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Eindhoven University of Technology

Department of Mathematics and Computing Science
Computing Science Section

Computing Science Notes

Delay-insensitive Directed Trace Structures Satisfy the Foam Rubber Wrapper Postulate

by

Tom Verhoeff
Huub M.J.L. Schols

85/04

COMPUTING SCIENCE NOTES

This is a series of notes of the Computing Science Section of the Department of Mathematics and Computing Science of Eindhoven University of Technology.
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Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB EINDHOVEN
The Netherlands

editor: M.L. Potters

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0 Abstract

In [JTU] Udding defines $\mathbf{C}_4$, the class of delay-insensitive directed trace structures. Schols defines the foam rubber wrapper postulate in [HS]. This postulate is a formalization of the foam rubber wrapper principle defined by Molnar, Fang, and Rosenberger in [MFR]. In this paper we prove that a directed trace stucture that is a $\mathbf{C}_4$, satisfies the foam rubber wrapper postulate and has absence of danger of transmission inference (the reverse is proven in [HS]). Furthermore we show that absence of danger of transmission interference, which is explicitly required in the definition of $\mathbf{C}_4$, is superfluous in order to prove that a directed trace structure satisfies the foam rubber wrapper postulate.

1 Notations

We explain the notation, that we use for variable-binding constructs. Universal quantification is denoted by

$(\mathbf{A}l:D:E)$

where **A** is the quantifier, l denotes a list of bound variables, D denotes a predicate, and E denotes the quantified expression. D and E contain - in general - variables from l. D indicates the domain of the bound variables. E is quantified for variable values that satisfy D. Existential quantification is denoted analogously, using quantifier **E** instead of **A**. By

$\{l:D:E\}$

we denote the set of all values of E obtained by substituting for all variables in l values that satisfy D. By

$(\mathbf{S}l:D:A)$

we denote the sum of all elements of $\{l:D:A\}$, where A denotes the quantified arithmetic expression. In all notations the domain D is omitted when obvious from the context.

For expressions E and G an expression of the form $E \Rightarrow G$ is often proved in a number of steps by the introduction of intermediate expressions. For instance, we can prove $E \Rightarrow G$ by proving $E = F$ and $F \Rightarrow G$ for some expression F. In order to prevent that the reader has to perform a string comparison to establish the (for the argument essential) sameness of the two occurrences of F, we represent proofs like this as follows

$$\begin{array}{ll} & E \\ = & \{\text{ hint why } E = F \ \} \\ & F \\ \Rightarrow & \{\text{ hint why } F \Rightarrow G \ \} \\ & G \end{array}$$

These notions have been adopted from [EWD] and [JTU].

2 Trace Theory

2.0 Introduction

We present an introduction to trace theory, which is sufficient for our purposes. An extended description can be found in [MR], [RSU], and [JvdS].

2.0.0 Traces and directed trace structures

An alphabet is a finite set of symbols. For an alphabet A, A^* denotes the set of all finite-length sequences of elements of A, including the empty sequence, which is denoted by ε. A trace is a finite-length sequence of symbols. A directed trace structure T is a triple $<\mathbf{i}T,\mathbf{o}T,\mathbf{t}T>$, where $\mathbf{i}T$ is the input alphabet of T, $\mathbf{o}T$ is the output alphabet of T, and $\mathbf{t}T$ is the trace set of T. $\mathbf{i}T$ and $\mathbf{o}T$ are disjoint. We denote $\mathbf{i}T \cup \mathbf{o}T$ by $\mathbf{a}T$, the alphabet of T. $\mathbf{t}T$ is a subset of $(\mathbf{a}T)^*$. Elements of $\mathbf{i}T$ are called input symbols of T, or symbols of type input. Elements of $\mathbf{o}T$ are called output symbols of T, or symbols of type output. Elements of $\mathbf{t}T$ are called traces of T.

Note

Unless stated otherwise, small and capital letters near the end of the Latin alphabet denote traces and directed trace structures respectively. Small and capital letters near the beginning of the Latin alphabet denote symbols and alphabets respectively.

end of note

2.0.1 Directed traces and partially directed traces

We may postfix symbols with an exclamation point or a question mark. Symbols a, $a!$, and $a?$ are three distinct symbols. For alphabet A, $A!$ denotes the set $\{a : a \in A : a!\}$ and $A?$ denotes $\{a : a \in A : a?\}$. A, $A!$, and $A?$ are three disjoint sets. Elements of $(A! \cup A?)^*$ are called directed traces and elements of $(A \cup A! \cup A?)^*$ are called partially directed traces. Elements of A^* are referred to as traces. Notice that all traces and all directed traces are partially directed traces. For directed trace structure T, $T!?$ denotes $(\mathbf{o}T)! \cup (\mathbf{i}T)?$.

2.0.2 Operations

Definition 2.0 (prefix)

For partially directed trace t the set of all prefixes of t, denoted by $\mathbf{pref}(t)$, is the trace set

$\{u,w : t = uw : u\}$

end of definition

(Concatenation is denoted by juxtaposition).
We extend this operator to trace sets.

Definition 2.1 (prefix-closure)

For trace set T the prefix-closure of T, denoted by $\mathbf{pref}(T)$, is the trace set

$\{t,u : t \in T \wedge u \in \mathbf{pref}(t) : u\}$

end of definition

We denote the length of a partially directed trace t by $\mathbf{l}(t)$:

Definition 2.2 (length)

(i) $\mathbf{l}(\varepsilon) = 0$

(ii) for partially directed trace t and symbol a
$\mathbf{l}(ta) = \mathbf{l}(t) + 1$

end of definition

The projection of a partially directed trace t on an alphabet A is denoted by $t \upharpoonright A$:

Definition 2.3 (projection)

For alphabet A

(i) $\varepsilon \upharpoonright A = \varepsilon$

(ii) for partially directed trace t and symbol a, such that $a \in A$,
$(ta) \upharpoonright A = (t \upharpoonright A)a$

(iii) for partially directed trace t and symbol a, such that $a \notin A$,
$(ta) \upharpoonright A = t \upharpoonright A$.

end of definition

For a partially directed trace t and a symbol a we denote $\mathbf{l}(t \upharpoonright \{a\})$ by $\#_a t$. We define the function direct, denoted by **dir**, which maps partially directed traces on partially directed traces :

Definition 2.4 (direct)

For alphabet A and partially directed trace u we define $\mathbf{dir}(A,u)$ recursively :

(i) $\mathbf{dir}(A,\varepsilon) = \varepsilon$

(ii) for partially directed trace t and symbol a, such that $a \in A$,
$\mathbf{dir}(A,ta) = \mathbf{dir}(A,t)a!a?$

(iii) for partially directed trace t and symbol a, such that $a \notin A$,
$\mathbf{dir}(A,ta) = \mathbf{dir}(A,t)a$

end of definition

2.0.3 Undirecting

Definition 2.5 (immediate undirect)

For partially directed traces t and u, t is an immediate undirect of u, denoted by t **iundirect** u, iff

$(\mathbf{E}a, x, y_0, y_1, z$

$\quad : (y_0 y_1) \upharpoonright \{a?\} = \varepsilon$

$\quad : ((t = x\, y_0\, a\, y_1\, z) \wedge (u = x\, a!\, y_0\, y_1\, a?\, z)) \vee ((t = x\, y_0\, a\, y_1) \wedge (u = x\, a!\, y_0\, y_1))$

$)$

end of definition

The reflexive and transitive closure of immediate undirect is a partial order called undirect. We denote this partial order among partially directed traces t and u by t **undirect** u.

2.1 Notions related to the FRW-postulate

The notions introduced in this section have been adopted from [HS].

Note
In the remainder of this section S and T denote directed trace structures, such that $\mathbf{i}S$ equals $\mathbf{o}T$ and $\mathbf{o}S$ equals $\mathbf{i}T$.
end of note

Definition 2.6 (absence of deadlock)
Traces s and t, such that $s \in \mathbf{t}S$ and $t \in \mathbf{t}T$, have absence of deadlock, denoted by $s\,\mathbf{nodeadlock}\,t$, iff

$$(\mathbf{A}a,b,s_0,t_0 : a \in \mathbf{i}T \wedge b \in \mathbf{i}S \wedge s_0 b \in \mathbf{pref}(s) \wedge t_0 a \in \mathbf{pref}(t) : \#_a s_0 > \#_a t_0 \vee \#_b t_0 > \#_b s_0)$$

end of definition

Traces s of S and t of T have absence of deadlock if and only if for all symbols a in $\mathbf{i}T$, which equals $\mathbf{o}S$, and b in $\mathbf{i}S$, which equals $\mathbf{o}T$, and for any natural numbers i and j, such that $1 \leq i \leq \#_a t$ and $1 \leq j \leq \#_b s$, in s the j-th input of b is preceded by the i-th output of a or in t the i-th input of a is preceded by the j-th output of b.

Definition 2.7 (composable)
Traces s and t, such that $s \in \mathbf{t}S$ and $t \in \mathbf{t}T$, are composable, denoted by $\mathbf{c}(s,t)$, iff

$$(\mathbf{A}a : a \in \mathbf{i}T : \#_a s \geq \#_a t) \wedge (\mathbf{A}b : b \in \mathbf{i}S : \#_b t \geq \#_b s) \wedge s\,\mathbf{nodeadlock}\,t$$

end of definition

A trace of S and a trace of T can be seen as observations of the same communication, if each input occurring in one of them, occurs in the other as output and they have absence of deadlock. Such traces we call composable.

