A Taylor series expansion method for calculation of the radiation integral for a circular aperture

by suppressing the excitation of parasitic modes such as parallel port modes [14].

D. Gain

Simulated gain vs. frequency is shown in Fig. 5. The gain values at the design frequency are 8.0 dBi and 8.4 dBi for port 1 and port 2, respectively. The 1 dB gain-drop bandwidth for port 1 is 22.7% and even higher for port 2 (37.6%). This is in agreement with the return loss bandwidth in Fig. 2. The measured gain of the antenna at 5.8 GHz is 7.63 dBi and 7.8 dBi for port 1 and port 2, respectively. These results agree well with the calculated values.

IV. Conclusion

Using a new configuration of coupling slots, the design and measured results for an aperture-coupled dual linearly polarized circular microstrip patch antenna at C-band have been presented. The antenna exhibits measured 10 dB RL bandwidth of 14.3% and 24.5% for the two polarizations. A parametric study of isolation and axial ratio with respect to slot locations is also carried out. It is shown that the isolation can be improved with a suitable selection of the slot locations. Over the frequency band of interest, an isolation greater than 28 dB is observed between the two ports. The prototype antenna has yielded 7.6 dBi gain, 70° 3-dB beamwidth and a cross-polarization level typically better than 18 dB. The high gain, broad beamwidth, low cross polarization, and high isolation between the input ports make the antenna quite suitable for RF applications at C-band in general.

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A Taylor Series Expansion Method for Calculation of the Radiation Integral for a Circular Aperture

Sava V. Savov

***Abstract*—The Taylor series expansion method developed in this paper is an efficient algorithm for performing the popular Jacobi series expansion for the radiation integral describing the operation of reflector and other circular aperture antennas. This approach results in a Bessel series expansion for the radiation integral. The method is then applied for simplicity to different examples of a source function with rotational symmetry.**

***Index Terms*—Aperture antennas, Hankel transforms, reflector antennas, series (mathematics).**

I. Introduction

It is well-known that the reflector antennas have many applications: radio-astronomy, satellite communications, etc. A problem that has to be solved in the theoretical explanation of their operation, provided that the current distribution (source function) is known, is the mathematical description of the far electric field in the focal plane (the radiation pattern). This problem can be solved by use of the equivalence theorem to find the electric and magnetic current distributions in the aperture plane (finite circular aperture). The exact solution of the problem needs a vector form representation, but in a paraxial approximation the solution can be expressed in a scalar form (Kirchhoff approximation) that is widely used for the co-polarized field. The mathematical problem is related to double integration procedure over the area of a circular aperture. The challenge is to find a suitable fast and accurate numerical method [1]–[3]. One possible numerical solution of this integral is based on the application of the fast Fourier transform method in rectangular coordinates [4], [5]. However, this method makes a stair-case approximation of the circular aperture, that introduces an additional numerical error.

Another interesting numerical technique for performing the double integration in polar coordinates is the Jacobi series expansion (JSE)

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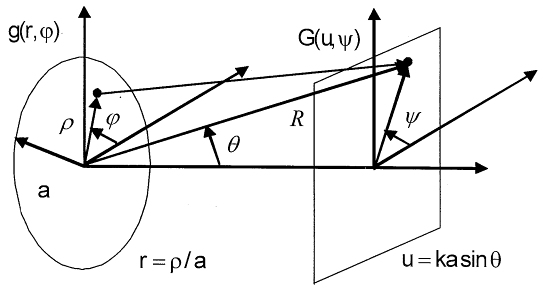

Fig. 1. Far-field $G(u, \psi)$, radiated by a circular aperture with radius a, excited by a general source $g(r, \varphi)$.

method [6]–[8]. This approach is based on the representation of the radial source function in terms of classical circle polynomials, introduced by Zernike [9]. The method leads to the Bessel series expansion (BSE) for the radiation pattern, which usually involves a fast convergent series. The main computational problem here is related to the calculation of the coefficients by a single numerical integration. This is generally a time-consuming procedure (due to the fact, that the circular polynomials have to be also generated).

In this paper an alternative efficient procedure for the calculation of the coefficients of this series, so-called Taylor series expansion (TSE) method, is explored. The proposed method is attractive especially in the case when the aperture distribution is an analytic function whose Taylor series expansion is known.

