

Positivity and ellipticity

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Positivity and ellipticity

A.F.M. ter Elst¹, Derek W. Robinson and Yueping Zhu²

Centre for Mathematics and its Applications
Mathematical Sciences Institute
Australian National University
Canberra, ACT 0200
Australia

Abstract

We consider partial differential operators $H = -\operatorname{div}(C\nabla)$ in divergence form on \mathbf{R}^d with a positive-semidefinite, symmetric, matrix C of real L_∞ -coefficients. First, we prove that one can define H as a self-adjoint operator on $L_2(\mathbf{R}^d)$ such that the corresponding semigroup extends as a positive, contraction semigroup to all the L_p -spaces. Secondly, we establish that H is strongly elliptic if and only if the distribution kernel of the semigroup satisfies appropriate small time lower bounds. Thirdly, we analyze degenerate operators satisfying the subellipticity condition

$$H \geq \mu \Delta^{1-\gamma} - \nu I$$

for some $\mu > 0$, $\nu \geq 0$ and $\gamma \in [0, 1)$, where Δ denotes the usual Laplacian, and derive large time Gaussian bounds from a local condition of strict positivity.

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Home institutions:

1. Department of Mathematics
and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
2. Department of Mathematics
Nantong Teachers' University
Nantong, 226007
Jiangsu Province
P.R. China

1 Introduction

In the theory of partial differential equations a key role is played by positive second-order operators in divergence form, i.e., operators

$$H = - \sum_{i,j=1}^d \partial_i c_{ij} \partial_j \quad (1)$$

where $\partial_i = \partial/\partial x_i$, the coefficients c_{ij} are real L_∞ -functions and the corresponding matrix $C = (c_{ij})$ is symmetric and positive-definite almost-everywhere. In particular the classical Nash–De Giorgi theory analyzes operators of the this type under the strong ellipticity assumption

$$C \geq \mu I > 0 \quad (2)$$

almost-everywhere. The principal result of this theory is the local Hölder continuity of weak solutions of the associated elliptic and parabolic equations. In Nash’s approach [Nas] the Hölder continuity of the elliptic solution is derived as a corollary of continuity of the parabolic solution and the latter is established by an iterative argument from good upper and lower bounds on the fundamental solution. Aronson [Aro] subsequently improved Nash’s bounds and proved that the fundamental solution of the parabolic equation, the heat kernel, satisfies Gaussian upper and lower bounds. Specifically the kernel K of the semigroup S is a symmetric function over $\mathbf{R}^d \times \mathbf{R}^d$ satisfying bounds

$$a' G_{b',t}(x - y) \leq K_t(x; y) \leq a G_{b,t}(x - y) \quad (3)$$

uniformly for all $x, y \in \mathbf{R}^d$ and $t > 0$ where $G_{b,t}(x) = t^{-d/2} e^{-b|x|^2/t}$ and $a, a', b, b' > 0$. These bounds give qualitatively correct estimates both locally and globally for the heat kernel.

One important implication of the strong ellipticity assumption (2) is that H can be precisely defined as a self-adjoint operator on $L_2(\mathbf{R}^d)$ by quadratic form techniques. Specifically one can define the quadratic form h on $L_2(\mathbf{R}^d)$ by

$$h(\varphi) = \sum_{i,j=1}^d \int_{\mathbf{R}^d} dx \overline{(\partial_i \varphi)(x)} c_{ij}(x) (\partial_j \varphi)(x) \quad (4)$$

with domain $D(h) = \bigcap_{i=1}^d D(\partial_i) = D(\Delta^{1/2})$ where Δ denotes the self-adjoint Laplacian, i.e., $\Delta = -\sum_{i=1}^d \partial_i^2$, on $L_2(\mathbf{R}^d)$. Then h is positive, symmetric, densely-defined and, as a direct consequence of (2), it is also closed. Therefore there is a unique, positive, self-adjoint operator H with $D(H) \subset D(h)$ canonically associated with h . In particular $(\varphi, H\varphi) = h(\varphi)$ for all $\varphi \in D(H)$. This quadratic form definition provides the starting point of the Nash–De Giorgi theory.

