

On some special theta functions

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MATHEMATICS

ON SOME SPECIAL THETA FUNCTIONS

BY

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(Communicated by Prof. H. FREUDENTHAL at the meeting of September 27, 1958)

1. *Preliminaries*

In this paper we shall use the usual notation $\{I, -r, v\}$ for the class of modular forms of dimension $-r$ for the group I , with multiplier system v .

$I[1]$ is the modular group, i.e. the group of integral matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. We use the same notation for the associated group of transformations $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$.

We write:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$I[N]$ is the subgroup of $I[1]$ defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm I \pmod{N}$.

$I_0[N]$ and $I^0[N]$ are the subgroups of $I[1]$ defined by $c \equiv 0 \pmod{N}$ and $b \equiv 0 \pmod{N}$ respectively.

I_\wp is the group generated by U^2 and T .

The functions η , ϑ_{gh} and G_k are defined as follows:

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \quad (\text{Im } \tau > 0).$$

$$\vartheta_{gh}(\tau) = \sum_{n=-\infty}^{+\infty} (-1)^{nh} e^{\pi i \tau (n + 1/2g)^2} \quad (\text{Im } \tau > 0, g \text{ and } h \text{ integers}).$$

$$G_k(\tau) = \frac{1}{2\zeta(k)} \sum'_{m_1, m_2} (m_1 \tau + m_2)^{-k} \quad (\text{Im } \tau > 0, k \text{ even}).$$

If S is the matrix (s_{ij}) we shall write the quadratic form $\sum_{i,j=1}^n s_{ij} x_i x_j$ as $x' S x$.

2. *The theta functions*

It is well known that the theta functions ϑ_{00} , ϑ_{01} and ϑ_{10} are modular forms for the groups I_\wp , $I^0[2]$ and $I_0[2]$ respectively. These groups are subgroups of index 3 of the full modular group $I[1]$. From these functions one can form modular forms for the group $I[1]$. A well known example is the fact that $\vartheta_{01}^{4k} + \vartheta_{10}^{4k} + (-1)^k \vartheta_{00}^{4k} = F_k$ where $F_k \in \{I[1], -2k, v\}$. Here v is determined by $v(U) = (-1)^k$.

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We shall now give a more general definition of theta functions and try to construct modular forms for $I[1]$ with these. We consider an integral positive definite quadratic form $x'Sx$ in n variables and two arbitrary vectors a and b and consider the function

$$\sum_{x \equiv a \pmod{1}} e^{\pi i x' S x} e^{2\pi i x' S b} \quad (\text{Im } \tau > 0).$$

The factor $e^{\pi i x' S x}$ enters into the sum when we consider the transformation $\tau \rightarrow \tau + 1$. We introduce a vector w with the property

$$x'Sx \equiv 2x'Sw \pmod{2}$$

for all integral x . It is easily seen that this is possible and that we can take $w=0$ if $x'Sx$ is an even form.

We shall require that S has determinant 1. Then the transformation $\tau \rightarrow \frac{-1}{\tau}$ maps the function into another function of the same type multiplied by a factor which depends on a and b . We now change our definition so that these transformation formulae no longer depend on a and b . In this way we come to the definition

$$(2.1) \quad \vartheta\left(\tau \left| \begin{matrix} a \\ b \end{matrix} \right.\right) = e^{-\pi i(a'Sb + 2a'Sw + w'Sw)} \sum_{x \equiv a + w \pmod{1}} e^{\pi i x' S x} e^{2\pi i x' S(b+w)}.$$

The transformation formulae are

$$(2.2) \quad \vartheta\left(\tau \left| \begin{matrix} a+g \\ b+h \end{matrix} \right.\right) = e^{\pi i(a'Sh - b'Sg)} (-1)^{g'Sg + h'Sh + g'Sh} \vartheta\left(\tau \left| \begin{matrix} a \\ b \end{matrix} \right.\right)$$

for integral vectors g and h .

