# Remarks on Hermitian matrices

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### 1. INTRODUCTION

The following research problem was proposed in 1959 by Olga Taussky Todd [5]: If the eigenvalues $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_n$ of the hermitian matrices $A$ and $B$ are given, what can be said about the eigenvalues of the Jordan product $AB+BA$?

W. Gilbert Strang [4] gave what can be considered to be a complete solution, indicating best possible bounds for the eigenvalues of $AB+BA$. And quite recently D. W. Nicholson [3], unaware of Strang's work, published an incomplete solution for the same problem.

The present paper obtains Strang's results by an alternative method, and in a slightly more manageable form. The key is Theorem 2.1 on the values of the inner product $(Ax,x)$; this theorem might have some independent interest.

Getting bounds for the eigenvalues of $AB+BA$ means getting bounds for $\mathrm{Re}(Ax,Bx)$ if $x$ runs through the set of vectors of length 1. Strang also obtained a result for the possible values of the complex number $(Ax,Bx)$ itself, if bounds for the eigenvalues of $A$ and $B$ are given, but he restricted himself to the special case $0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_n\leqslant 1$, $0\leqslant\beta_1\leqslant\cdots\leqslant\beta_n\leqslant 1$. We treat the general case here (Theorem 3.1), and we show that the result specializes to Strang's if the bounds are 0 and 1.

The author is indebted to Professor Olga Taussky Todd for stimulating him to publish this paper. It is essentially the material he communicated to her in 1962 as a comment on Strang's paper and as an alternative solution to her problem.

NOTATION. All matrices are complex $n$ by $n$ matrices, where $n$ is a fixed positive integer, and all vectors are column vectors with $n$ complex entries.

If $p$ and $q$ are real numbers, the notation $[p,q]$ indicates the point with coordinates $p$ and $q$ in some fixed coordinate plane.



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If $p$ and $q$ are real, $p \leqslant q$, then $C(p,q)$ is the circle with diameter from $\lceil p,0 \rceil$ to $\lceil q,0 \rceil$ [center $\left\lceil \frac{1}{2}(p+q),0 \right\rceil$ and radius $\frac{1}{2}(q-p)$]. The interior is denoted by $C^i(p,q)$, and the closed disk (the union of $C$ and $C^i$) by $C^c(p,q)$. In the case $p=q$, $C^i$ is empty, and both $C$ and $C^c$ degenerate to a single point.

As usual, $(x,y)$ is the inner product of $x$ and $y$, and $\|x\|=(x,x)^{1/2}$.

## 2. KEY THEOREM

Let $A$ be a hermitian matrix, and let $x$ be a unit vector. We can split $Ax$ in a vector along $x$ and a vector orthogonal to $x$:

$$Ax = \alpha x + \gamma e, \tag{2.1}$$

where $\|e\|=1$, $(e,x)=0$, and $\gamma$ is real. It follows that $\alpha$ is real,

$$\alpha = (Ax,x), \qquad \gamma = \pm \|Ax - \alpha x\|.$$

What can be said about the point $\lceil \alpha,\gamma \rceil$ if the eigenvalues $\alpha_1,\ldots,\alpha_n$ (with $\alpha_1 \leqslant \cdots \leqslant \alpha_n$) are given? The following theorem says that the range of this point is the set $K$, depicted in Fig. 1 and defined by

$$K = C^c(\alpha_1,\alpha_n) \setminus \{ C^i(\alpha_1,\alpha_2) \cup C^i(\alpha_2,\alpha_3) \cup \cdots \cup C^i(\alpha_{n-1},\alpha_n) \}. \tag{2.2}$$

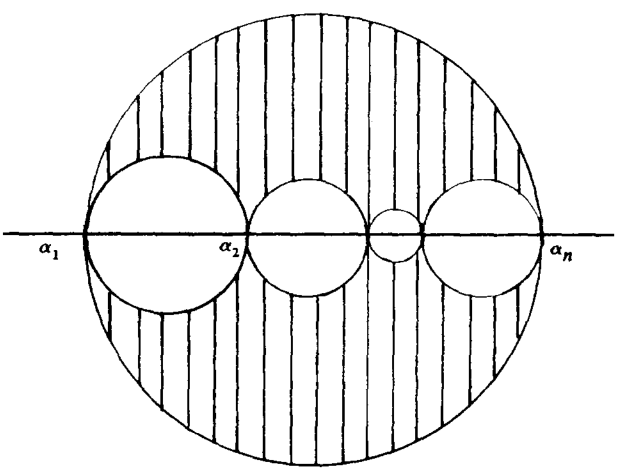


FIG. 1.

