

## Remarks on Hermitian matrices

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## Remarks on Hermitian Matrices

N. G. de Bruijn

*Eindhoven University of Technology  
Department of Mathematics  
Eindhoven, The Netherlands*

Submitted by Olga Taussky Todd

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### 1. INTRODUCTION

The following research problem was proposed in 1959 by Olga Taussky Todd [5]: If the eigenvalues  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$  of the hermitian matrices  $A$  and  $B$  are given, what can be said about the eigenvalues of the Jordan product  $AB + BA$ ?

W. Gilbert Strang [4] gave what can be considered to be a complete solution, indicating best possible bounds for the eigenvalues of  $AB + BA$ . And quite recently D. W. Nicholson [3], unaware of Strang's work, published an incomplete solution for the same problem.

The present paper obtains Strang's results by an alternative method, and in a slightly more manageable form. The key is Theorem 2.1 on the values of the inner product  $(Ax, x)$ ; this theorem might have some independent interest.

Getting bounds for the eigenvalues of  $AB + BA$  means getting bounds for  $\operatorname{Re}(Ax, Bx)$  if  $x$  runs through the set of vectors of length 1. Strang also obtained a result for the possible values of the complex number  $(Ax, Bx)$  itself, if bounds for the eigenvalues of  $A$  and  $B$  are given, but he restricted himself to the special case  $0 < \alpha_1 < \dots < \alpha_n < 1$ ,  $0 < \beta_1 < \dots < \beta_n < 1$ . We treat the general case here (Theorem 3.1), and we show that the result specializes to Strang's if the bounds are 0 and 1.

The author is indebted to Professor Olga Taussky Todd for stimulating him to publish this paper. It is essentially the material he communicated to her in 1962 as a comment on Strang's paper and as an alternative solution to her problem.

**NOTATION.** All matrices are complex  $n$  by  $n$  matrices, where  $n$  is a fixed positive integer, and all vectors are column vectors with  $n$  complex entries.

If  $p$  and  $q$  are real numbers, the notation  $[p, q]$  indicates the point with coordinates  $p$  and  $q$  in some fixed coordinate plane.

If  $p$  and  $q$  are real,  $p \leq q$ , then  $C(p, q)$  is the circle with diameter from  $[p, 0]$  to  $[q, 0]$  [center  $[\frac{1}{2}(p+q), 0]$  and radius  $\frac{1}{2}(q-p)$ ]. The interior is denoted by  $C^i(p, q)$ , and the closed disk (the union of  $C$  and  $C^i$ ) by  $C^c(p, q)$ . In the case  $p = q$ ,  $C^i$  is empty, and both  $C$  and  $C^c$  degenerate to a single point.

As usual,  $(x, y)$  is the inner product of  $x$  and  $y$ , and  $\|x\| = (x, x)^{1/2}$ .

## 2. KEY THEOREM

Let  $A$  be a hermitian matrix, and let  $x$  be a unit vector. We can split  $Ax$  in a vector along  $x$  and a vector orthogonal to  $x$ :

$$Ax = \alpha x + \gamma e, \tag{2.1}$$

where  $\|e\| = 1$ ,  $(e, x) = 0$ , and  $\gamma$  is real. It follows that  $\alpha$  is real,

$$\alpha = (Ax, x), \quad \gamma = \pm \|Ax - \alpha x\|.$$

What can be said about the point  $[\alpha, \gamma]$  if the eigenvalues  $\alpha_1, \dots, \alpha_n$  (with  $\alpha_1 \leq \dots \leq \alpha_n$ ) are given? The following theorem says that the range of this point is the set  $K$ , depicted in Fig. 1 and defined by

$$K = C^c(\alpha_1, \alpha_n) \setminus \{C^i(\alpha_1, \alpha_2) \cup C^i(\alpha_2, \alpha_3) \cup \dots \cup C^i(\alpha_{n-1}, \alpha_n)\}. \tag{2.2}$$