$$(2.3) \quad \vartheta\left(\tau \left| \begin{matrix} -a \\ -b \end{matrix} \right.\right) = e^{4\pi i w'Sw} \vartheta\left(\tau \left| \begin{matrix} a \\ b \end{matrix} \right.\right),$$

$$(2.4) \quad \vartheta\left(\tau + 1 \left| \begin{matrix} a \\ b \end{matrix} \right.\right) = e^{\pi i w'Sw} \vartheta\left(\tau \left| \begin{matrix} a \\ a+b \end{matrix} \right.\right),$$

$$(2.5) \quad \vartheta\left(\frac{-1}{\tau} \left| \begin{matrix} a \\ b \end{matrix} \right.\right) = \tau^{\frac{n}{2}} e^{-\frac{\pi i n}{4}} e^{-2\pi i w'Sw} \vartheta\left(\tau \left| \begin{matrix} b \\ -a \end{matrix} \right.\right).$$

Using $(UT)^3 = -I$ we find from these relations:

$$\vartheta\left(\tau \left| \begin{matrix} a \\ b \end{matrix} \right.\right) = e^{\pi i \left\{ \frac{n}{4} - w'Sw \right\}} \vartheta\left(\tau \left| \begin{matrix} a \\ b \end{matrix} \right.\right).$$

As S is positive definite we have $\vartheta\left(\tau \left| \begin{matrix} -w \\ -w \end{matrix} \right.\right) \neq 0$ and hence $\frac{n}{4} \equiv w'Sw \pmod{2}$. If S is an even form we find the condition $n \equiv 0 \pmod{8}$ (cf. [1]).

In some cases it is easier to work with modular functions than modular forms. We therefore define

$$(2.6) \quad \varphi\left(\tau \left| \begin{matrix} a \\ b \end{matrix} \right.\right) = \vartheta\left(\tau \left| \begin{matrix} a \\ b \end{matrix} \right.\right) \cdot \eta^{-n}(\tau).$$

We have:

$$(2.7) \quad \varphi\left(\tau+1 \middle| \begin{matrix} a \\ b \end{matrix}\right) = e^{\frac{\pi i n}{6}} \varphi\left(\tau \middle| \begin{matrix} a \\ a+b \end{matrix}\right),$$

$$(2.8) \quad \varphi\left(\frac{-1}{\tau} \middle| \begin{matrix} a \\ b \end{matrix}\right) = e^{-\frac{\pi i n}{2}} \varphi\left(\tau \middle| \begin{matrix} b \\ -a \end{matrix}\right).$$

From (2.7) and (2.8) we find for $L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma[1]$ the transformation formula

$$(2.9) \quad \varphi\left(L\tau \middle| \begin{matrix} a \\ b \end{matrix}\right) = \varepsilon(L) \cdot \varphi\left(\tau \middle| \begin{matrix} \alpha a + \gamma b \\ \beta a + \delta b \end{matrix}\right).$$

Here $\varepsilon(L)$ is a character of the group $\Gamma[1]$. If we write $\xi = e^{\frac{\pi i n}{6}}$ we have (cf. [2])

$$(2.10) \quad \varepsilon(L) = \begin{cases} \xi^{(\alpha+\delta)\gamma - \beta\delta(\gamma^2-1) - 3\gamma} & \text{if } \gamma \equiv 1 \pmod{2}, \\ \xi^{(\alpha+\delta)\gamma - \beta\delta(\gamma^2-1) + 3\delta - 3 - 3\gamma\delta} & \text{if } \delta \equiv 1 \pmod{2}. \end{cases}$$

We shall now restrict ourselves to the case where there is an integer N such that Na and Nb are integral vectors. Then, by (2.9) and (2.2), we have, if $L \equiv \pm I \pmod{N}$:

$$(2.11) \quad \varphi\left(L\tau \middle| \begin{matrix} a \\ b \end{matrix}\right) = \varepsilon(L) \varrho(L, a, b) \varphi\left(\tau \middle| \begin{matrix} a \\ b \end{matrix}\right).$$

Hence φ is a modular function of $\{I[N], 0, \varepsilon\varrho\}$.