Definition 2.8 (directed resultant)
For a directed trace x and traces s and t, such that $s \in \mathbf{t}S$, $t \in \mathbf{t}T$, and $\mathbf{c}(s,t)$, x is a directed resultant of s and t, denoted by $x\,\mathbf{dres}(s,t)$, iff

$$((x=\varepsilon)\wedge(s=\varepsilon)\wedge(t=\varepsilon))$$

$$\vee\,(\mathbf{E}a,s_0,x_0:(s=s_0a)\wedge x_0\mathbf{dres}(s_0,t):(a\in\mathbf{o}S\Rightarrow(x=x_0a!))\wedge(a\in\mathbf{i}S\Rightarrow(x=x_0a?)))$$

$$\vee\,(\mathbf{E}a,t_0,x_0:(t=t_0a)\wedge x_0\mathbf{dres}(s,t_0):(a\in\mathbf{o}T\Rightarrow(x=x_0a!))\wedge(a\in\mathbf{i}T\Rightarrow(x=x_0a?)))$$

end of definition

In a directed resultant x we use $a!$ or $a?$ to indicate that this occurrence of a in x originates from the output alphabet of a directed trace structure or the input alphabet of a directed trace structure respectively. Notice that for each pair of composable traces a set of directed resultants is defined.

Property 2.0
For directed traces x and y and composable traces s and t, such that $(xy)\mathbf{dres}(s,t)$, and for symbol a

$$\#_{a!}x \geq \#_{a?}x$$

end of property

Each input in a directed resultant is preceded by its corresponding output.

Definition 2.9 (resultant)
For traces s, t, and z, such that $s \in \mathbf{t}S$, $t \in \mathbf{t}T$, $z \in (\mathbf{a}S \cup \mathbf{a}T)^*$, and $\mathbf{c}(s,t)$, z is a resultant of s and t, denoted by $z\,\mathbf{res}(s,t)$, iff

$$(\mathbf{E}x : x\,\mathbf{dres}(s,t) : z\,\mathbf{undirect}\,x)$$

end of definition

Notice that since the alphabets of S and T are equal, $\mathbf{a}S \cup \mathbf{a}T$ equals $\mathbf{a}S$ and $\mathbf{a}T$. For each pair of composable traces a set of resultants is defined. A resultant is a minimal element with respect to the partial order **undirect**, since a resultant is an element of $(\mathbf{a}S \cup \mathbf{a}T)^*$.

Definition 2.10 (composite)
The composite of two directed trace structures S and T, such that $\mathbf{i}S = \mathbf{o}T$ and $\mathbf{o}S = \mathbf{i}T$, which is denoted by $S \circledcirc T$, is the directed trace structure

$$<\mathbf{i}S \cap \mathbf{o}T, \mathbf{o}S \cap \mathbf{i}T, \mathbf{pref}(\{s,t,z : s \in \mathbf{t}S \wedge t \in \mathbf{t}T \wedge \mathbf{c}(s,t) \wedge z\,\mathbf{res}(s,t) : z\})>$$

end of definition

For a directed trace structure T, $<\mathbf{i}T, \mathbf{o}T, \mathbf{t}T>$, $\overline{T}$ denotes its complement, i.e. $<\mathbf{o}T, \mathbf{i}T, \mathbf{t}T>$.

Property 2.1

$$\overline{(\overline{T})} = T$$

end of property

Property 2.2

$$\mathbf{t}T \subseteq \mathbf{t}(T \circledcirc \overline{T})$$

end of property

Definition 2.11 (foam rubber wrapper postulate)
A directed trace structure T does justice to the foam rubber wrapper principle, iff

$$T = T \circledcirc \overline{T}$$

end of definition

For an explanation of our notion of the foam rubber wrapper principle we refer to [HS] and [MFR]. Our notion "foam rubber wrapper principle" equals the notion "FRW-postulate", that Molnar, Fang, and Rosenberger use in [MFR].

2.2 Notions related to $\mathbf{C}_4$

The notions introduced in this section have been adopted from [JTU]. $\mathbf{C}_4$ is the class of delay-insensitive directed trace structures.

Definition 2.12 ($\mathbf{C}_4$)
A directed trace structure T is an element of $\mathbf{C}_4$ if it satisfies the requirements $\mathbf{R}_0$ through $\mathbf{R}_5$:

($\mathbf{R}_0$) $\mathbf{i}T \cup \mathbf{o}T = \mathbf{a}T$

($\mathbf{R}_1$) $\mathbf{i}T$ is prefix-closed and nonempty

($\mathbf{R}_2$) for trace s and symbol $a \in \mathbf{a}T$
$saa \notin \mathbf{t}T$

($\mathbf{R}_3$) for traces s and t, and for symbols $a \in \mathbf{a}T$ and $b \in \mathbf{a}T$ of the same type
$(sabt \in \mathbf{t}T) = (sbat \in \mathbf{t}T)$

($\mathbf{R}_4$) for traces s and t, and for symbols $a \in \mathbf{a}T$, $b \in \mathbf{a}T$, and $c \in \mathbf{a}T$
with b of another type than a and c
$(sabtc \in \mathbf{t}T \wedge sbat \in \mathbf{t}T) \Rightarrow (sbatc \in \mathbf{t}T)$

($\mathbf{R}_5$) for trace s and symbols $a \in \mathbf{a}T$ and $b \in \mathbf{a}T$ of different types
$(sa \in \mathbf{t}T \wedge sb \in \mathbf{t}T) \Rightarrow (sab \in \mathbf{t}T)$

end of definition

Definition 2.13 (from)
For directed trace structure T, and for composable traces $t \in \mathbf{t}T$ and $u \in \mathbf{t}\overline{T}$ we define $\mathbf{from}(t,u)$ as

$$\{x : x \in (\mathbf{o}T \cap \mathbf{i}\overline{T})^* \wedge (\mathrm{A}a : a \in \mathbf{o}T \cap \mathbf{i}\overline{T} : \#_a x = \#_a t - \#_a u) : x\}$$

end of definition

Since $\mathbf{from}(t,u)$ is nonempty and the lengths of the traces in $\mathbf{from}(t,u)$ are equal, we define $\mathbf{l}(\mathbf{from}(t,u))$ as the length of the traces in $\mathbf{from}(t,u)$.

Definition 2.14 (mismatches)
For directed trace structure T, and for composable traces $t \in \mathbf{t}T$ and $u \in \mathbf{t}\overline{T}$, we define

$$\mathbf{mm}(t,u) = \mathbf{l}(\mathbf{from}(t,u)) + \mathbf{l}(\mathbf{from}(u,t))$$

end of definition

Property 2.3

For directed trace structure T, traces t_0, t_1, u_0, and u_1, such that $t_0 \in \mathbf{t}T$, $t_1 \in \mathbf{t}T$, $u_0 \in \mathbf{t}\overline{T}$, $u_1 \in \mathbf{t}\overline{T}$, $\mathbf{c}(t_0,u_0)$, and $\mathbf{c}(t_1,u_1)$, and directed traces x and y, such that $x\,\mathbf{dres}(t_0,u_0)$ and $y\,\mathbf{dres}(t_1,u_1)$,

$$(\mathbf{A}a : a \in (T!? \cup \overline{T}!?) : \#_a x = \#_a y) \Rightarrow (\mathbf{mm}(t_0,u_0) = \mathbf{mm}(t_1,u_1))$$

end of property

Udding uses a definition of composability of traces that differs from our definition, cf. [JTU]. In [HS] is proven that these definitions are equivalent.

3 Directed trace structures that satisfy the foam rubber wrapper postulate

Determining whether a directed trace structure satisfies the foam rubber wrapper postulate, comes to checking whether the resultants of its traces are elements of its trace set. Resultants are obtained by applying directed resultant and immediate undirect operators. In section 3.0 some tools are presented which are used in the subsequent sections. We deal with directed resultant and immediate undirect operations in section 3.1. In section 3.2 mathematical induction on the number of immediate undirect operations is applied. Theorems 3.0 and 3.1 in section 3.3 are conclusions drawn from the results of the previous sections.

Note
For the remainder of this chapter U denotes a directed trace structure that satisfies $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{R}_3$, $\mathbf{R}_4$, and $\mathbf{R}_5$.
end of note

3.0 Preparation

In order to indicate the type of symbols with respect to a directed trace structure we introduce the notion postfix type. Notice that for a symbol a, $a!$ and $a?$ are symbols, not concatenations of symbols and an exclamation point or a question mark respectively.

Definition 3.0 (postfix type)
For directed trace structure T and trace t, such that $t \in \mathbf{t}T$, the trace denoted by $\mathbf{postf}(T,t)$, in which the symbols in t are postfixed by their type with respect to T, is defined by:

(i) $\mathbf{postf}(T,\varepsilon) = \varepsilon$

(ii) for trace u and symbol a, such that $ua \in \mathbf{t}T$ and $a \in \mathbf{o}T$
$\mathbf{postf}(T,ua) = \mathbf{postf}(T,u)a!$

(iii) for trace u and symbol a, such that $ua \in \mathbf{t}T$ and $a \in \mathbf{i}T$
$\mathbf{postf}(T,ua) = \mathbf{postf}(T,u)a?$

end of definition

Property 3.0 is derived from the definition of directed resultant, composability, and postfix type.