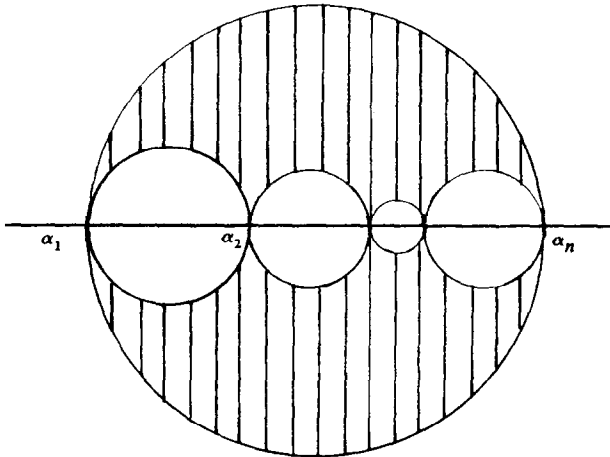


FIG. 1.

For the definition of  $K$  the multiplicity of an eigenvalue does not matter, since  $C^t(\alpha_k, \alpha_{k+1})$  is empty if  $\alpha_k = \alpha_{k+1}$ . Also note that if there are just *two* distinct eigenvalues  $\alpha_1$  and  $\alpha_n$ , then  $K$  equals the circumference  $C(\alpha_1, \alpha_n)$ .

**THEOREM 2.1.** *We have  $[\alpha, \gamma] \in K$ , and every point of  $K$  can be attained by suitable selection of  $x$  if  $A$  is fixed.*

*Proof.* By unitary transformation we can get  $A$  and  $x$  in the form

$$A = \begin{bmatrix} \alpha_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \alpha_n \end{bmatrix}, \quad x = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}.$$

This transformation does not affect the inner products  $(x, x)$ ,  $(Ax, x)$ ,  $(Ax, Ax)$ . We have

$$\begin{aligned} \|x\|^2 &= |\xi_1|^2 + \dots + |\xi_n|^2 = 1, \\ (Ax, x) &= \alpha_1 |\xi_1|^2 + \dots + \alpha_n |\xi_n|^2, \\ (Ax, Ax) &= \alpha_1^2 |\xi_1|^2 + \dots + \alpha_n^2 |\xi_n|^2. \end{aligned}$$

If we define  $Q$  by  $Q = [(Ax, x), (Ax, Ax)]$ , then the point  $Q$  belongs to the convex hull of the set

$$\{[\alpha_1, \alpha_1^2], \dots, [\alpha_n, \alpha_n^2]\}.$$

Moreover, every point of this convex hull is a possibility for  $Q$ : we just have to take appropriate nonnegative weights with sum 1, and to take  $|\xi_1|^2, \dots, |\xi_n|^2$  equal to those weights.

In the  $u, v$ -plane we consider the transformation  $f$  given by

$$f([u, v]) = [u, u^2 + v^2].$$

The inverse image of the convex hull turns out to be  $K$ . The theorem now follows from the identity

$$(Ax, Ax) = (Ax, x)^2 + \|Ax - (Ax, x)x\|^2. \quad \blacksquare$$

REMARK 1. By unitary transformation it follows that if  $x$  is fixed, then every point of  $K$  can be attained by suitable selection of  $A$  (with the prescribed eigenvalues).

REMARK 2. Theorem 2.1 generalizes a very well-known theorem in elasticity theory, where it is restricted to real vectors and matrices, and  $n=3$ . There the  $A$  is a (symmetrical) stress tensor;  $\alpha x$  and  $\gamma e$  are the normal and shear stress vectors associated with a surface element orthogonal to  $x$ . The figure with the circles is named after O. Mohr, who discussed it in 1882. It is treated in many textbooks on elasticity theory (e.g. [2]). For a derivation different from the usual ones see [1].