For the definition of $K$ the multiplicity of an eigenvalue does not matter, since $C^i(\alpha_k, \alpha_{k+1})$ is empty if $\alpha_k = \alpha_{k+1}$. Also note that if there are just *two* distinct eigenvalues $\alpha_1$ and $\alpha_n$, then $K$ equals the circumference $C(\alpha_1, \alpha_n)$.

THEOREM 2.1. *We have* $\lceil \alpha, \gamma \rceil \in K$, *and every point of* $K$ *can be attained by suitable selection of* $x$ *if* $A$ *is fixed.*

*Proof.* By unitary transformation we can get $A$ and $x$ in the form

$$A = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix}, \qquad x = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}.$$

This transformation does not affect the inner products $(x,x)$, $(Ax,x)$, $(Ax,Ax)$. We have

$$\|x\|^2 = |\xi_1|^2 + \cdots + |\xi_n|^2 = 1,$$

$$(Ax,x) = \alpha_1 |\xi_1|^2 + \cdots + \alpha_n |\xi_n|^2,$$

$$(Ax,Ax) = \alpha_1^2 |\xi_1|^2 + \cdots + \alpha_n^2 |\xi_n|^2.$$

If we define $Q$ by $Q = \lceil (Ax,x), (Ax,Ax) \rceil$, then the point $Q$ belongs to the convex hull of the set

$$\{ \lceil \alpha_1, \alpha_1^2 \rceil, \ldots, \lceil \alpha_n, \alpha_n^2 \rceil \}.$$

Moreover, every point of this convex hull is a possibility for $Q$: we just have to take appropriate nonnegative weights with sum 1, and to take $|\xi_1|^2, \ldots, |\xi_n|^2$ equal to those weights.

In the $u, v$-plane we consider the transformation $f$ given by

$$f(\lceil u, v \rceil) = \lceil u, u^2 + v^2 \rceil.$$

The inverse image of the convex hull turns out to be $K$. The theorem now follows from the identity

$$(Ax,Ax) = (Ax,x)^2 + \|Ax - (Ax,x)x\|^2. \qquad \blacksquare$$

REMARK 1. By unitary transformation it follows that if $x$ is fixed, then every point of $K$ can be attained by suitable selection of $A$ (with the prescribed eigenvalues).

REMARK 2. Theorem 2.1 generalizes a very well-known theorem in elasticity theory, where it is restricted to real vectors and matrices, and $n=3$. There the $A$ is a (symmetrical) stress tensor; $\alpha x$ and $\gamma e$ are the normal and shear stress vectors associated with a surface element orthogonal to $x$. The figure with the circles is named after O. Mohr, who discussed it in 1882. It is treated in many textbooks on elasticity theory (e.g. [2]). For a derivation different from the usual ones see [1].

REMARK 3. The only thing we shall need about $K$ in the next sections is

$$C(\alpha_1,\alpha_n)\subset K\subset C^c(\alpha_1,\alpha_n).$$

## 3. VALUES OF $(Ax,Bx)$

We next consider two hermitian matrices $A$ and $B$, both with prescribed eigenvalues, $\alpha_1,\dots,\alpha_n$ for $A$ (with $\alpha_1\leqslant\dots\leqslant\alpha_n$), $\beta_1,\dots,\beta_n$ for $B$ (with $\beta_1\leqslant\dots\leqslant\beta_n$). We ask for the possible values of $(Ax,Bx)$ with $\|x\|=1$. In particular we are interested in the values of $\operatorname{Re}(Ax,Bx)$ for the following reason. The so-called Jordan product $AB+BA$ is hermitian, we have $\frac{1}{2}((AB+BA)x,x)=\operatorname{Re}(Ax,Bx)$, and the set of all values of $((AB+BA)x,x)$ (with $\|x\|=1$) is the interval whose endpoints are the smallest and the largest eigenvalue of $AB+BA$.

According to Sec. 2 we have

$$Ax=\alpha x+\gamma e,\qquad Bx=\beta x+\delta e'$$

with $(x,e)=(x,e')=1$, $\|x\|=\|e\|=\|e'\|=1$, and

$$[\alpha,\gamma]\in C^c(\alpha_1,\alpha_n),\qquad [\beta,\delta]\in C^c(\beta_1,\beta_n). \tag{3.1}$$

Since $(Ax,Bx)=\alpha\beta+\gamma\delta(e,e')$, we are interested in the products $\alpha\beta$ and $\gamma\delta$. We can get those products just using the circumferences of the circles, as shown in the following lemma.

LEMMA. *If $\alpha$, $\beta$, $\gamma$, $\delta$ satisfy* (3.1), *and if* $-1 \leqslant \varepsilon \leqslant 1$, *then* $\alpha'$, $\beta'$, $\gamma'$, $\delta'$ *exist with*

$$\alpha'\beta' = \alpha\beta, \qquad \gamma'\delta' = \varepsilon\gamma\delta,$$

$$\lceil \alpha', \gamma' \rceil \in C(\alpha_1, \alpha_n), \qquad \lceil \beta', \delta' \rceil \in C(\beta_1, \beta_n).$$

*Proof.* For reasons of symmetry we may assume that $\gamma$, $\delta$, $\varepsilon$ are all $\geqslant 0$.