Property 3.0

For directed trace structure T, composable traces t and u, such that $t \in \mathbf{t}T$ and $u \in \mathbf{t}\overline{T}$, and directed trace x, such that $x \in (T!? \cup \overline{T}!?)^*$,

$$\begin{aligned} x\,\mathbf{dres}(t,u) = (&(x \upharpoonright T!? = \mathbf{postf}(T,t)) \wedge (x \upharpoonright \overline{T}!? = \mathbf{postf}(\overline{T},u)) \\ &\wedge (\mathbf{A}a,y : y \in \mathbf{pref}(x) : \#_{a!}y \geq \#_{a?}y) \\ ) \end{aligned}$$

end of property

In lemma 3.0 we deal with absence of danger of computation interference, i.e. all symbols on their way between composable traces can be received. For a definition of these notions we refer to [JTU]. Our notion "absence of danger of computation interference" equals the notion "absence of computation interference" that Udding uses in [JTU].

Lemma 3.0

For directed trace structure T, that satisfies $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{R}_3$, $\mathbf{R}_4$, and $\mathbf{R}_5$, and for composable traces t and u, such that $t \in \mathbf{t}T$ and $u \in \mathbf{t}\overline{T}$,

$$(\mathbf{A}a : a \in \mathbf{o}T \cap \mathbf{i}\overline{T} \wedge (\#_a t > \#_a u) : ua \in \mathbf{t}\overline{T})$$

Proof

This is a theorem which is proved by Udding [JTU, p.45 and pp.49-58]; T and $\overline{T}$ satisfy all conditions of connectable directed trace structures except $\mathbf{R}_2$; Udding does not use $\mathbf{R}_2$ to prove that theorem.

end of lemma

Note

Udding refers to $\mathbf{R}_2$, [JTU, p.50]. He uses $\mathbf{R}_2$ to prove absence of (danger of) transmission interference only, not to prove absence of danger of computation interference.

end of note

3.1 Interchanging adjacent symbols in directed resultants

Lemmata 3.1 and 3.2 deal with interchanging adjacent symbols in directed resultants. In order to derive them we present some properties and lemmata. Properties 3.1.0 and 3.1.1 are derived from property 3.0. Property 3.1.2 is derived from property 3.0 and the definitions of composability, directed resultant, and postfix type.

The occurrences of the symbols interchanged in properties 3.1.0 and 3.1.1 originate from distinct directed trace structures.

Property 3.1.0
For traces t and u, such that $t \in \mathbf{t}U$, $u \in \mathbf{t}\overline{U}$, and $\mathbf{c}(t,u)$, directed traces x and y, and symbols a and b of distinct types

(i) $(xa!b!y)\,\mathbf{dres}(t,u) = (xb!a!y)\,\mathbf{dres}(t,u)$

(ii) $(xa?b?y)\,\mathbf{dres}(t,u) = (xb?a?y)\,\mathbf{dres}(t,u)$

end of property

Property 3.1.1
For traces t and u, such that $t \in \mathbf{t}U$, $u \in \mathbf{t}\overline{U}$, and $\mathbf{c}(t,u)$, directed traces x and y, and distinct symbols a and b of the same type

$(xa!b?y)\,\mathbf{dres}(t,u) = (xb?a!y)\,\mathbf{dres}(t,u)$

end of property

Property 3.1.2 expresses that prefixes of directed resultants of composable traces be directed resultants of composable prefixes of those traces..

Property 3.1.2
For traces t and u, such that $t \in \mathbf{t}U$, $u \in \mathbf{t}\overline{U}$, and $\mathbf{c}(t,u)$, and directed trace x, such that $x\,\mathbf{dres}(t,u)$,

$$
\begin{array}{l}
(\mathbf{A}x_0 : x_0 \in \mathbf{pref}(x) \\
\quad : (\mathbf{E}t_0,u_0 : t_0 \in \mathbf{pref}(t) \wedge u_0 \in \mathbf{pref}(u) \\
\qquad\qquad \wedge (x_0 \restriction U!? = \mathbf{postf}(U,t_0)) \wedge (x_0 \restriction \overline{U}!? = \mathbf{postf}(\overline{U},u_0)) \\
\qquad\quad : \mathbf{c}(t_0,u_0) \wedge x_0\,\mathbf{dres}(t_0,u_0) \\
\quad) \\
)
\end{array}
$$

end of property

In lemmata 3.1.0 and 3.1.1 we deal with interchanging occurrences, that originate from the same directed trace structure, of symbols of the same type. In lemma 3.1.0 we treat occurrences, that originate from the directed trace structure in which the symbols are outputs.

Lemma 3.1.0

For directed traces x and y, such that $xy \in (U!? \cup \overline{U}!?)^*$, and symbols a and b of the same type

$$(\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0, u_0) : (xa!b!y)\,\mathbf{dres}(t_0, u_0))$$

$$= (\mathbf{E} t, u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t, u) : (xb!a!y)\,\mathbf{dres}(t, u))$$

Proof

Given directed traces x and y, such that $xy \in (U!? \cup \overline{U}!?)^*$, and symbols a and b, such that $a \in \mathbf{o}U$ and $b \in \mathbf{o}U$. We derive :

$$(\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0, u_0) : (xa!b!y)\,\mathbf{dres}(t_0, u_0))$$

= { property 3.0, definition of directed resultant, and calculus }

$$\begin{aligned}
&(\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \\
&\qquad : ((xa!b!y) \upharpoonright U!? = \mathbf{postf}(U, t_0)) \wedge ((xa!b!y) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U}, u_0)) \\
&\qquad \wedge (\mathbf{A} c, z : z \in \mathbf{pref}(xa!b!y) : \#_{c!} z \geq \#_{c?} z) \\
&)
\end{aligned}$$

= { calculus, $a! \in (\mathbf{o}U)!$, and $b! \in (\mathbf{o}U)!$ }

$$\begin{aligned}
&(\mathbf{E} t_0, t_1, t_2, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (t_0 = t_1 ab t_2) \wedge (x \upharpoonright U!? = \mathbf{postf}(U, t_1)) \\
&\qquad\qquad : ((xa!b!y) \upharpoonright U!? = \mathbf{postf}(U, t_1 ab t_2)) \wedge ((xy) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U}, u_0)) \\
&\qquad\qquad \wedge (\mathbf{A} c, z : z \in \mathbf{pref}(xb!a!y) : \#_{c!} z \geq \#_{c?} z) \\
&)
\end{aligned}$$

= { U satisfies $\mathbf{R}_3$, $a! \in (\mathbf{o}U)!$, $b! \in (\mathbf{o}U)!$, definition of composability, and definition of postfix type
}

$$\begin{aligned}
&(\mathbf{E} t_1, t_2, u_0 : (t_1 ba t_2) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (x \upharpoonright U!? = \mathbf{postf}(U, t_1)) \\
&\qquad : ((xb!a!y) \upharpoonright U!? = \mathbf{postf}(U, t_1 ba t_2)) \wedge ((xb!a!y) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U}, u_0)) \\
&\qquad \wedge (\mathbf{A} c, z : z \in \mathbf{pref}(xb!a!y) : \#_{c!} z \geq \#_{c?} z) \\
&)
\end{aligned}$$

= { calculus, property 3.0, and definition of directed resultant }

$$(\mathbf{E} t, u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t, u) : (xb!a!y)\,\mathbf{dres}(t, u))$$

For symbols a en b, such that $a \in \mathbf{o}\overline{U}$ and $b \in \mathbf{o}\overline{U}$, the proof is analogous.

end of lemma

Lemma 3.1.1 is the counterpart of lemma 3.1.0 : occurrences, that originate from the directed trace structure in which the symbols are input symbols, are treated.

Lemma 3.1.1

For directed traces x and y, such that $xy \in (U!? \cup \overline{U}!?)^*$, and symbols a and b of the same type

$$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) : (xa?b?y)\,\mathbf{dres}\,(t_0,u_0))$$

$$= (\mathbf{E}t,u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t,u) : (xb?a?y)\,\mathbf{dres}\,(t,u))$$

The proof of this lemma is analogous to the proof of lemma 3.1.0.

end of lemma

Lemma 3.1.2 deals with interchanging occurrences, that originate from the same directed trace structure, of symbols of distinct types in one way: output backward and input forward.

Lemma 3.1.2

For directed traces x and y, such that $xy \in (U!? \cup \overline{U}!?)^*$, and symbols a and b of distinct types

$$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) : (xa!b?y)\,\mathbf{dres}\,(t_0,u_0))$$

$$\Rightarrow (\mathbf{E}t,u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t,u) : (xb?a!y)\,\mathbf{dres}\,(t,u))$$

Proof

We prove this lemma by mathematical induction on the length of y. Given directed traces x and y, such that $xy \in (U!? \cup \overline{U}!?)^*$, and symbols a and b, such that $a \in \mathbf{o}U$ and $b \in \mathbf{i}U$.