REMARK 3. The only thing we shall need about  $K$  in the next sections is

$$C(\alpha_1, \alpha_n) \subset K \subset C^c(\alpha_1, \alpha_n).$$

### 3. VALUES OF $\langle Ax, Bx \rangle$

We next consider two hermitian matrices  $A$  and  $B$ , both with prescribed eigenvalues,  $\alpha_1, \dots, \alpha_n$  for  $A$  (with  $\alpha_1 \leq \dots \leq \alpha_n$ ),  $\beta_1, \dots, \beta_n$  for  $B$  (with  $\beta_1 < \dots < \beta_n$ ). We ask for the possible values of  $\langle Ax, Bx \rangle$  with  $\|x\|=1$ . In particular we are interested in the values of  $\text{Re}\langle Ax, Bx \rangle$  for the following reason. The so-called Jordan product  $AB+BA$  is hermitian, we have  $\frac{1}{2}\langle (AB+BA)x, x \rangle = \text{Re}\langle Ax, Bx \rangle$ , and the set of all values of  $\langle (AB+BA)x, x \rangle$  (with  $\|x\|=1$ ) is the interval whose endpoints are the smallest and the largest eigenvalue of  $AB+BA$ .

According to Sec. 2 we have

$$Ax = \alpha x + \gamma e, \quad Bx = \beta x + \delta e'$$

with  $\langle x, e \rangle = \langle x, e' \rangle = 1$ ,  $\|x\| = \|e\| = \|e'\| = 1$ , and

$$[\alpha, \gamma] \in C^c(\alpha_1, \alpha_n), \quad [\beta, \delta] \in C^c(\beta_1, \beta_n). \tag{3.1}$$

Since  $\langle Ax, Bx \rangle = \alpha\beta + \gamma\delta\langle e, e' \rangle$ , we are interested in the products  $\alpha\beta$  and  $\gamma\delta$ . We can get those products just using the circumferences of the circles, as shown in the following lemma.

LEMMA. If  $\alpha, \beta, \gamma, \delta$  satisfy (3.1), and if  $-1 \leq \epsilon \leq 1$ , then  $\alpha', \beta', \gamma', \delta'$  exist with

$$\alpha' \beta' = \alpha \beta, \quad \gamma' \delta' = \epsilon \gamma \delta,$$

$$[\alpha', \gamma'] \in C(\alpha_1, \alpha_n), \quad [\beta', \delta'] \in C(\beta_1, \beta_n).$$

*Proof.* For reasons of symmetry we may assume that  $\gamma, \delta, \epsilon$  are all  $\geq 0$ .

Let  $\phi$  denote the nonnegative function such that  $[u, \phi(u)] \in C(\alpha_1, \alpha_n)$  for  $\alpha_1 \leq u \leq \alpha_n$ , and  $\psi$  the similar function for the other circle. We note that  $\gamma \leq \phi(\alpha), \delta \leq \psi(\beta)$ .

In the  $u, v$ -plane we consider the rectangle  $\alpha_1 \leq u \leq \alpha_n, \beta_1 \leq v \leq \beta_n$ , and the hyperbola  $uv = \alpha\beta$ . Since  $[\alpha, \beta]$  belongs to the rectangle, the hyperbola has a point on the boundary too. If  $[u, v]$  moves along the hyperbola from  $[\alpha, \beta]$  to such a boundary point, the product  $\phi(u)\psi(v)$  moves to zero. It follows that on this hyperbola  $\phi(u)\psi(v)$  attains every value between  $\phi(\alpha)\psi(\beta)$  and 0. In particular  $\alpha'$  and  $\beta'$  exist such that  $\phi(\alpha')\psi(\beta') = \epsilon\gamma\delta$ . Now take  $\gamma' = \phi(\alpha'), \delta' = \psi(\beta')$ . ■