Our intention is to analyze the operator H , formally given by (1), without the strong ellipticity assumption (2). Therefore the first difficulty is to define H in a precise fashion. One can still introduce the form h by the above definition but there is no reason for the form to be closable. Hence there is no evident method of defining H as a positive self-adjoint operator. We tackle this problem by a ‘viscosity’ method discussed in Section 2. In particular we argue that H can be defined by monotonic approximation with strongly elliptic operators. Then in Section 3 we establish that the corresponding viscosity operator

is strongly elliptic if and only if it has a kernel satisfying the Aronson bounds (3). In fact it suffices that the kernel, viewed as a distribution, satisfies local lower bounds with a singularity $t^{-d/2}$. Then in Section 4 we discuss the structure of mildly degenerate operators, i.e., operators for which the strong ellipticity condition fails because the coefficients have possible zeros. In this context one can deduce large t upper bounds on the kernel from local lower bounds.

2 The viscosity operator

Let l be the closed quadratic form associated with the Laplacian Δ , i.e.,

$$l(\varphi) = \sum_{i=1}^d \|\partial_i \varphi\|_2^2 = \|\Delta^{1/2} \varphi\|_2^2$$

with $D(l) = D(\Delta^{1/2})$. Then for each $\varepsilon \in \langle 0, 1 \rangle$ define h_ε by $D(h_\varepsilon) = D(h) = D(l)$ and

$$h_\varepsilon(\varphi) = h(\varphi) + \varepsilon l(\varphi)$$

where h denotes the form given by (4). Recall that we assume throughout this paper that the coefficients c_{ij} are real L_∞ -functions and the corresponding matrix $C = (c_{ij})$ is symmetric and positive-definite almost-everywhere.

Since h is positive the form h_ε satisfies the strong ellipticity condition

$$h_\varepsilon(\varphi) \geq \varepsilon l(\varphi) \tag{5}$$

for all $\varphi \in D(h)$. In addition it satisfies the upper bounds

$$h_\varepsilon(\varphi) \leq (1 + \|C\|) l(\varphi) \tag{6}$$

where $\|C\|$ is the essential supremum of the matrix norm of $C(x) = (c_{ij}(x))$. It follows immediately from (5) and (6) that h_ε is closed. Therefore there is a positive self-adjoint operator H_ε canonically associated with h_ε . The operator H_ε is the strongly elliptic operator with coefficients $C + \varepsilon I$. But $\varepsilon \mapsto h_\varepsilon(\varphi)$ decreases monotonically as ε decreases for each $\varphi \in D(h)$. Therefore it follows from a result of Kato, [Kat] Theorem VIII.3.11, that the H_ε converge in the strong resolvent sense, as $\varepsilon \rightarrow 0$, to a positive self-adjoint operator H_0 which we will refer to as the **viscosity operator** with coefficients $C = (c_{ij})$. This procedure gives a precise meaning to the formal operator H given by (1). Let h_0 denote the form associated with H_0 , i.e., $D(h_0) = D(H_0^{1/2})$ and $h_0(\varphi) = \|H_0^{1/2} \varphi\|_2^2$.

If h is closable then the viscosity operator H_0 is the operator associated with the closure \bar{h} . More generally one has the following.

Proposition 2.1 *The following are valid.*

- I. $D(h_0) \supseteq D(h)$ and $h_0(\varphi) \leq h(\varphi)$ for all $\varphi \in D(h)$.
- II. $h_0(\varphi) = h(\varphi)$ for all $\varphi \in D(h)$ if and only if h is closable and then $h_0 = \bar{h}$, the closure of h .

Proof It follows from [Kat] Theorem VIII.3.11 that $D(h_0) = D(H_0^{1/2}) \supseteq D(H_\varepsilon^{1/2}) = D(\Delta^{1/2}) = D(h)$ and $(\lambda I + H_\varepsilon)^{1/2}\varphi$ converges weakly to $(\lambda I + H_0)^{1/2}\varphi$ as $\varepsilon \rightarrow 0$ for all $\varphi \in D(h)$. Therefore

$$\begin{aligned} \lambda \|\varphi\|_2^2 + h_0(\varphi) &= \|(\lambda I + H_0)^{1/2}\varphi\|_2^2 \leq \liminf_{\varepsilon \rightarrow 0} \|(\lambda I + H_\varepsilon)^{1/2}\varphi\|_2^2 \\ &= \lambda \|\varphi\|_2^2 + \inf_{\varepsilon \in (0,1]} h_\varepsilon(\varphi) = \lambda \|\varphi\|_2^2 + h(\varphi) \end{aligned}$$

for all $\varphi \in D(h)$. This establishes the first statement of the proposition.