Here $\varrho(L, a, b)$ is a root of unity which depends on L, a and b in a complicated way. We introduce the following notation:

$$A = N^2 a' S a, \quad B = N^2 b' S b, \quad \text{and} \quad C = 2N^2 a' S b.$$

We take $L \equiv I \pmod{N}$, and write:

$$L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Then we find:

$$(2.12) \quad \varrho = e^{\frac{\pi i}{N}(\beta' A + \delta' C - \gamma' B)} (-1)^{(\alpha' + \alpha' \beta' + \beta' \gamma) A + (\gamma' + \gamma' \delta' + \delta' \gamma) B}.$$

If N is odd this reduces to

$$\varrho = e^{\frac{\pi i}{N}(\beta' A - \gamma' B)} e^{\frac{\pi i}{N} \delta' C}.$$

3. Relations for the theta functions

We wish to find linear combinations of the functions φ which are a modular function for $\Gamma[1]$. It is clear that we can restrict the a and b to vectors of a lattice generated by two given vectors a_0 and b_0 . Now if the function of $\Gamma[1]$ is not identically zero it generally must have the form $\varphi|T[K, A]$ where $T[K, A]$ is a general Hecke-operator introduced by WOHLFAHRT ([3]). From the theory of these we see that a necessary condition is that there is a character of $\Gamma[1]$ which is equal to $\varepsilon\varrho$ on the subgroup $\Gamma[N]$. In this way we find restrictions for the vectors a and b

that are to be used. We shall distinguish different values of N and give some necessary conditions for the existence of linear combinations which belong to $\Gamma[1]$.

The value of the character for U^N must be $\varepsilon(U^N)\varrho(U^N, a, b)$. This gives us the condition

$$nN^2 + 6A(N+1) \equiv 0 \pmod{\{2N \cdot (6, N)\}}.$$

We distinguish

1) $N \equiv 1 \pmod{2}$.

Then the above condition is equivalent with $n \equiv 0 \pmod{2}$ and $A \equiv 0 \pmod{N}$. Using $\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$ and $\begin{pmatrix} 1+N & N \\ N & 1-N \end{pmatrix}$ we find in the same way: $B \equiv 0 \pmod{N}$ and $C \equiv 0 \pmod{2N}$. In this case $\varrho = 1$.

2) $N \equiv 0 \pmod{4}$ or $N \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{2}$.

In the same way as above we find $A \equiv B \equiv C \equiv 0 \pmod{2N}$ and $\varrho = 1$.

3) $N \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{2}$.

We find $A \equiv B \equiv C \equiv N \pmod{2N}$ and $\varrho = (-1)^{\delta' + \delta' - \nu'}$.

In some cases there are different characters of $\Gamma[1]$ which are equal to $\varepsilon\varrho$ on $\Gamma[N]$. To get linear combinations of the functions φ which are not identically zero and belong to $\Gamma[1]$ we must consider a function φ and the largest subgroup of $\Gamma[1]$ for which it is a modular function and then find a character of $\Gamma[1]$ which has the same value on this subgroup. Even then we can sometimes find two or more such characters. In 4 we shall give an example of this case.

4. Examples

We shall give some examples of the relations one can find for the functions φ . We consider three different cases, namely

1) $a=b=0$, 2) $a=0$, b arbitrary, and 3) a and b linearly independent.

4.1. $a=b=0$. In this case $\varphi(\tau) = \varphi\left(\tau \middle| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is already a modular function for the group $\Gamma[1]$. From (2.3) we see that $\vartheta\left(\tau \middle| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) = (-1)^{n\vartheta}\left(\tau \middle| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and hence if $\varphi \neq 0$, n must be even. In fact if $n \leq 11$ we have $\vartheta\left(\tau \middle| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) = 0$, except if $n=8$ and S is an even form.

For $n=12$ and $x'Sx = 23x_1^2 + 2 \sum_{i=2}^{12} x_i^2 + 10x_1x_2 + 2 \sum_{i=2}^{11} x_i x_{i+1}$, we have $\vartheta\left(\tau \middle| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) = - \sum_{x=(0, \pm 0, \pm 0, \dots, 0, \pm) \pmod{1}} e^{\pi i x' S x} e^{\pi i x_1} = 24\eta^{12}(\tau)$ (cf. [4]).