THEOREM 3.1. If  $x$  is fixed,  $\|x\| = 1$ , then the set of all possible values for  $(Ax, Bx)$  ( $A$  and  $B$  with the prescribed eigenvalues) is equal to the set of all  $\alpha\beta + \gamma\delta\zeta$  with

$$[\alpha, \gamma] \in C^c(\alpha_1, \alpha_n), \quad [\beta, \delta] \in C^c(\beta_1, \beta_n), \quad \zeta \in \mathbb{C}, \quad |\zeta| \leq 1 \quad (3.2)$$

and also identical to the set of all  $\alpha\beta + \gamma\delta\zeta$  with

$$[\alpha, \gamma] \in C(\alpha_1, \alpha_n), \quad [\beta, \delta] \in C(\beta_1, \beta_n), \quad \zeta \in \mathbb{C}, \quad |\zeta| = 1. \quad (3.3)$$

*Proof.* It follows from the lemma (take  $\epsilon = |\zeta|$ ) that the characterizations by means of (3.2) and (3.3) are equivalent.

Since  $(Ax, Bx) = \alpha\beta + \gamma\delta(e, e')$  and  $|(e, e')| \leq 1$ , every  $(Ax, Bx)$  lies in the set given by (3.2).

Next let  $\alpha, \beta, \gamma, \delta, \zeta$  satisfy (3.3). We can find hermitian matrices  $A$  and  $B_1$  [cf. Remark 1 at the end of Sec. 2 and note that  $C(\alpha_1, \alpha_n) \subset K$ ] such that  $Ax = \alpha x + \gamma e, B_1 x = \beta x + \delta e'$ , where  $e$  and  $e'$  have length 1,  $(x, e) = (x, e') = 0$ . Next we can find a unitary matrix  $U$  such that  $Ux = x$  and  $Ue' = \bar{\zeta}e$ . Fixing  $B$  by  $B = UB_1U^*$ , we obtain  $Bx = \beta x + \delta\bar{\zeta}e, (Ax, Bx) = \alpha\beta + \gamma\delta\zeta$ . ■

**THEOREM 3.2.** *The set of all possible values for  $\operatorname{Re}(Ax, Bx)$  is equal to the set of all  $\alpha\beta + \gamma\delta$  with*

$$[\alpha, \gamma] \in C(\alpha_1, \alpha_n), \quad [\beta, \delta] \in C(\beta_1, \beta_n). \quad (3.4)$$

*Proof.* We can repeat the proof of Theorem 3.1, now with restriction to real values of  $\zeta$ . In (3.3) we can write  $\zeta = \pm 1$  instead of  $|\zeta| = 1$ . The value  $\zeta = -1$  can be replaced by  $\zeta = +1$  if we just change the sign of  $\delta$ . ■

#### 4. PRELIMINARIES

As a preparation to Sec. 5, we take positive numbers  $a$  and  $b$  and we ask for the set

$$\{\operatorname{Re}(1 + \eta)(1 + \bar{\zeta}) \mid \eta \in \mathbb{C}, \zeta \in \mathbb{C}, |\eta| = a, |\zeta| = b\}. \quad (4.1)$$

We shall use the identity

$$2 \operatorname{Re}(1 + \eta)(1 + \bar{\zeta}) = 1 - |\eta|^2 - |\zeta|^2 + |1 + \eta + \zeta|^2. \quad (4.2)$$

The range of  $\eta + \zeta$  is the ring between the circles with center 0 and radii  $|a - b|$  and  $a + b$  (boundary included). Therefore, the range of  $|1 + \eta + \zeta|$  is the closed interval  $[V, a + b + 1]$ , where  $V = a - b - 1$  if  $a \geq b + 1$ ,  $V = 0$  if  $|a - b| \leq 1 \leq a + b$ ,  $V = a + b - 1$  if  $a + b \leq 1$ ,  $V = b - a - 1$  if  $b \geq a + 1$ . By (4.2) we infer that the set (4.1) is the closed interval  $[W, (1 + a)(1 + b)]$ , where

$$W = \begin{cases} (1 - a)(1 + b) & (a \geq b + 1), \\ \frac{1}{2}(1 - a^2 - b^2) & (|a - b| \leq 1 \leq a + b), \\ (1 - a)(1 - b) & (a + b \leq 1), \\ (1 + a)(1 - b) & (b \geq a + 1). \end{cases}$$