II. Radiation Integral—Jacobi Series Expansion

A brief treatment of the JSE method follows. From the approximation used in the *physical optics*, it is well-known that the scalar field in the far zone of a circular aperture of a radius a (Fig. 1) can be presented by the following *radiation integral*, obtained by a double Fourier transform, written in polar coordinates (ρ, φ) [1]

$$G(u, \psi) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g(r, \varphi) e^{-jur\cos(\varphi - \psi)} r\,dr\,d\varphi \tag{1}$$

where $g(r, \varphi)$ is a general scalar *source function* in terms of the normalized radial variable $r = \rho/a$. The other parameters are as follows: $u = ka \cdot \sin\theta$ is a frequency parameter, $k = 2\pi/\lambda$ is the free-space wavenumber (λ is the wavelength) and (θ, ψ) are the spherical angles of the observation point (Fig. 1). Let us assume that the source function is represented by the Fourier series

$$g(r, \varphi) = \sum_{m=0}^{\infty} R_m(r) e^{jm\varphi}. \tag{2}$$

By substitution and integration over the angle φ [2] we obtain

$$G(u, \psi) = \sum_{m=0}^{\infty} (-1)^m e^{jm\psi} \tilde{G}_m(u) \tag{3}$$

where the *finite Hankel transform* (FHT) of the radial source function is defined by the following:

$$\tilde{G}_m(u) = \int_0^1 R_m(r) J_m(ur) r\,dr \tag{4}$$

and $J_m(.)$ are Bessel's functions of first kind, m-th order. The simplest case is a source function with rotational symmetry $(m = 0)$ when the radiation pattern does not depend on ψ.

A popular method for calculation of the FHT transform is the JSE method [6]–[8]. Here, the radial function is presented in terms of *modified circle polynomials* [9], [10]

$$F_n^m(r) = r^m P_n^{(m,0)}(1 - 2r^2) \quad (n = 2k,\ m = 2l) \tag{5}$$

where $P_n^{(\alpha,\beta)}(.)$ are Jacobi polynomials. In the case of rotational symmetry they are

$$F_k^0(r) = P_k(1 - 2r^2) \tag{6}$$

where $P_k(.)$ are Legendre polynomials of order k. These functions obey the following *orthogonality* [1]:

$$\int_0^1 F_n^m(r) F_l^m(r) r\,dr = \frac{\delta_{nl}}{2(m + 2n + 1)} \tag{7}$$

where $\delta_{nl} = \{1, n = l \mid 0, n \neq l\}$ are Kronecker's symbols. Now the radial source function can be expanded in the following JSE series:

$$R_m(r) = \sum_{k=0}^{\infty} \beta_k^m F_k^m(r) \tag{8}$$

where the unknown coefficients are determined by using the orthogonality

$$\beta_k^m = 2(m + 2k + 1) \int_0^1 R_m(r) F_k^m(r) r\,dr. \tag{9}$$

After an application of the identity [1]

$$\int_0^1 F_k^m(r) J_m(ur) r\,dr = \frac{J_{m+2k+1}(u)}{u} \tag{10}$$

the following final BSE expression is obtained for the FHT transform:

$$\tilde{G}_m(u) = \sum_{k=0}^{\infty} \beta_k^m \frac{J_{m+2k+1}(u)}{u}. \tag{11}$$

The calculation of the coefficients in (9) generally is a time-consuming operation [6]–[8] which is the main *disadvantage* of the JSE method.

III. Radiation Integral—Novel Taylor Series Expansion

Here an alternative representation of the coefficients $[\beta_k^m]$ in the BSE expansion (11) is developed. The TSE algorithm is based on the following Taylor series representation of the radial function

$$R_m(r) = \sum_{n=0}^{\infty} \tau_n^m r^n, \text{ with } \tau_n^m = \frac{R_m^{(n)}(0)}{n!}. \tag{12}$$

Substitution into (4) yields

$$\tilde{G}_m(u) = \sum_{n=0}^{\infty} \tau_n^m G_n^m(u) \tag{13}$$

TABLE I
FIRST NONZERO COEFFICIENTS $\sigma_{k,n}^0$

k \ n	0	1	2	3	4	5	6
0	1.0000	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500
1			-0.5000	-0.5143	-0.5000	-0.4762	-0.4500
2					0.1667	0.2165	0.2500
3							-0.0500

where new notations are introduced

$$G_n^m(u) = \int_0^1 r^{n+1} J_m(ur)dr. \quad (14)$$

The key point here is that the integral (14) can be found in a *closed-form*.

We use the following identity [10]

$$\int_0^z t^\mu J_\nu(t)dt = \frac{z^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \times \sum_{k=0}^{\infty} \frac{(\nu+2k+1)\Gamma\left(\frac{\nu-\mu+1}{2}+k\right)}{\Gamma\left(\frac{\nu+\mu+3}{2}+k\right)} J_{\nu+2k+1}(z) \quad (15)$$

then letting $t = ur$, $z = u$, $\mu = n+1$, $\nu = m$ and applying the main property of the Γ function $\Gamma(z+1) = z\Gamma(z)$, after algebraic manipulations, the following series expansion is obtained

$$G_n^m(u) = \sum_{k=0}^{\infty} \sigma_{k,n}^m \frac{J_{m+2k+1}(u)}{u} \quad (16)$$