If, however, $h_0(\varphi) = h(\varphi)$ for all $\varphi \in D(h)$, then h_0 is a closed extension of h . Hence h is closable. Conversely if h is closable then $h_0 = \bar{h}$ by the last statement of [Kat] Theorem VIII.3.11. \square

The viscosity operator H_0 generates a self-adjoint contraction semigroup $S^{(0)}$ on $L_2(\mathbf{R}^d)$ which can also be constructed by approximation. Since H_0 is defined as the strong resolvent limit of the strongly elliptic self-adjoint operators H_ε associated with the closed forms h_ε the semigroup $S^{(0)}$ is the strong limit of the self-adjoint contraction semigroups $S^{(\varepsilon)}$ generated by the H_ε . In particular $S_t^{(\varepsilon)}$ converges strongly, on $L_2(\mathbf{R}^d)$, to $S_t^{(0)}$ and the convergence is uniform for t in finite intervals. Note that each h_ε is a Dirichlet form, i.e., it satisfies the Beurling–Deny criteria (see, for example, [Dav2] Section 1.3). Specifically the positive quadratic form h on L_2 is a Dirichlet form if it satisfies the following two conditions:

1. $\varphi \in D(h)$ implies $|\varphi| \in D(h)$ and $h(|\varphi|) \leq h(\varphi)$,
2. $\varphi \in D(h)$ implies $\varphi \wedge \mathbf{1} \in D(h)$ and $h(\varphi \wedge \mathbf{1}) \leq h(\varphi)$.

The primary result of the Beurling–Deny theory is that h is a Dirichlet form if and only if the semigroup S generated by the corresponding operator H on L_2 is positive and extends from $L_2 \cap L_p$ to a contraction semigroup on L_p for all $p \in [1, \infty]$. Therefore the semigroup $S^{(\varepsilon)}$ generated by H_ε is positive and extends to a positive contraction semigroup, also denoted by $S^{(\varepsilon)}$, on each of the spaces $L_p(\mathbf{R}^d)$ with $p \in [1, \infty]$. Since $S^{(\varepsilon)}$ converges strongly to $S^{(0)}$ on $L_2(\mathbf{R}^d)$ it follows that $S^{(0)}$ is positive but it is not evident that it extends to a contraction semigroup on the L_p -spaces, i.e., it is not evident that h_0 is a Dirichlet form. But this is a consequence of the following characterization of Dirichlet forms.

Lemma 2.2 *Let h denote the quadratic form associated with a positive self-adjoint operator H on $L_2(\mathbf{R}^d)$ and S the contraction semigroup generated by H . Further let $H_s = s^{-1}(I - S_s)$, for $s > 0$, with h_s the corresponding bounded quadratic form. The following conditions are equivalent.*

- I. h is a Dirichlet form.
- II. h_s is a Dirichlet form for each $s \in (0, 1]$.

Proof I \Rightarrow II. Let $S^{(s)}$ denote the self-adjoint contraction semigroup generated by H_s on L_2 . Then

$$S_t^{(s)}\varphi = e^{-s^{-1}t} \sum_{n=0}^{\infty} \frac{(s^{-1}t)^n}{n!} S_{ns}\varphi \quad .$$

Since, by Condition I, the semigroup S is positive and extends to an L_p -contractive semigroup it follows immediately that $S^{(s)}$ has similar properties. Hence h_s must be a Dirichlet form.

II \Rightarrow I. Since the h_s are Dirichlet forms

$$s^{-1}(|\varphi|, (I - S_s)|\varphi|) \leq s^{-1}(\varphi, (I - S_s)\varphi)$$

for all $\varphi \in L_2$ and $s \in \langle 0, 1 \rangle$. Then the first property characterizing a Dirichlet form follows by taking the supremum over s . The second property follows similarly. \square