#### 5. APPLICATION

The result of Sec. 4 can be applied to find the set of all possible values of  $\operatorname{Re}(Ax, Bx)$ , i.e., the set of all  $\alpha\beta + \gamma\delta$  with (3.4). We have  $\alpha\beta + \gamma\delta =$

$\text{Re}[(\alpha + i\gamma)(\beta - i\delta)]$ , and  $\alpha + i\gamma$  runs through the circle  $C(\alpha_1, \alpha_n)$ ,  $\beta + i\delta$  through  $C(\beta_1, \beta_n)$  (we now consider the plane of those circles as the complex plane).

If  $\frac{1}{2}(\alpha_1 + \alpha_n)$  happens to be zero, the range of  $(\alpha + i\gamma)(\beta - i\delta)$  is obtained from  $C(\beta_1, \beta_n)$  by multiplication, and the range of its real part is

$$\left[ \frac{1}{2}(\alpha_n - \alpha_1)\beta_1, \frac{1}{2}(\alpha_n - \alpha_1)\beta_n \right].$$

If  $\frac{1}{2}(\beta_1 + \beta_n) = 0$  we have a similar result. If both  $\alpha_1 + \alpha_n$  and  $\beta_1 + \beta_n$  are different from zero, we put

$$\frac{\alpha_n - \alpha_1}{|\alpha_1 + \alpha_n|} = a, \quad \frac{\beta_n - \beta_1}{|\beta_1 + \beta_n|} = b,$$

and the answer is supplied by Sec. 4. The range is

$$\begin{aligned} & [\rho W, \rho(1+a)(1+b)] \quad (\rho > 0), \\ & [\rho(1+a)(1+b), \rho W] \quad (\rho < 0), \end{aligned}$$

where  $\rho = (\alpha_1 + \alpha_n)(\beta_1 + \beta_n)/4$ .

## 6. SPECIALIZATION TO STRANG'S RESULT

In [4] Strang gives the set of values of  $(Ax, Bx)$  in an explicit form in the case that  $\alpha_1 = \beta_1 = 0$ ,  $\alpha_n = \beta_n = 1$ . We can do this here too, on the basis of (3.2). Now  $\alpha$  and  $\beta$  run through the interval  $[0, 1]$ , and  $\gamma^2 \leq \alpha(1 - \alpha)$ ,  $\delta^2 \leq \beta(1 - \beta)$ . If  $\alpha\beta$  is kept constant,  $\alpha\beta = p$  say, the product  $\gamma\delta$  runs between  $-q$  and  $+q$ , where  $q = p^{1/2} - p$ . In the complex plane we now have to find the union of the closed circular disks with center  $p$  and radius  $p^{1/2} - p$ , if  $p$  varies from 0 to 1. Taking coordinates  $x, y$  in the complex plane we find, by classical analytic geometry, the region given by

$$-\frac{1}{8} \leq x \leq 1, \quad 27y^2 \leq (x-1)^2(8x+1). \tag{6.1}$$

Note that the circle  $(x-p)^2 + y^2 = (p^{1/2} - p)^2$  lies entirely inside (6.1), and that as long as  $p > \frac{1}{16}$ , it touches the boundary at the points with  $x = \frac{1}{2}(3p^{1/2} - 1)$ . For values of  $p$  less than  $\frac{1}{16}$  the circles lie in the interior of the region.



It is easy to check that (6.1) is equivalent to Strang's remarkable representation in terms of polar coordinates  $r, \phi$ :

$$0 \leq r \leq \left[ \frac{\cos \frac{1}{3} \pi}{\cos \frac{1}{3} (\pi - |\phi|)} \right]^3 \quad (-\pi \leq \phi \leq \pi).$$

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