where new coefficients are introduced

$$\sigma_{k,n}^m = (m+2k+1) \frac{\Gamma\left(\frac{n+m}{2}+1\right)\Gamma\left(\frac{m-n}{2}+k\right)}{\Gamma\left(\frac{n+m}{2}+2+k\right)\Gamma\left(\frac{m-n}{2}\right)} \quad (n < m) \quad (17)$$

and

$$\sigma_{k,n}^m = (-1)^k(m+2k+1) \times \frac{\Gamma\left(\frac{n+m}{2}+1\right)\Gamma\left(\frac{n-m}{2}+1\right)}{\Gamma\left(\frac{n+m}{2}+2+k\right)\Gamma\left(\frac{n-m}{2}+1-k\right)} \quad (n \geq m).$$

By substitution of (16) into (13), the FHT takes a form of the BSE expression, already defined in (11). This leads to the following *new closed-form* expression for the coefficients

$$\beta_k^m = \sum_{n=0}^{\infty} \sigma_{k,n}^m \tau_n^m = \sum_{n=m+2k}^{\infty} \sigma_{k,n}^m \tau_n^m. \quad (18)$$

The last equation results from the fact that $\sigma_{k,n}^m = 0$ for $n < m+2k$ in (17), due to the property $1/\Gamma(-l) = 0$ $(l = 0, 1, 2, \ldots)$. The new coefficients $[\sigma_{k,n}^m]$ from (17) can be defined as *universal coefficients*—see Table I for the special case of rotational symmetry, while the coefficients $[\tau_n^m]$ are *source-specific coefficients* [see (12)].

This novel method of calculation of the radiation integral is based on the (12), (17), and (18). The main *advantage* of the TSE algorithm for calculation of the coefficients in (11) is that the numerical integration is replaced by a simple matrix multiplication.

IV. NUMERICAL EXAMPLES

We will consider here for simplicity only a case with rotational symmetry specified by the source function

$$g(r) \equiv R_0(r) = \cos(\delta r) \quad (19)$$

where δ is a real parameter of the source distribution.

In this particular case the integral (9) for the coefficients $[\beta_k^0]$ *cannot* be found in a closed-form and the only way out is a numerical integration. A better alternative is to use the TSE procedure explained above—the Taylor series of this elementary source function is well-known [10]

$$R_0(r) = \sum_{l=0}^{\infty} \frac{(-1)^l \delta^{2l}}{(2l)!} r^{2l} \quad (20)$$

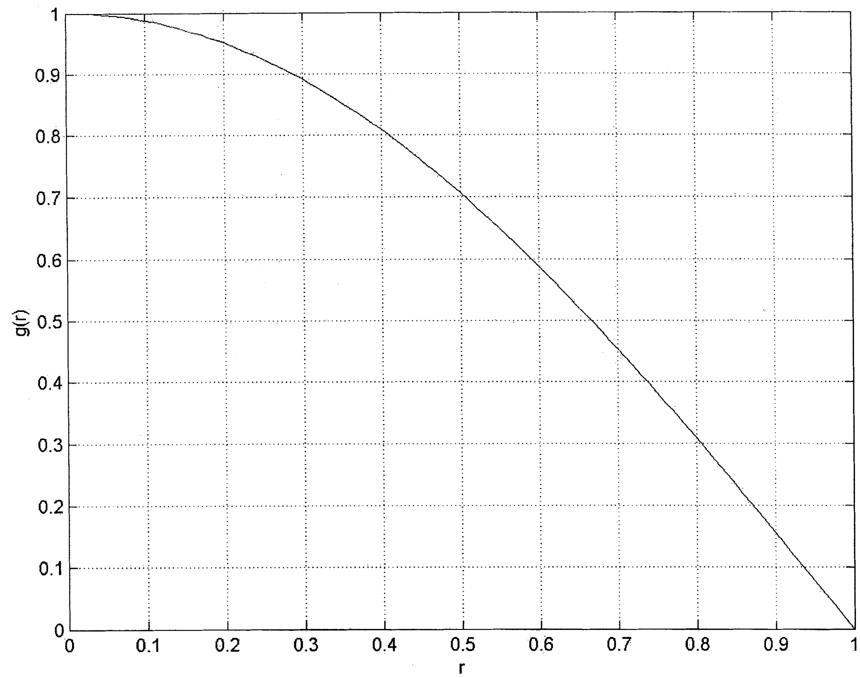

Fig. 2. Source excitation of a circular aperture with radius $a = 2\lambda$: $g(r) = \cos(\delta r)$, $\delta = \pi/2$.