Now let us return to the discussion of the viscosity operator. Since the approximating forms h_ε are Dirichlet forms it follows from Lemma 2.2 that

$$s^{-1}(|\varphi|, (I - S_s^{(\varepsilon)})|\varphi|) \leq s^{-1}(\varphi, (I - S_s^{(\varepsilon)})\varphi)$$

for all $\varphi \in L_2(\mathbf{R}^d)$ and $s \in \langle 0, 1 \rangle$. But $S^{(\varepsilon)}$ converges strongly to $S^{(0)}$ as $\varepsilon \rightarrow 0$. Hence

$$s^{-1}(|\varphi|, (I - S_s^{(0)})|\varphi|) \leq s^{-1}(\varphi, (I - S_s^{(0)})\varphi)$$

for all $\varphi \in L_2(\mathbf{R}^d)$ and $s \in \langle 0, 1 \rangle$. Similarly one deduces that

$$s^{-1}(\varphi \wedge \mathbf{1}, (I - S_s^{(0)})\varphi \wedge \mathbf{1}) \leq s^{-1}(\varphi, (I - S_s^{(0)})\varphi)$$

for all $\varphi \in L_2(\mathbf{R}^d)$ and $s \in \langle 0, 1 \rangle$. Thus we obtain the following observation by another application of Lemma 2.2.

Lemma 2.3 *The form h_0 associated to the viscosity operator H_0 is a Dirichlet form. Consequently $S^{(0)}$ extends to a positive contraction semigroup on each of the L_p -spaces.*

It follows from the positivity and contractivity that the viscosity semigroup $S^{(0)}$ satisfies

$$0 \leq S_t^{(0)} \mathbf{1} \leq \mathbf{1} \tag{7}$$

for all $t > 0$ on $L_\infty(\mathbf{R}^d)$. Now let (\cdot, \cdot) denote the duality pairing between the L_p -spaces. Then if $\varphi \in L_2(\mathbf{R}^d)$ it follows that $|\varphi|^2 \in L_1(\mathbf{R}^d)$ and $(\varphi, \varphi) = (\mathbf{1}, |\varphi|^2) = (|\varphi|^2, \mathbf{1})$. Therefore one deduces from (7) that

$$\|\varphi\|_2^2 = (\varphi, \varphi) \geq (S_t^{(0)} \mathbf{1}, |\varphi|^2) = (|\varphi|^2, S_t^{(0)} \mathbf{1}) \tag{8}$$

for all $t > 0$. Note that the strongly elliptic approximants $S^{(\varepsilon)}$ satisfy the stronger condition $S_t^{(\varepsilon)} \mathbf{1} = \mathbf{1}$ and for these semigroups one has equality in the last expression. This can be established by an approximation argument.

3 Strongly elliptic operators

If the coefficients of H satisfy the strong ellipticity condition (2) then the kernel of the corresponding semigroup satisfies the Aronson Gaussian bounds (3). These Gaussian bounds in fact characterize the strongly elliptic operators among the broader class of second-order divergence form operators. More is true. Strong ellipticity can be characterized by local lower bounds.

Theorem 3.1 *Let H_0 be the viscosity operator with coefficients $C = (c_{ij})$ and $K^{(0)}$ the distribution kernel of the contraction semigroup $S^{(0)}$ generated by H_0 . The following conditions are equivalent.*

- I. *There is a $\mu > 0$ such that $C \geq \mu I$ almost everywhere.*
- II. *There is a $\mu > 0$ such that $H_0 \geq \mu \Delta$ in the quadratic form sense on $L_2(\mathbf{R}^d)$.*
- III. *$K_t^{(0)}$ is a bounded function satisfying the Aronson Gaussian bounds (3).*
- IV. *There are $a, r > 0$ such that*

$$K_t^{(0)}(x; y) \geq a t^{-d/2}$$

for all $t \in \langle 0, 1 \rangle$ and $x, y \in \mathbf{R}^d$ with $|x - y| \leq r t^{1/2}$.

Moreover, if the conditions are satisfied then the form h is closed and $h_0 = h$.

Proof Condition I implies that $H_\varepsilon \geq (\mu + \varepsilon) \Delta \geq \mu \Delta$. Therefore $(\lambda I + H_\varepsilon)^{-1} \leq (\lambda I + \mu \Delta)^{-1}$ for all $\lambda > 0$. Then in the limit $\varepsilon \rightarrow 0$ one has

$$(\lambda I + H_0)^{-1} \leq (\lambda I + \mu \Delta)^{-1}$$

for all $\lambda > 0$ which is equivalent to Condition II.