Induction hypothesis

$$(\mathbf{A}y_0 : \mathbf{l}(y_0) < \mathbf{l}(y) :$$

$$: (\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) : (xa!b?y_0)\,\mathbf{dres}\,(t_0,u_0))$$

$$\Rightarrow (\mathbf{E}t,u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t,u) : (xb?a!y_0)\,\mathbf{dres}\,(t,u))$$

$$)$$

Base : $\mathbf{l}(y)=0$

We derive :

$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) : (xa!b?y)\,\mathbf{dres}(t_0,u_0))$

$=$ $\{ y=\varepsilon$, since $\mathbf{l}(y)=0 \}$

$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) : (xa!b?)\,\mathbf{dres}(t_0,u_0))$

$=$ $\{$ calculus and property 3.0 $\}$

$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0)$

$\quad : ((xa!b?) \restriction U!? = \mathbf{postf}(U,t_0)) \wedge ((xa!b?) \restriction \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$

$\quad \wedge (xa!b?)\,\mathbf{dres}(t_0,u_0)$

$)$

$=$ $\{$ calculus, $a! \in (\mathbf{o}U)!$, $b? \in (\mathbf{i}U)?$, and definition of composability $\}$

$(\mathbf{E}t_0,t_1,u_0 : t_0 \in \mathbf{t}U \wedge (t_0 = t_1ab) \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_1ab,u_0)$

$\quad : (x \restriction U!? = \mathbf{postf}(U,t_1)) \wedge (x \restriction \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$

$\quad \wedge (xa!b?)\,\mathbf{dres}(t_1ab,u_0)$

$)$

$\Rightarrow$ $\{$ property 3.1.2 and calculus $\}$

$(\mathbf{E}t_1,u_0 : (t_1ab) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_1ab,u_0) : \mathbf{c}(t_1,u_0) \wedge x\,\mathbf{dres}(t_1,u_0))$

$\Rightarrow$ $\{$ definition of composability, $b \in \mathbf{i}U$, and $\mathbf{t}U$ is prefix closed $\}$

$(\mathbf{E}t_1,u_0 : (t_1a) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} : \#_b t_1 < \#_b u_0 \wedge \mathbf{c}(t_1,u_0) \wedge x\,\mathbf{dres}(t_1,u_0))$

$=$ $\{$ lemma 3.0 using property 2.1, and calculus $\}$

$(\mathbf{E}t_1,u_0 : (t_1a) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} : \#_b t_1 < \#_b u_0 \wedge \mathbf{c}(t_1,u_0) \wedge (t_1b) \in \mathbf{t}U \wedge x\,\mathbf{dres}(t_1,u_0))$

$=$ $\{ b \in \mathbf{i}U$, definition of composability, and definition of directed resultant $\}$

$(\mathbf{E}t_1,u_0 : (t_1a) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} : \mathbf{c}(t_1b,u_0) \wedge (t_1b) \in \mathbf{t}U \wedge (xb?)\,\mathbf{dres}(t_1b,u_0))$

$\Rightarrow$ $\{U$ satisfies $\mathbf{R}_5$, $a \in \mathbf{o}U$, $b \in \mathbf{i}U$, definition of composability,

and definition of directed resultant using property 2.0

$\}$

$(\mathbf{E}t_1,u_0 : u_0 \in \mathbf{t}\overline{U} : \mathbf{c}(t_1ba,u_0) \wedge (t_1ba) \in \mathbf{t}U \wedge (xb?a!)\,\mathbf{dres}(t_1ba,u_0))$

$\Rightarrow$ $\{ y=\varepsilon$ and calculus $\}$

$(\mathbf{E}t,u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t,u) : (xb?a!y)\,\mathbf{dres}(t,u))$

Step : $\mathbf{l}(y)>0$

We distinguish four cases :

(0) $(\mathbf{E}y_0,c : c \in \mathbf{o}\overline{U} : y = y_0c!)$

(1) $(\mathbf{E}y_0,c : c \in \mathbf{i}\overline{U} : y = y_0c?)$

(2) $(\mathbf{E}y_0,c : c \in \mathbf{o}U : y = y_0c!)$

(3) $(\mathbf{E}y_0,c : c \in \mathbf{i}U : y = y_0c?)$

Case (0) : $(\mathbf{E}y_0,c : c \in \mathbf{o}\overline{U} : y = y_0 c!)$

$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) : (xa!b?y)\,\mathbf{dres}(t_0,u_0))$

$\wedge(\mathbf{E}y_0,c : c \in \mathbf{o}\overline{U} : y = y_0 c!)$

$=$ $\{$ calculus and property 3.0 $\}$

$(\mathbf{E}c,t_0,u_0,y_0$

$: c \in \mathbf{o}\overline{U} \wedge t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0 c!)$

$: (xa!b?y_0 c!)\,\mathbf{dres}(t_0,u_0) \wedge \mathbf{c}(t_0,u_0)$

$\wedge((xa!b?y_0 c!) \upharpoonright U!? = \mathbf{postf}(U,t_0)) \wedge ((xa!b?y_0 c!) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$

$)$

$=$ $\{$ calculus $\}$

$(\mathbf{E}c,t_0,u_0,u_1,y_0$

$: c \in \mathbf{o}\overline{U} \wedge t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (u_0 = u_1 c) \wedge (y = y_0 c!)$

$: (xa!b?y_0 c!)\,\mathbf{dres}(t_0,u_1 c) \wedge \mathbf{c}(t_0,u_1 c)$

$\wedge((xa!b?y_0) \upharpoonright U!? = \mathbf{postf}(U,t_0)) \wedge ((xa!b?y_0) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_1))$

$)$

$\Rightarrow$ $\{$ property 3.1.2 and calculus $\}$

$(\mathbf{E}c,t_0,u_1,y_0 : c \in \mathbf{o}\overline{U} \wedge t_0 \in \mathbf{t}U \wedge (u_1 c) \in \mathbf{t}\overline{U} \wedge (y = y_0 c!)$

$: (xa!b?y_0)\,\mathbf{dres}(t_0,u_1) \wedge ((xa!b?y_0) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_1))$

$)$

$\Rightarrow$ $\{$ induction hypothesis $\}$

$(\mathbf{E}c,t_1,u_1,u_2,y_0 : c \in \mathbf{o}\overline{U} \wedge t_1 \in \mathbf{t}U \wedge (u_1 c) \in \mathbf{t}\overline{U} \wedge u_2 \in \mathbf{t}\overline{U} \wedge (y = y_0 c!)$

$: (xb?a!y_0)\,\mathbf{dres}(t_1,u_2) \wedge ((xa!b?y_0) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_1))$

$)$

$=$ $\{$ property 3.0 and calculus $\}$

$(\mathbf{E}c,t_1,u_1,u_2,y_0 : c \in \mathbf{o}\overline{U} \wedge t_1 \in \mathbf{t}U \wedge (u_1 c) \in \mathbf{t}\overline{U} \wedge u_2 \in \mathbf{t}\overline{U} \wedge (y = y_0 c!)$

$: (xb?a!y_0)\,\mathbf{dres}(t_1,u_2) \wedge ((xb?a!y_0) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_2))$

$\wedge((xa!b?y_0) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_1))$

$)$

$\Rightarrow$ $\{$ calculus, definition of postfix type, and $(b?a!) \upharpoonright \overline{U}!? = \varepsilon$ $\}$

$(\mathbf{E}c,t_1,u_1,u_2,y_0 : c \in \mathbf{o}\overline{U} \wedge t_1 \in \mathbf{t}U \wedge (u_1 c) \in \mathbf{t}\overline{U} \wedge u_2 \in \mathbf{t}\overline{U} \wedge (y = y_0 c!)$

$: (xb?a!y_0)\,\mathbf{dres}(t_1,u_2) \wedge (u_1 = u_2)$

$)$

$=$ $\{$ calculus, definition of composability, and definition of directed resultant $\}$

$(\mathbf{E}c,t_1,u_1,y_0 : c \in \mathbf{o}\overline{U} \wedge t_1 \in \mathbf{t}U \wedge (u_1 c) \in \mathbf{t}\overline{U} \wedge (y = y_0 c!)$

$: \mathbf{c}(t_1,u_1,c) \wedge (xb?a!y_0 c!)\,\mathbf{dres}(t_1,u_1 c)$

$)$

$\Rightarrow$ $\{$ calculus $\}$

$(\mathbf{E}t,u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t,u) : (xb?a!y)\,\mathbf{dres}(t,u))$

