enschede, the Netherlands), S. L. Lee (University of Alberta) and the proposer.*

Additionally, Lee provides the generalization

\[
\det |A_{rs}| = (-1)^{k(2n+k+1)/2} \det |B_{ij}|,
\]

where

\[
A_{rs} = 1/[2(r-s+k+1)]!, \quad r, s = 1, 2, \cdots, n-k+1, \quad k = 0, 1, \cdots, n
\]

\((1/p! = 0 \text{ for } p < 0)\), and

\[
B_{ij} = \beta_{2(n+2-i-j)}, \quad i, j = 1, 2, \cdots, k,
\]

where

\[
\beta_{2m} = (2^m - 2)B_{2m}/(2m)!.
\]

His proof uses Sylvester's identity and induction.

**A Matrix Stability Problem**

*Problem 80-3*, by K. Sourisseau (University of Minnesota) and M. F. Doherty (University of Massachusetts).

Let

\[
J = \begin{bmatrix}
A_1 & B_1 \\
C_2 & A_2 & B_2 \\
& \ddots & \ddots \\
& & C_n & A_n & B_n \\
& & & \ddots & \ddots \\
& & & & \ddots & B_{N-1} \\
& & & & & C_N & A_N
\end{bmatrix}
\]