4.2. $a=0$. In this case the functions φ are modular functions for the group generated by U and $\Gamma[N]$. If $N=2$ we have

$$\varphi\left(\tau \middle| \begin{smallmatrix} 0 \\ b \end{smallmatrix}\right) + \varphi\left(\tau \middle| \begin{smallmatrix} b \\ 0 \end{smallmatrix}\right) + \varphi\left(\tau \middle| \begin{smallmatrix} b \\ b \end{smallmatrix}\right) = \Phi(\tau) \text{ with } \Phi \in \{\Gamma[1], 0, \nu\}.$$

Here a necessary condition is that $b'Sb$ is an integer. As example we take for S the unit matrix. If one of the components of b is 0 we have $\varphi = 0$. Therefore we can assume that all components of b are $\frac{1}{2}$. Then $n \equiv 0 \pmod{4}$.

After multiplying by η^{4k} the relation becomes

$$(4.1) \quad \partial_{01}^{4k} + \partial_{10}^{4k} + (-1)^k \partial_{00}^{4k} = F_k \varepsilon \{ \Gamma[1], -2k, v \}.$$

We have: $F_1=0$; $F_2=2G_4$; $F_3=-48\eta^{12}$; $F_4=2G_8$.

This is a well known result.

If $N=3$ we have

$$\varphi\left(\tau \left| \begin{smallmatrix} 0 \\ b \end{smallmatrix} \right.\right) + \varphi\left(\tau \left| \begin{smallmatrix} b \\ 0 \end{smallmatrix} \right.\right) + \varphi\left(\tau \left| \begin{smallmatrix} b \\ b \end{smallmatrix} \right.\right) + \varphi\left(\tau \left| \begin{smallmatrix} -b \\ b \end{smallmatrix} \right.\right) = \Phi(\tau) \text{ with } \Phi \in \{ \Gamma[1], 0, v \}.$$

Here a necessary condition is that $b'Sb \equiv 0 \pmod{3}$ and $n \equiv 0 \pmod{2}$. If we take as S the unit matrix we again have $\varphi=0$ if one of the components of $b=0$. It is easily seen that we can therefore restrict ourselves to vectors b with all components equal to $\frac{1}{2}$. Then we have $n \equiv 0 \pmod{6}$ and as

$$\sum_{x \equiv 1 \pmod{1}} e^{\pi i x^2} e^{2\pi i^3/x} = -\sqrt{3} \eta(3\tau)$$

we find a relation which can be written as

$$(4.2) \quad \eta^{6k} | T(3) \in \{ \Gamma[1], -3k, v \},$$

where $T(3)$ is a generalized Hecke-operator (cf. [3]).

For $k=1$ we have $\eta^6 | T(3) = 0$, for $k=2$: $\eta^{12}(\tau) | T(3) = c \cdot \eta^{12}(\tau)$.

Finally we give an example with $N=5$, $n=2$. Let S be the unit matrix and $a=0$, $b = \frac{1}{5}(1, 2)$. Then

$$\begin{aligned} \vartheta\left(\tau \left| \begin{smallmatrix} a \\ b \end{smallmatrix} \right.\right) &= -i \left\{ \sum_{x \equiv 1 \pmod{1}} e^{\pi i x^2} e^{2\pi i^3/10x} \right\} \left\{ \sum_{x \equiv 1 \pmod{1}} e^{\pi i x^2} e^{2\pi i^3/10x} \right\} = \\ &= -4i \left\{ \sum_{n=0}^{\infty} e^{\frac{\pi i x(2n+1)^2}{4}} \cos \frac{3(2n+1)}{10} \pi \right\} \left\{ \sum_{n=0}^{\infty} e^{\frac{\pi i x(2n+1)^2}{4}} \cos \frac{(2n+1)\pi}{10} \right\} \end{aligned}$$

$$\text{and } \varphi(\tau) = \frac{\vartheta\left(\tau \left| \begin{smallmatrix} a \\ b \end{smallmatrix} \right.\right)}{\eta^2(\tau)}.$$