end of case

Case (1) : $(\mathbf{E}y_0,c : c\in\mathbf{i}\overline{U} : y = y_0c?\,)$

$(\mathbf{E}t_0,u_0 : t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_0,u_0) : (xa!b?y\,)\mathbf{dres}(t_0,u_0))$
$\wedge(\mathbf{E}c,y_0 : c\in\mathbf{i}\overline{U} : y = y_0c?\,)$
$=$ { calculus and property 3.0 }
$(\mathbf{E}c,t_0,u_0,y_0 : c\in\mathbf{i}\overline{U}\wedge t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge(y = y_0c?\,)\wedge\mathbf{c}(t_0,u_0)$
$: (xa!b?y_0c?\,)\mathbf{dres}(t_0,u_0)\wedge((xa!b?y_0c?\,)\upharpoonright U!? = \mathbf{postf}(U,t_0))$
$\wedge((xa!b?y_0c?\,)\upharpoonright\overline{U}!? = \mathbf{postf}(\overline{U},u_0))$
$)$
$=$ { calculus and property 2.0 }
$(\mathbf{E}c,t_0,u_0,u_1,y_0$
$: c\in\mathbf{i}\overline{U}\wedge t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge(u_0 = u_1c\,)\wedge(y = y_0c?\,)$
$: \#_{c!}(xa!b?y_0c?\,)\geq\#_{c?}(xa!b?y_0c?\,)\wedge(xa!b?y_0c?\,)\mathbf{dres}(t_0,u_1c\,)\wedge\mathbf{c}(t_0,u_1c\,)$
$\wedge((xa!b?y_0)\upharpoonright U!? = \mathbf{postf}(U,t_0))\wedge((xa!b?y_0)\upharpoonright\overline{U}!? = \mathbf{postf}(\overline{U},u_1))$
$)$
$\Rightarrow$ { property 3.1.2 and calculus }
$(\mathbf{E}c,t_0,u_1,y_0 : c\in\mathbf{i}\overline{U}\wedge t_0\in\mathbf{t}U\wedge(u_1c\,)\in\mathbf{t}\overline{U}\wedge(y = y_0c?\,)$
$: \#_{c!}(xa!b?y_0) > \#_{c?}(xa!b?y_0)\wedge(xa!b?y_0)\mathbf{dres}(t_0,u_1)$
$\wedge((xa!b?y_0)\upharpoonright\overline{U}!? = \mathbf{postf}(\overline{U},u_1))$
$)$
$\Rightarrow$ { induction hypothesis and calculus }
$(\mathbf{E}c,t_1,u_1,u_2,y_0 : c\in\mathbf{i}\overline{U}\wedge t_1\in\mathbf{t}U\wedge(u_1c\,)\in\mathbf{t}\overline{U}\wedge u_2\in\mathbf{t}\overline{U}\wedge(y = y_0c?\,)$
$: \#_{c!}(xb?a!y_0) > \#_{c?}(xb?a!y_0)\wedge(xb?a!y_0)\mathbf{dres}(t_1,u_2)$
$\wedge((xa!b?y_0)\upharpoonright\overline{U}!? = \mathbf{postf}(\overline{U},u_1))$
$)$
$=$ { property 3.0 and calculus }
$(\mathbf{E}c,t_1,u_1,u_2,y_0$
$: c\in\mathbf{i}\overline{U}\wedge t_1\in\mathbf{t}U\wedge(u_1c\,)\in\mathbf{t}\overline{U}\wedge u_2\in\mathbf{t}\overline{U}\wedge(y = y_0c?\,)$
$: \#_{c!}(xb?a!y_0) > \#_{c?}(xb?a!y_0)\wedge(xb?a!y_0)\mathbf{dres}(t_1,u_2)$
$\wedge((xb?a!y_0)\upharpoonright\overline{U}!? = \mathbf{postf}(\overline{U},u_2))\wedge((xa!b?y_0)\upharpoonright\overline{U}!? = \mathbf{postf}(\overline{U},u_1))$
$)$
$\Rightarrow$ { calculus, definition of postfix type, and $(b?a!)\upharpoonright\overline{U}!? = \varepsilon$ }
$(\mathbf{E}c,t_1,u_1,u_2,y_0 : c\in\mathbf{i}\overline{U}\wedge t_1\in\mathbf{t}U\wedge(u_1c\,)\in\mathbf{t}\overline{U}\wedge u_2\in\mathbf{t}\overline{U}\wedge(y = y_0c?\,)$
$: \#_c t_1 > \#_c u_2\wedge(xb?a!y_0)\mathbf{dres}(t_1,u_2)\wedge(u_1 = u_2)$
$)$
$=$ { calculus, definition of composability, and definition of directed resultant }
$(\mathbf{E}c,t_1,u_1,y_0 : c\in\mathbf{i}\overline{U}\wedge t_1\in\mathbf{t}U\wedge(u_1c\,)\in\mathbf{t}\overline{U}\wedge(y = y_0c?\,)$
$: \mathbf{c}(t_1,u_1c\,)\wedge(xb?a!y_0c!\,)\mathbf{dres}(t_1,u_1c\,)$
$)$
$\Rightarrow$ { calculus }
$(\mathbf{E}t,u : t\in\mathbf{t}U\wedge u\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t,u) : (xb?a!y\,)\mathbf{dres}(t,u))$

end of case

Case (2) : $(\mathbf{E}y_0,c : c \in \mathbf{o}U : y = y_0c!)$

$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) : (xa!b?y)\mathbf{dres}(t_0,u_0))$
$\wedge(\mathbf{E}c,y_0 : c \in \mathbf{o}U : y = y_0c!)$

= { calculus and property 3.0 }

$(\mathbf{E}c,t_0,u_0,y_0 : c \in \mathbf{o}U \wedge t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c!) \wedge \mathbf{c}(t_0,u_0)$
$\quad : (xa!b?y_0c!)\mathbf{dres}(t_0,u_0) \wedge ((xa!b?y_0c!)\upharpoonright U!? = \mathbf{postf}(U,t_0))$
$\quad \wedge((xa!b?y_0c!)\upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$
$)$

= { calculus }

$(\mathbf{E}c,t_0,t_1,u_0,y_0 : c \in \mathbf{o}U \wedge t_0 \in \mathbf{t}U \wedge (t_0 = t_1c) \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c!) \wedge \mathbf{c}(t_1c,u_0)$
$\quad : (xa!b?y_0c!)\mathbf{dres}(t_1c,u_0) \wedge ((xa!b?y_0)\upharpoonright U!? = \mathbf{postf}(U,t_1))$
$\quad \wedge((xa!b?y_0)\upharpoonright \overline{U} = \mathbf{postf}(\overline{U},u_0))$
$)$

⇒ { property 3.1.2 and calculus }

$(\mathbf{E}c,t_1,u_0,y_0 : c \in \mathbf{o}U \wedge (t_1c) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c!)$
$\quad : (xa!b?y_0)\mathbf{dres}(t_1,u_0) \wedge ((xa!b?y_0)\upharpoonright U!? = \mathbf{postf}(U,t_1))$
$\quad \wedge((xa!b?y_0)\upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$
$)$

⇒ { induction hypothesis }

$(\mathbf{E}c,t_1,t_2,u_0,u_1,y_0 : c \in \mathbf{o}U \wedge (t_1c) \in \mathbf{t}U \wedge t_2 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge u_1 \in \mathbf{t}\overline{U} \wedge (y = y_0c!)$
$\quad : (xb?a!y_0)\mathbf{dres}(t_2,u_1) \wedge ((xa!b?y_0)\upharpoonright U!? = \mathbf{postf}(U,t_1))$
$\quad \wedge((xa!b?y_0)\upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$
$)$

= { property 3.0 and calculus }

$(\mathbf{E}c,t_1,t_2,u_0,u_1,y_0$
$\quad : c \in \mathbf{o}U \wedge (t_1c) \in \mathbf{t}U \wedge t_2 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge u_1 \in \mathbf{t}\overline{U} \wedge (y = y_0c!)$
$\quad : (xb?a!y_0)\mathbf{dres}(t_2,u_1)$
$\quad \wedge((xb?a!y_0)\upharpoonright U!? = \mathbf{postf}(U,t_2)) \wedge ((xb?a!y_0)\upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_1))$
$\quad \wedge((xa!b?y_0)\upharpoonright U!? = \mathbf{postf}(U,t_1)) \wedge ((xa!b?y_0)\upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$
$)$

⇒ { calculus, definition of postfix type, and $(b?a!)\upharpoonright \overline{U}!? = \varepsilon$ }

$(\mathbf{E}c,t_1,t_2,t_3,t_4,u_0,u_1,y_0$
$\quad : c \in \mathbf{o}U \wedge (t_1c) \in \mathbf{t}U \wedge t_2 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge u_1 \in \mathbf{t}\overline{U} \wedge (y = y_0c!)$
$\quad : (xb?a!y_0)\mathbf{dres}(t_2,u_1) \wedge (t_1 = t_3abt_4) \wedge (t_2 = t_3bat_4) \wedge (u_0 = u_1)$
$)$

= { calculus }

$$(\mathbf{E}c,t_3,t_4,u_0,y_0 : c \in \mathbf{o}U \wedge (t_3abt_4c) \in \mathbf{t}U \wedge (t_3bat_4) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c!)$$

$$: (xb?a!y_0)\mathbf{dres}(t_3bat_4, u_0)$$

$$)$$

$\Rightarrow$ { U satisfies $\mathbf{R}_4$ }

$$(\mathbf{E}c,t_3,t_4,u_0,y_0 : c \in \mathbf{o}U \wedge (t_3bat_4c) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c!)$$

$$: (xb?a!y_0)\mathbf{dres}(t_3bat_4, u_0)$$

$$)$$

$=$ { definition of composability, definition of directed resultant, and calculus }

$$(\mathbf{E}c,t_3,t_4,u_0,y_0 : c \in \mathbf{o}U \wedge (t_3bat_4c) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c!)$$

$$: \mathbf{c}(t_3bat_4c, u) \wedge (xb?a!y_0c!)\mathbf{dres}(t_3bat_4c, u)$$

$$)$$

$\Rightarrow$ { calculus }

$$(\mathbf{E}t,u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t,u) : (xb?a!y)\mathbf{dres}(t,u))$$

end of case

Case (3): $(\mathbf{E}y_0,c : c \in \mathbf{i}U : y = y_0c?)$

$$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) : (xa!b?y)\mathbf{dres}(t_0,u_0))$$