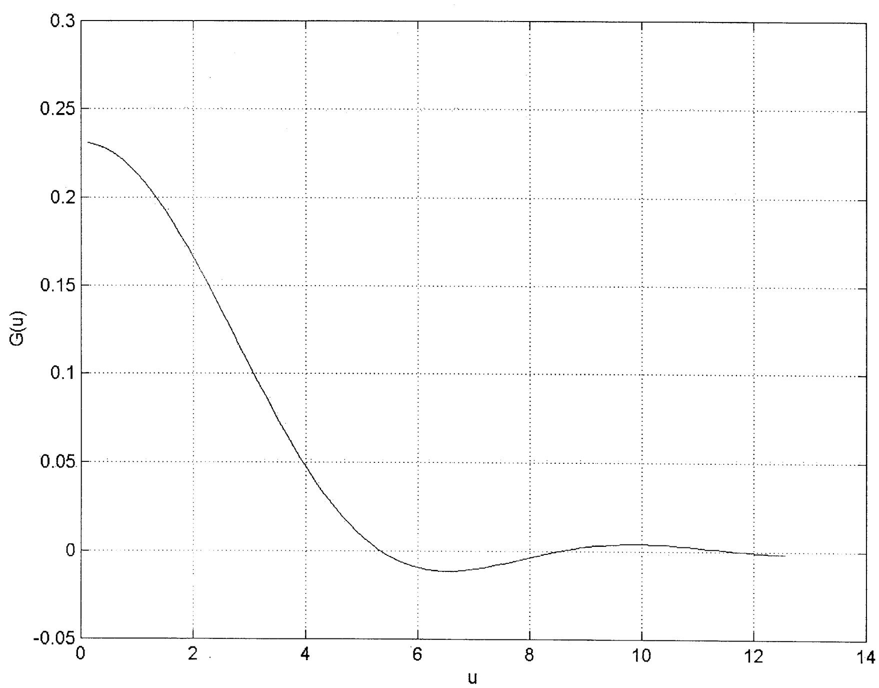

Fig. 3. Radiation pattern $G(u)$ of a circular aperture with a source excitation, shown in Fig. 2, calculated by TSE method.

which means that the coefficients in (12) are

$$\tau_n^0 = \begin{cases} 0, & n = 2l+1 \\ \frac{(-1)^l \delta^{2l}}{(2l)!}, & n = 2l. \end{cases} \tag{21}$$

Now, two particular cases are examined.

$\delta = 0$. From (19) one can easily obtain

$$g(r) = 1 \tag{22}$$

which is the well-known *uniform excitation* case. For this particular case the only nonzero coefficient in (21) is $\tau_0^0 = 1$, which means that $\beta_0^0 = 1$ in (18). The BSE expansion (11) gives the following expected result for the radiation pattern

$$\tilde{G}_0(u) = \frac{J_1(u)}{u} \tag{23}$$

which is called *Airy function*. The last result can be easily obtained after a direct analytical solution of the integral in (4) [10].

$\delta = \pi/2$. The source excitation $g(r)$ is shown in Fig. 2. The radiation pattern is calculated by (11), (17) (second expression), and (18) for $m = 0$ and it is shown in Fig. 3 for a radius $a = 2\lambda$. It is similar to Airy function, but the main maximum is a little wider, and the side-lobes are lower. The values of the first few coefficients for this case found by (18) are:$\beta_0^0 = 4.627 \times 10^{-1}$, $\beta_1^0 = 4.990 \times 10^{-1}$, $\beta_2^0 = 3.732 \times 10^{-2}$, $\beta_3^0 = 9.547 \times 10^{-4}$, $\beta_4^0 = 1.226 \times 10^{-5}$, $\beta_5^0 = 9.454 \times 10^{-8}$, $\beta_6^0 = 4.862 \times 10^{-10}$ etc.—the remaining coefficients are negligible. They are in an excellent agreement with those, reported in [1], calculated by numerical integration.

Now a few words have to be said about the convergence of the series in (11) and (18).The limits for both series are taken to be N which means that less than $N + 1$ terms are taken into account. For the example reported above the case of $N = 10$ is chosen—that is more than enough to meet the required accuracy (absolute error $\varepsilon = 10^{-4}$). The series (20) is *convergent for every value* of the argument r, but the *rate* of convergence decreases with the increase of the parameter δ.

V. Conclusion

In this communication, a novel variant of the JSE method for the calculation of the radiation integral of a circular aperture is proposed—the so called *TSE method*. Provided that the Taylor series expansion of the radial source function (or its polynomial approximation) is known analytically, the coefficients $[\beta_k^n]$ in the BSE expression of the radiation field (11) can be calculated efficiently by simple matrix multiplication (18). In that way we avoid the time-consuming numerical integration for the same coefficients that are used in (9), by involving the matrix of the known universal coefficients $[\sigma_{k,n}^m]$ (17).

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