Let $\phi$ denote the nonnegative function such that $\lceil u, \phi(u) \rceil \in C(\alpha_1, \alpha_n)$ for $\alpha_1 \leqslant u \leqslant \alpha_n$, and $\psi$ the similar function for the other circle. We note that $\gamma \leqslant \phi(\alpha)$, $\delta \leqslant \psi(\beta)$.

In the $u,v$-plane we consider the rectangle $\alpha_1 \leqslant u \leqslant \alpha_n$, $\beta_1 \leqslant v \leqslant \beta_n$, and the hyperbola $uv = \alpha\beta$. Since $\lceil \alpha, \beta \rceil$ belongs to the rectangle, the hyperbola has a point on the boundary too. If $\lceil u, v \rceil$ moves along the hyperbola from $\lceil \alpha, \beta \rceil$ to such a boundary point, the product $\phi(u)\psi(v)$ moves to zero. It follows that on this hyperbola $\phi(u)\psi(v)$ attains every value between $\phi(\alpha)\psi(\beta)$ and 0. In particular $\alpha'$ and $\beta'$ exist such that $\phi(\alpha')\psi(\beta') = \varepsilon\gamma\delta$. Now take $\gamma' = \phi(\alpha')$, $\delta' = \psi(\beta')$. ■

THEOREM 3.1. *If $x$ is fixed, $\|x\| = 1$, then the set of all possible values for $(Ax, Bx)$ ($A$ and $B$ with the prescribed eigenvalues) is equal to the set of all $\alpha\beta + \gamma\delta\zeta$ with*

$$\lceil \alpha, \gamma \rceil \in C^c(\alpha_1, \alpha_n), \quad \lceil \beta, \delta \rceil \in C^c(\beta_1, \beta_n), \qquad \zeta \in \mathbb{C}, \quad |\zeta| \leqslant 1 \qquad (3.2)$$

*and also identical to the set of all $\alpha\beta + \gamma\delta\zeta$ with*

$$\lceil \alpha, \gamma \rceil \in C(\alpha_1, \alpha_n), \quad \lceil \beta, \delta \rceil \in C(\beta_1, \beta_n), \qquad \zeta \in \mathbb{C}, \quad |\zeta| = 1. \qquad (3.3)$$

*Proof.* It follows from the lemma (take $\varepsilon = |\zeta|$) that the characterizations by means of (3.2) and (3.3) are equivalent.

Since $(Ax, Bx) = \alpha\beta + \gamma\delta(e, e')$ and $|(e, e')| \leqslant 1$, every $(Ax, Bx)$ lies in the set given by (3.2).

Next let $\alpha$, $\beta$, $\gamma$, $\delta$, $\zeta$ satisfy (3.3). We can find hermitian matrices $A$ and $B_1$ [cf. Remark 1 at the end of Sec. 2 and note that $C(\alpha_1, \alpha_n) \subset K$] such that $Ax = \alpha x + \gamma e$, $B_1 x = \beta x + \delta e'$, where $e$ and $e'$ have length 1, $(x, e) = (x, e') = 0$. Next we can find a unitary matrix $U$ such that $Ux = x$ and $Ue' = \bar{\zeta} e$. Fixing $B$ by $B = UB_1U^*$, we obtain $Bx = \beta x + \delta\bar{\zeta} e$, $(Ax, Bx) = \alpha\beta + \gamma\delta\zeta$. ■

THEOREM 3.2. *The set of all possible values for* $\mathrm{Re}(Ax,Bx)$ *is equal to the set of all* $\alpha\beta+\gamma\delta$ *with*

$$\lceil \alpha,\gamma \rceil \in C(\alpha_1,\alpha_n), \qquad \lceil \beta,\delta \rceil \in C(\beta_1,\beta_n). \tag{3.4}$$

*Proof.* We can repeat the proof of Theorem 3.1, now with restriction to real values of $\zeta$. In (3.3) we can write $\zeta=\pm 1$ instead of $|\zeta|=1$. The value $\zeta=-1$ can be replaced by $\zeta=+1$ if we just change the sign of $\delta$. ■

## 4. PRELIMINARIES

As a preparation to Sec. 5, we take positive numbers $a$ and $b$ and we ask for the set

$$\left\{\mathrm{Re}\,(1+\eta)(1+\bar{\zeta})\,\middle|\,\eta\in\mathbb{C},\ \zeta\in\mathbb{C},\ |\eta|=a,\ |\zeta|=b\right\}. \tag{4.1}$$