$$\wedge (\mathbf{E}c,y_0 : c \in \mathbf{i}U : y = y_0c?)$$

$=$ { calculus and property 3.0 }

$$(\mathbf{E}c,t_0,u_0,y_0$$

$$: c \in \mathbf{i}U \wedge t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c?) \wedge \mathbf{c}(t_0,u_0)$$

$$: (xa!b?y_0c?)\mathbf{dres}(t_0,u_0)$$

$$\wedge ((xa!b?y_0c?) \upharpoonright U!? = \mathbf{postf}(U,t_0)) \wedge ((xa!b?y_0c?) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$$

$$)$$

$=$ { calculus and property 2.0 }

$$(\mathbf{E}c,t_0,t_1,u_0,y_0$$

$$: c \in \mathbf{i}U \wedge t_0 \in \mathbf{t}U \wedge (t_0 = t_1c) \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c?) \wedge \mathbf{c}(t_1c,u_0)$$

$$: \#_{c!}(xa!b?y_0c?) \geq \#_{c?}(xa!b?y_0c?) \wedge (xa!b?y_0c?)\mathbf{dres}(t_1c,u_0)$$

$$\wedge ((xa!b?y_0) \upharpoonright U!? = \mathbf{postf}(U,t_1)) \wedge ((xa!b?y_0) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$$

$$)$$

$\Rightarrow$ { property 3.1.2 and calculus }

$$(\mathbf{E}c,t_1,u_0,y_0$$

$$: c \in \mathbf{i}U \wedge (t_1c) \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge (y = y_0c?)$$

$$: \#_{c!}(xa!b?y_0) > \#_{c?}(xa!b?y_0) \wedge (xa!b?y_0)\mathbf{dres}(t_1,u_0)$$

$$\wedge ((xa!b?y_0) \upharpoonright U!? = \mathbf{postf}(U,t_1)) \wedge ((xa!b?y_0) \upharpoonright \overline{U}!? = \mathbf{postf}(\overline{U},u_0))$$

$$)$$

$\Rightarrow$ { induction hypothesis and calculus }

$$\begin{array}{l}
(\mathbf{E}c,t_1,t_2,u_0,u_1,y_0\\
\quad :c\in\mathbf{i}U\wedge(t_1c)\in\mathbf{t}U\wedge t_2\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge u_1\in\mathbf{t}\overline{U}\wedge(y=y_0c?)\\
\quad :\#_{c!}(xb?a!y_0)>\#_{c?}(xb?a!y_0)\wedge(xb?a!y_0)\mathbf{dres}(t_2,u_1)\\
\qquad \wedge((xa!b?y_0)\upharpoonright U!?=\mathbf{postf}(U,t_1))\wedge((xa!b?y_0)\upharpoonright\overline{U}!?=\mathbf{postf}(\overline{U},u_0))\\
)
\end{array}$$

= { property 3.0 and calculus }

$$\begin{array}{l}
(\mathbf{E}c,t_1,t_2,u_0,u_1,y_0\\
\quad :c\in\mathbf{i}U\wedge(t_1c)\in\mathbf{t}U\wedge t_2\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge u_1\in\mathbf{t}\overline{U}\wedge(y=y_0c?)\\
\quad :\#_{c!}(xb?a!y_0)>\#_{c?}(xb?a!y_0)\wedge(xb?a!y_0)\mathbf{dres}(t_2,u_1)\\
\qquad \wedge((xb?a!y_0)\upharpoonright U!?=\mathbf{postf}(U,t_2))\wedge((xb?a!y_0)\upharpoonright\overline{U}!?=\mathbf{postf}(\overline{U},u_1))\\
\qquad \wedge((xa!b?y_0)\upharpoonright U!?=\mathbf{postf}(U,t_1))\wedge((xa!b?y_0)\upharpoonright\overline{U}!?=\mathbf{postf}(\overline{U},u_0))\\
)
\end{array}$$

$\Rightarrow$ { calculus, definition of postfix type, and $(b?a!)\upharpoonright\overline{U}!?=\varepsilon$ }

$$\begin{array}{l}
(\mathbf{E}c,t_1,t_2,t_3,t_4,u_0,u_1,y_0\\
\quad :c\in\mathbf{i}U\wedge(t_1c)\in\mathbf{t}U\wedge t_2\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge u_1\in\mathbf{t}\overline{U}\wedge(y=y_0c?)\\
\quad :\#_c u_1>\#_c t_2\wedge(xb?a!y_0)\mathbf{dres}(t_2,u_1)\wedge(t_1=t_3abt_4)\wedge(t_2=t_3bat_4)\wedge(u_0=u_1)\\
)
\end{array}$$

= { calculus and property 3.0 }

$$\begin{array}{l}
(\mathbf{E}c,t_3,t_4,u_0,y_0:c\in\mathbf{i}U\wedge(t_3bat_4)\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge(y=y_0c?)\\
\qquad\qquad :\#_c u_0>\#_c(t_3bat_4)\wedge(xb?a!y_0)\mathbf{dres}(t_3bat_4,u_0)\wedge\mathbf{c}(t_3bat_4,u_0)\\
)
\end{array}$$

= { lemma 3.0 using property 2.1, and definition of composability }

$$\begin{array}{l}
(\mathbf{E}c,t_3,t_4,u_0,y_0:c\in\mathbf{i}U\wedge(t_3bat_4c)\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge(y=y_0c?)\\
\qquad\qquad :(xb?a!y_0)\mathbf{dres}(t_3bat_4,u_0)\wedge\mathbf{c}(t_3bat_4c,u_0)\\
)
\end{array}$$

= { definition of directed resultant }

$$\begin{array}{l}
(\mathbf{E}c,t_3,t_4,u_0,y_0:c\in\mathbf{i}U\wedge(t_3bat_4c)\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge(y=y_0c?)\\
\qquad\qquad :(xb?a!y_0c?)\mathbf{dres}(t_3bat_4c,u_0)\wedge\mathbf{c}(t_3bat_4c,u_0)\\
)
\end{array}$$

$\Rightarrow$ { calculus }

$$(\mathbf{E}t,u:t\in\mathbf{t}U\wedge u\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t,u):(xb?a!y)\mathbf{dres}(t,u))$$

end of case

end of step

For symbols a and b, such that $a\in\mathbf{o}\overline{U}$ and $b\in\mathbf{i}\overline{U}$, the proof is analogous.

end of lemma

We combine the results obtained thus far into two lemmata. In lemma 3.1 we move output occurrences backward. Input occurrences are put forward in lemma 3.2.

Lemma 3.1

For symbol a, such that $a \in \mathbf{a}U$, and partially directed traces y_0, y_1, y_2, and y_3, such that $(y_0 y_1 y_2 y_3) \in (\mathbf{a}U \cup U!? \cup \bar{U}!?)^*$ and $(y_1 y_2) \upharpoonright \{a?\} = \varepsilon$,

$$(\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t_0, u_0) : (\mathbf{dir}(\mathbf{a}U, y_0 a! y_1 y_2 a? y_3))\,\mathbf{dres}(t_0, u_0))$$

$$\Rightarrow (\mathbf{E} t, u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t, u) : (\mathbf{dir}(\mathbf{a}U, y_0 y_1 a! y_2 a? y_3))\,\mathbf{dres}(t, u))$$

Proof

Given a, y_0, y_1, y_2, and y_3, such that $a \in \mathbf{a}U$, $(y_0 y_1 y_2 y_3) \in (\mathbf{a}U \cup U!? \cup \bar{U}!?)^*$, and $(y_1 y_2) \upharpoonright \{a?\} = \varepsilon$. For shortness we introduce some abbreviations : x_0, x_1, and x_2 denote $\mathbf{dir}(\mathbf{a}U, y_0)$, $\mathbf{dir}(\mathbf{a}U, y_1)$, and $\mathbf{dir}(\mathbf{a}U, y_2 a? y_3)$ respectively. We proof this lemma by mathematical induction on the length of x_1.

Induction hypothesis

$$(\mathbf{A} w_0, w_1 : (w_0 w_1 = x_1) \wedge \mathbf{l}(w_0) < \mathbf{l}(x_1)$$

$$: (\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t_0, u_0) : (x_0 a! w_0 w_1 x_2)\,\mathbf{dres}(t_0, u_0))$$

$$\Rightarrow (\mathbf{E} t, u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t, u) : (x_0 w_1 a! w_2)\,\mathbf{dres}(t, u))$$

$$)$$

Base : $\mathbf{l}(x_1) = 0$

We derive :

$$(\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t_0, u_0) : (\mathbf{dir}(\mathbf{a}U, y_0 a! y_1 y_2 a? y_3))\,\mathbf{dres}(t_0, u_0))$$

$= \{$ definitions of direct, x_0, x_1, and x_2, using $a! \notin \mathbf{a}U$ $\}$

$$(\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t_0, u_0) : (x_0 a! x_1 x_2)\,\mathbf{dres}(t_0, u_0))$$

$= \{ x_1 = \varepsilon$, since $\mathbf{l}(x_1) = 0 \}$

$$(\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t_0, u_0) : (x_0 x_1 a! x_2)\,\mathbf{dres}(t_0, u_0))$$

Step : $\mathbf{l}(x_1) > 0$

We derive :

$$(\mathbf{E} t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t_0, u_0) : (\mathbf{dir}(\mathbf{a}U, y_0 a! y_1 y_2 a? y_3))\,\mathbf{dres}(t_0, u_0))$$
$$\wedge (\mathbf{E} w, b : b \in (U!? \cup \bar{U}!?) : x_1 = wb)$$