$\varphi \in \{ \Gamma_0[5], 0, v \}$ where v is a character of $\Gamma[1]$. The relation is

$$\vartheta\left(\tau \left| \begin{smallmatrix} 0 \\ b \end{smallmatrix} \right.\right) + \vartheta\left(\tau \left| \begin{smallmatrix} b \\ b \end{smallmatrix} \right.\right) + \vartheta\left(\tau \left| \begin{smallmatrix} 2b \\ b \end{smallmatrix} \right.\right) + \vartheta\left(\tau \left| \begin{smallmatrix} 3b \\ b \end{smallmatrix} \right.\right) + \vartheta\left(\tau \left| \begin{smallmatrix} 4b \\ b \end{smallmatrix} \right.\right) + \vartheta\left(\tau \left| \begin{smallmatrix} b \\ 0 \end{smallmatrix} \right.\right) = 0.$$

4.3. a and b independent.

In all the examples S will be the unit matrix.

i) $N=2$, $n=3$, $a = (\frac{1}{2}, \frac{1}{2}, 0)$, $b = (0, \frac{1}{2}, \frac{1}{2})$.

$\vartheta\left(\tau \left| \begin{smallmatrix} a \\ b \end{smallmatrix} \right.\right) = -\vartheta_{00}(\tau) \vartheta_{01}(\tau) \vartheta_{10}(\tau) = -2\eta^3(\tau)$. Hence $\varphi=2$ which is a trivial modular function for $\Gamma[1]$.

ii) $N=2$, $n=8$, $a = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0)$, $b = (0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Then the theta function is $\vartheta_{01}^4 \vartheta_{10}^4 \in \{ \Gamma_{\vartheta}, -4, v \}$.

The relation we find is

$$(4.3) \quad -\vartheta_{01}^4 \vartheta_{10}^4 + \vartheta_{00}^4 \vartheta_{10}^4 + \vartheta_{00}^4 \vartheta_{01}^4 = G_4.$$

This can be derived from (4.1).

iii) $N=2$, $n=15$, $a=\frac{1}{2}(0, 1, \dots, 1)$, $b=\frac{1}{2}(1, 1, 1, 1, 1, 1, 0, \dots, 0)$.

The theta function is $-\vartheta_{10}^5 \vartheta_{00}^5 \vartheta_{01}^0 \in \left\{ \Gamma[2], -\frac{15}{2}, v \right\}$. In this case there are two characters of $\Gamma[1]$ which are equal to the multiplier of φ on $\Gamma[2]$. Therefore we find two operators mapping φ into a modular function of $\Gamma[1]$. The two relations can be written as follows:

$$\begin{aligned} \text{Define } F &= -\vartheta_{00}^4 \vartheta_{01}^8 + \vartheta_{10}^4 \vartheta_{00}^8 + \vartheta_{01}^4 \vartheta_{10}^8 \\ G &= \vartheta_{01}^4 \vartheta_{00}^8 - \vartheta_{00}^4 \vartheta_{10}^8 + \vartheta_{10}^4 \vartheta_{01}^8. \end{aligned}$$

Then F and G are modular forms of $\{\Gamma, -6, 1\}$ where Γ is the subgroup of index 2 of $\Gamma[1]$ generated by U^2 and UT .

We have

$$(4.4) \quad \begin{cases} F+G=48\eta^{12} \\ F-G=-2G_6. \end{cases}$$

iv) $N=3$, $n=6$, $a=\frac{1}{3}(0, 0, 0, 1, 1, 1)$, $b=\frac{1}{3}(1, 1, 1, 0, 0, 0)$.

$$\text{Then } \vartheta\left(\tau \begin{vmatrix} a \\ b \end{vmatrix}\right) = e^{\frac{\pi i}{3}} \sqrt{3} \eta^3(3\tau) \eta^3\left(\frac{\tau}{3}\right).$$

In this case φ is mapped into 0 by the operator and the relation we find can be written as $\eta^6|T(3)=0$.

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