We shall use the identity

$$2\,\mathrm{Re}\,(1+\eta)(1+\bar{\zeta})=1-|\eta|^2-|\zeta|^2+|1+\eta+\zeta|^2. \tag{4.2}$$

The range of $\eta+\zeta$ is the ring between the circles with center 0 and radii $|a-b|$ and $a+b$ (boundary included). Therefore, the range of $|1+\eta+\zeta|$ is the closed interval $[V,a+b+1]$, where $V=a-b-1$ if $a\geqslant b+1$, $V=0$ if $|a-b|\leqslant 1\leqslant a+b$, $V=a+b-1$ if $a+b\leqslant 1$, $V=b-a-1$ if $b\geqslant a+1$. By (4.2) we infer that the set (4.1) is the closed interval $[W,(1+a)(1+b)]$, where

$$W=\begin{cases}(1-a)(1+b) & (a\geqslant b+1),\\ \tfrac{1}{2}(1-a^2-b^2) & (|a-b|\leqslant 1\leqslant a+b),\\ (1-a)(1-b) & (a+b\leqslant 1),\\ (1+a)(1-b) & (b\geqslant a+1).\end{cases}$$

## 5. APPLICATION

The result of Sec. 4 can be applied to find the set of all possible values of $\mathrm{Re}(Ax,Bx)$, i.e., the set of all $\alpha\beta+\gamma\delta$ with (3.4). We have $\alpha\beta+\gamma\delta=$

$\mathrm{Re}[(\alpha+i\gamma)(\beta-i\delta)]$, and $\alpha+i\gamma$ runs through the circle $C(\alpha_1,\alpha_n)$, $\beta+i\delta$ through $C(\beta_1,\beta_n)$ (we now consider the plane of those circles as the complex plane).

If $\frac{1}{2}(\alpha_1+\alpha_n)$ happens to be zero, the range of $(\alpha+i\gamma)(\beta-i\delta)$ is obtained from $C(\beta_1,\beta_n)$ by multiplication, and the range of its real part is

$$\left[\tfrac{1}{2}(\alpha_n-\alpha_1)\beta_1,\ \tfrac{1}{2}(\alpha_n-\alpha_1)\beta_n\right].$$

If $\frac{1}{2}(\beta_1+\beta_n)=0$ we have a similar result. If both $\alpha_1+\alpha_n$ and $\beta_1+\beta_n$ are different from zero, we put

$$\frac{\alpha_n-\alpha_1}{|\alpha_1+\alpha_n|}=a, \qquad \frac{\beta_n-\beta_1}{|\beta_1+\beta_n|}=b,$$

and the answer is supplied by Sec. 4. The range is

$$\begin{aligned}&[\rho W,\rho(1+a)(1+b)] \quad (\rho>0),\\ &[\rho(1+a)(1+b),\rho W] \quad (\rho<0),\end{aligned}$$

where $\rho=(\alpha_1+\alpha_n)(\beta_1+\beta_n)/4$.

## 6. SPECIALIZATION TO STRANG'S RESULT

In [4] Strang gives the set of values of $(Ax,Bx)$ in an explicit form in the case that $\alpha_1=\beta_1=0$, $\alpha_n=\beta_n=1$. We can do this here too, on the basis of (3.2). Now $\alpha$ and $\beta$ run through the interval $[0,1]$, and $\gamma^2\leqslant\alpha(1-\alpha)$, $\delta^2\leqslant\beta(1-\beta)$. If $\alpha\beta$ is kept constant, $\alpha\beta=p$ say, the product $\gamma\delta$ runs between $-q$ and $+q$, where $q=p^{1/2}-p$. In the complex plane we now have to find the union of the closed circular disks with center $p$ and radius $p^{1/2}-p$, if $p$ varies from 0 to 1. Taking coordinates $x,y$ in the complex plane we find, by classical analytic geometry, the region given by

$$-\tfrac{1}{8}\leqslant x\leqslant 1, \qquad 27y^2\leqslant(x-1)^2(8x+1). \tag{6.1}$$

Note that the circle $(x-p)^2+y^2=(p^{1/2}-p)^2$ lies entirely inside (6.1), and that as long as $p\geqslant\frac{1}{16}$, it touches the boundary at the points with $x=\frac{1}{2}(3p^{1/2}-1)$. For values of $p$ less than $\frac{1}{16}$ the circles lie in the interior of the region.

It is easy to check that (6.1) is equivalent to Strang's remarkable representation in terms of polar coordinates $r$, $\phi$:

$$0 \leqslant r \leqslant \left\{ \frac{\cos\frac{1}{3}\pi}{\cos\frac{1}{3}(\pi - |\phi|)} \right\}^3 \qquad (-\pi \leqslant \phi \leqslant \pi).$$

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