$= \{$ definitions of direct, x_0, x_1, and x_2, using $a! \notin \mathbf{a}U$ $\}$

$$(\mathbf{E} b, t_0, u_0, w : b \in (U!? \cup \bar{U}!?) \wedge t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t_0, u_0) \wedge (x_1 = wb)$$
$$: (x_0 a! w b x_2)\,\mathbf{dres}(t_0, u_0)$$
$$)$$

$\Rightarrow \{$ induction hypothesis $\}$

$$(\mathbf{E} b, t_1, u_1, w : b \in (U!? \cup \bar{U}!?) \wedge t_1 \in \mathbf{t}U \wedge u_1 \in \mathbf{t}\bar{U} \wedge \mathbf{c}(t_1, u_1) \wedge (x_1 = wb)$$
$$: (x_0 w a! b x_2)\,\mathbf{dres}(t_1, u_1)$$
$$)$$

We distinguish two cases :

(0) $a \in \mathbf{o}U$

(1) $a \in \mathbf{o}\overline{U}$

Case (0) : $a \in \mathbf{o}U$

We use case-analysis :

(0.0) $b \in (\mathbf{o}\overline{U})!$

(0.1) $b \in (\mathbf{o}U)!$

(0.2) $b \in (\mathbf{i}\overline{U})? \wedge (b \neq a?)$

(0.3) $b \in (\mathbf{i}\overline{U})? \wedge (b = a?)$

(0.4) $b \in (\mathbf{i}U)?$

Case (0.0) : $b \in (\mathbf{o}\overline{U})!$

$(\mathbf{E} b, c, t_1, u_1, w : (b = c!) \wedge c \in \mathbf{o}\overline{U} \wedge t_1 \in \mathbf{t}U \wedge u_1 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_1, u_1) \wedge (x_1 = wb)$
$: (x_0 w a! c! x_2) \mathbf{dres}(t_1, u_1)$
$)$

$=$ { property 3.1.0 (i) using a and c have distinct types, and calculus}
$(\mathbf{E} c, t_1, u_1, w : c \in \mathbf{o}\overline{U} \wedge t_1 \in \mathbf{t}U \wedge u_1 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_1, u_1) \wedge (x_1 = wc!)$
$: (x_0 w c! a! x_2) \mathbf{dres}(t_1, u_1)$
$)$

$=$ { calculus }
$(\mathbf{E} t, u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t, u) : (x_0 x_1 a! x_2) \mathbf{dres}(t, u))$

end of case

Case (0.1) : $b \in (\mathbf{o}U)!$

$(\mathbf{E} b, c, t_1, u_1, w : (b = c!) \wedge c \in \mathbf{o}U \wedge t_1 \in \mathbf{t}U \wedge u_1 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_1, u_1) \wedge (x_1 = wb)$
$: (x_0 w a! c! x_2) \mathbf{dres}(t_1, u_1)$
$)$

$=$ { lemma 3.1.0 using a and c have the same type, and calculus }
$(\mathbf{E} c, t, u, w : c \in \mathbf{o}U \wedge t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t, u) \wedge (x_1 = wc!)$
$: (x_0 w c! a! x_2) \mathbf{dres}(t, u)$
$)$

$=$ { calculus }
$(\mathbf{E} t, u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t, u) : (x_0 x_1 a! x_2) \mathbf{dres}(t, u))$

end of case

Case (0.2) : $b\in(\mathbf{i}\overline{U})?\wedge(b\neq a?)$

$(\mathbf{E}b,c,t_1,u_1,w$
$\quad:(b=c?)\wedge c\in\mathbf{i}\overline{U}\wedge(c\neq a)\wedge t_1\in\mathbf{t}U\wedge u_1\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_1,u_1)\wedge(x_1=wb)$
$\quad:(x_0wa!c?x_2)\mathbf{dres}(t_1,u_1)$
$)$
= { property 3.1.1 using a and c have the same type, and calculus }
$(\mathbf{E}c,t_1,u_1,w:c\in\mathbf{i}\overline{U}\wedge(c\neq a)\wedge t_1\in\mathbf{t}U\wedge u_1\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_1,u_1)\wedge(x_1=wc?)$
$\qquad:(x_0wc?a!x_2)\mathbf{dres}(t_1,u_1)$
$)$
= { calculus }
$(\mathbf{E}t,u:t\in\mathbf{t}U\wedge u\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t,u):(x_0x_1a!x_2)\mathbf{dres}(t,u))$

end of case

Case (0.3) : $b\in(\mathbf{i}\overline{U})?\wedge(b=a?)$

$(\mathbf{E}b,t_1,u_1,w:(b=a?)\wedge t_1\in\mathbf{t}U\wedge u_1\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_1,u_1)\wedge(x_1=wb)$
$\qquad:(x_0wa!a?x_2)\mathbf{dres}(t_1,u_1)$
$)$
= { property 2.0 and calculus }
$(\mathbf{E}t_1,u_1,w:t_1\in\mathbf{t}U\wedge u_1\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_1,u_1)\wedge(x_1=wa?)$
$\qquad:(x_0wa!a?x_2)\mathbf{dres}(t_1,u_1)\wedge\#_{a!}x_0\geq\#_{a?}x_0$
$)$
= { $x_1=\mathbf{dir}(\mathbf{a}U,y_1)$, $\#_{a?}y_1=0$, definitions of direct and prefix-closure }
$(\mathbf{E}t_1,u_1,w:t_1\in\mathbf{t}U\wedge u_1\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_1,u_1)\wedge(x_1=wa?)$
$\qquad:(x_0wa!a?x_2)\mathbf{dres}(t_1,u_1)\wedge\#_{a!}x_0\geq\#_{a?}x_0\wedge\#_{a!}x_1\geq\#_{a?}x_1$
$)$
= { calculus }
$(\mathbf{E}t_1,u_1,w:t_1\in\mathbf{t}U\wedge u_1\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_1,u_1)\wedge(x_1=wa?)$
$\qquad:(x_0wa!a?x_2)\mathbf{dres}(t_1,u_1)\wedge\#_{a!}x_0w>\#_{a?}x_0w$
$)$
= { $a!\in(\mathbf{o}U)!$, $a?\in(\mathbf{i}\overline{U})?$, definitions of composability and directed resultant }
$(\mathbf{E}t_1,u_1,w:t_1\in\mathbf{t}U\wedge u_1\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_1,u_1)\wedge(x_1=wa?)$
$\qquad:(x_0wa?a!x_2)\mathbf{dres}(t_1,u_1)$
$)$
= { calculus }
$(\mathbf{E}t,u:t\in\mathbf{t}U\wedge u\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t,u):(x_0x_1a!x_2)\mathbf{dres}(t,u))$

end of case

Case (0.4) : $b \in (\mathbf{i}U)?$

$(\mathbf{E}b, c, t_1, u_1, w : (b = c?) \wedge c \in \mathbf{i}U \wedge t_1 \in \mathbf{t}U \wedge u_1 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_1, u_1) \wedge (x_1 = wb)$
$\quad : (x_0 wa!c?x_2)\mathbf{dres}(t_1, u_1)$
$)$

$\Rightarrow$ { lemma 3.1.1 using a and c have distinct types, and calculus }

$(\mathbf{E}c, t, u, w : c \in \mathbf{i}U \wedge t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t, u) \wedge (x_1 = wc?)$
$\quad : (x_0 wc?a!x_2)\mathbf{dres}(t, u)$
$)$

$=$ { calculus }

$(\mathbf{E}t, u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t, u) : (x_0 x_1 a!x_2)\mathbf{dres}(t, u))$

end of case

end of case

Case (1) : $a \in \mathbf{o}\overline{U}$

The proof of this case is analogous to the proof of case (0).

end of case

end of lemma

Lemma 3.2

For symbol a, such that $a \in \mathbf{a}U$, and partially directed traces y_0, y_1, y_2, and y_3, such that $(y_0 y_1 y_2 y_3) \in (\mathbf{a}U \cup U!? \cup \overline{U}!?)^*$ and $(y_1 y_2) \upharpoonright \{a?\} = \varepsilon$,

$$(\mathbf{E}t_0, u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0, u_0) : (\mathbf{dir}(\mathbf{a}U, y_0 a!y_1 y_2 a?y_3))\mathbf{dres}(t_0, u_0))$$

$$\Rightarrow (\mathbf{E}t, u : t \in \mathbf{t}U \wedge u \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t, u) : (\mathbf{dir}(\mathbf{a}U, y_0 a!y_1 a?y_2 y_3))\mathbf{dres}(t, u))$$

Using lemma 3.1.1 instead of lemma 3.1.0 the proof of this lemma is analogous to the proof of lemma 3.1.

end of lemma

3.2 Undirecting directed resultants

In order to derive lemma 3.3 we define the notion D. It can be interpreted as the distance between two partially directed traces, that are ordered by the partial order undirect.

Definition 3.2.0 (D)
For partially directed traces x and y, such that $y\,\mathbf{undirect}\,x$,

$$\mathbf{D}(x,y)=(\mathrm{S}a::\#_{a!}x-\#_{a!}y)$$

end of definition

Property 3.2.0

(i) $\mathbf{D}(x,y)\geq 0$

(ii) $(\mathbf{D}(x,y)=0)\equiv(x=y)$

(iii) $y\,\mathbf{undirect}\,x\equiv(\mathbf{D}(x,y)=1)$

(iv) $(z\,\mathbf{undirect}\,y\wedge y\,\mathbf{undirect}\,x)\equiv(\mathbf{D}(x,y)+\mathbf{D}(y,z)=\mathbf{D}(x,z))$

end of property

Lemma 3.3
For traces t and u, such that $t\in\mathbf{t}U$, $u\in\mathbf{t}\overline{U}$, $\mathbf{c}(t,u)$, and $\mathbf{mm}(t,u)=0$, and partially directed traces x and y, such that $x\,\mathbf{dres}(t,u)$ and $y\,\mathbf{undirect}\,x$,

$$(\mathbf{E}t_0,u_0:t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_0,u_0)\wedge(\mathbf{mm}(t_0,u_0)=0):(\mathbf{dir}(\mathbf{a}U,y))\mathbf{dres}(t_0,u_0))$$

Proof
Given t, u, and x, such that $t\in\mathbf{t}U$, $u\in\mathbf{t}\overline{U}$, $\mathbf{c}(t,u)$, $\mathbf{mm}(t,u)=0$, and $x\,\mathbf{dres}(t,u)$. We proof this lemma for partially directed trace y, such that $y\,\mathbf{undirect}\,x$, by mathematical induction on $\mathbf{D}(x,y)$. Given y, such that $y\,\mathbf{undirect}\,x$.

Induction hypothesis

$$\begin{aligned}&(\mathbf{A}z:z\,\mathbf{undirect}\,x\wedge\mathbf{D}(x,z)<\mathbf{D}(x,y)\\&\quad:(\mathbf{E}t_0,u_0:t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_0,u_0)\wedge(\mathbf{mm}(t_0,u_0)=0):(\mathbf{dir}(\mathbf{a}U,z))\mathbf{dres}(t_0,u_0))\\&)\end{aligned}$$

Base : $\mathbf{D}(x,y)=0$
We derive :

$$
\begin{array}{ll}
 & \mathbf{D}(x,y)=0 \\
= & \{\ \text{property 3.2.0 (ii) and } x\,\mathbf{dres}(t,u)\ \} \\
 & (x=y)\wedge x\,\mathbf{dres}(t,u) \\
= & \{\ \text{calculus}\ \} \\
 & y\,\mathbf{dres}(t,u) \\
= & \{\ y=\mathbf{dir}(\mathbf{a}U,y)\ \text{due to the definitions of direct and directed resultant}\ \} \\
 & (\mathbf{dir}(\mathbf{a}U,y))\,\mathbf{dres}(t,u) \\
= & \{\ \text{calculus, using } \mathbf{mm}(t,u)=0\ \} \\
 & (\mathbf{E}t_0,u_0 : t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_0,u_0)\wedge(\mathbf{mm}(t_0,u_0)=0) : (\mathbf{dir}(\mathbf{a}U,y))\mathbf{dres}(t_0,u_0))
\end{array}
$$

Step : $\mathbf{D}(x,y)>0$

We derive :

$$
\begin{array}{ll}
 & \mathbf{D}(x,y)>0 \\
= & \{\ \text{definition of D, definition of immediate undirect,} \\
 & \ \ \text{definition of undirect, } x\,\mathbf{dres}(t,u)\text{, and calculus} \\
 & \} \\
 & (\mathbf{E}z : y\,\mathbf{iundirect}\,z\wedge z\,\mathbf{undirect}\,x : x\,\mathbf{dres}(t,u)) \\
= & \{\ \text{property 3.2.0 (iii) and (iv), and definition of immediate undirect}\ \} \\
 & (\mathbf{E}z : y\,\mathbf{iundirect}\,z\wedge z\,\mathbf{undirect}\,x\wedge\mathbf{D}(x,z)<\mathbf{D}(x,y) : x\,\mathbf{dres}(t,u)) \\
\Rightarrow & \{\ \text{induction hypothesis}\ \} \\
 & (\mathbf{E}t_0,u_0,z : t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_0,u_0)\wedge(\mathbf{mm}(t_0,u_0)=0)\wedge y\,\mathbf{iundirect}\,z \\
 & \qquad\qquad : (\mathbf{dir}(\mathbf{a}U,z))\mathbf{dres}(t_0,u_0) \\
 &) \\
= & \{\ \text{definition of immediate undirect using } \mathbf{mm}(t_0,u_0)=0\ \} \\
 & (\mathbf{E}t_0,u_0,z \\
 & \quad : t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_0,u_0)\wedge(\mathbf{mm}(t_0,u_0)=0) \\
 & \quad : (\mathbf{dir}(\mathbf{a}U,z))\mathbf{dres}(t_0,u_0) \\
 & \quad\ \wedge(\mathbf{E}a,y_0,y_1,y_2,y_3 : (y_1y_2)\upharpoonright\{a?\}=\varepsilon : (y=y_0y_1ay_2y_3)\wedge(z=y_0a!y_1y_2a?y_3)) \\
 &) \\
\Rightarrow & \{\ \text{lemmata 3.1 and 3.2, and calculus}\ \} \\
 & (\mathbf{E}a,t,t_0,u,u_0,y_0,y_1,y_2,y_3 \\
 & \quad : t\in\mathbf{t}U\wedge u\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t,u)\wedge t_0\in\mathbf{t}U\wedge u_0\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t_0,u_0)\wedge(\mathbf{mm}(t_0,u_0)=0) \\
 & \quad\ \wedge((y_1y_2)\upharpoonright\{a?\}=\varepsilon) \\
 & \quad : (\mathbf{dir}(\mathbf{a}U,y_0a!y_1y_2a?y_3))\mathbf{dres}(t_0,u_0)\wedge(\mathbf{dir}(\mathbf{a}U,y_0y_1a!a?y_2y_3))\mathbf{dres}(t,u) \\
 & \quad\ \wedge(y=y_0y_1ay_2y_3) \\
 &) \\
\Rightarrow & \{\ \text{property 2.3, and calculus}\ \} \\
 & (\mathbf{E}t,u : t\in\mathbf{t}U\wedge u\in\mathbf{t}\overline{U}\wedge\mathbf{c}(t,u)\wedge(\mathbf{mm}(t,u)=0) : (\mathbf{dir}(\mathbf{a}U,y))\mathbf{dres}(t,u))
\end{array}
$$

end of lemma

3.3 Concluding theorems

Theorem 3.0
Every directed trace structure U, that satisfies $\mathbf{R}_1, \mathbf{R}_3, \mathbf{R}_4,$ and $\mathbf{R}_5$ satisfies the foam rubber wrapper postulate.

Proof
Given a directed trace structure U, such that U satisfies $\mathbf{R}_1$, $\mathbf{R}_3$, $\mathbf{R}_4$, and $\mathbf{R}_5$, traces t, u, and z, such that $t \in \mathbf{t}U$, $u \in \mathbf{t}\overline{U}$, $\mathbf{c}(t,u)$, $z \in (\mathbf{a}U)^*$, and $z\,\mathbf{res}(t,u)$.
We derive :

true
$=$ { lemma 3.0, definition of mismatches, and $z\,\mathbf{res}(t,u)$ }
$(\mathbf{E}t_0,u_0 : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) \wedge (\mathbf{mm}(t_0,u_0)=0) : (z\,\mathbf{res}(t_0,u_0))$
$=$ { definition of resultant }
$(\mathbf{E}t_0,u_0,x : t_0 \in \mathbf{t}U \wedge u_0 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_0,u_0) \wedge (\mathbf{mm}(t_0,u_0)=0)$
$\quad : z\,\mathbf{undirect}\,x \wedge x\,\mathbf{dres}(t_1,u_1)$
$)$
$\Rightarrow$ { lemma 3.3 }
$(\mathbf{E}t_1,u_1 : t_1 \in \mathbf{t}U \wedge u_1 \in \mathbf{t}\overline{U} \wedge \mathbf{c}(t_1 u_1) \wedge (\mathbf{mm}(t_1,u_1)=0) : (\mathbf{dir}(\mathbf{a}U,z))\mathbf{dres}(t_1,u_1))$
$\Rightarrow$ { property 3.0 }
$(\mathbf{E}t_1 : t_1 \in \mathbf{t}U : \mathbf{dir}(\mathbf{a}U,z) \upharpoonright (\mathbf{a}U)!? = \mathbf{postf}(U,t_1))$
$=$ { $z \in (\mathbf{a}U)^*$, and definitions of postfix type, direct, and projection }
$(\mathbf{E}t_1 : t_1 \in \mathbf{t}U : z = t_1)$
$=$ { calculus }
$z \in \mathbf{t}U$

We conclude, since $\mathbf{t}U$ is prefix-closed due to $\mathbf{R}_1$, that $\mathbf{t}(U \circledcirc \overline{U}) \subseteq \mathbf{t}U$. Using property 2.2 we conclude that U satisfies the foam rubber wrapper postulate.

end of theorem

Theorem 3.1
Every directed trace structure that is a C4, satisfies he foam rubber wrapper postulate and has absence of danger of transmission interference.

Proof
In the context of delay-insensitive directed trace structures absence of danger of transmission interference equals $\mathbf{R}_2$. A directed trace structure, that is a C4, satisfies the foam rubber wrapper postulate on account of theorem 3.0, and has absence of danger of transmission interference on account of $\mathbf{R}_2$.

end of theorem

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