Conversely $h_\varepsilon(\varphi) \geq h(\varphi) \geq h_0(\varphi) \geq \mu l(\varphi)$ for all $\varphi \in D(\Delta^{1/2})$ by the first statement of Proposition 2.1 and Condition II. Next let $\varphi \in D(h) = D(h_\varepsilon)$, $k \in \mathbf{R}$ and $\xi \in \mathbf{R}^d$. Define φ_k by $\varphi_k(x) = e^{ikx \cdot \xi} \varphi(x)$. Then one calculates that

$$\lim_{k \rightarrow \infty} k^{-2} h_\varepsilon(\varphi_k) = \int_{\mathbf{R}^d} dx |\varphi(x)|^2 \left((\xi, C(x)\xi) + \varepsilon(\xi, \xi) \right) .$$

But

$$\lim_{k \rightarrow \infty} k^{-2} l(\varphi_k) = \int_{\mathbf{R}^d} dx |\varphi(x)|^2 |\xi|^2 .$$

Since $h_\varepsilon(\varphi_k) \geq \mu l(\varphi_k)$ one deduces that

$$\int_{\mathbf{R}^d} dx |\varphi(x)|^2 \left((\xi, C(x)\xi) + \varepsilon(\xi, \xi) \right) \geq \mu \int_{\mathbf{R}^d} dx |\varphi(x)|^2 |\xi|^2$$

for all $\varepsilon \in \langle 0, 1 \rangle$. Then in the limit $\varepsilon \rightarrow 0$ one concludes that $C \geq \mu I$ almost-everywhere.

Next II \Rightarrow III by the Nash–Aronson estimates and obviously III \Rightarrow IV. Finally we establish that IV \Rightarrow II by the following variation of an argument of Carlen, Kusuoka and Stroock [CKS].

By the contractivity of $S^{(0)}$ and spectral theory one has

$$h_0(\varphi) \geq t^{-1}(\varphi, (I - S_t^{(0)})\varphi)$$

for all $\varphi \in D(h_0)$ and all $t > 0$. Therefore it follows from (8) and self-adjointness of $S_t^{(0)}$ that

$$h_0(\varphi) \geq (2t)^{-1} \left((S_t^{(0)} \mathbf{1}, |\varphi|^2) + (|\varphi|^2, S_t^{(0)} \mathbf{1}) - (\varphi, S_t^{(0)} \varphi) - (S_t^{(0)} \varphi, \varphi) \right)$$

for all $\varphi \in D(h_0)$ and $t > 0$. Then this can be restated in terms of the kernel as

$$h_0(\varphi) \geq (2t)^{-1} \int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy K_t^{(0)}(x; y) |\varphi(x) - \varphi(y)|^2$$

for all $\varphi \in D(h_0)$ and $t > 0$. Next choose a smooth positive function ρ with support in $\langle -r, r \rangle$ such that $\rho \leq a$ and $\rho = a$ if $|x| \leq r/2$. Then it follows from Condition IV that

$$K_t^{(0)}(x; y) \geq t^{-d/2} \rho(|x - y|^2 t^{-1})$$

for all $x, y \in \mathbf{R}^d$ and all $t \in \langle 0, 1 \rangle$. Hence

$$\begin{aligned} h_0(\varphi) &\geq (2t)^{-1} \int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy t^{-d/2} \rho(|x - y|^2 t^{-1}) |\varphi(x) - \varphi(y)|^2 \\ &= t^{-1} \int_{\mathbf{R}^d} dx t^{-d/2} \rho(|x|^2 t^{-1}) \int_{\mathbf{R}^d} d\xi |\widehat{\varphi}(\xi)|^2 (1 - \cos \xi \cdot x) \\ &= t^{-1} \int_{\mathbf{R}^d} dx \rho(|x|^2) \int_{\mathbf{R}^d} d\xi |\widehat{\varphi}(\xi)|^2 (1 - \cos t^{1/2} \xi \cdot x) \\ &= 2 \int_{\mathbf{R}^d} d\xi |\widehat{\varphi}(\xi)|^2 \int_{\mathbf{R}^d} dx \rho(|x|^2) t^{-1} \sin^2(2^{-1} t^{1/2} \xi \cdot x) \end{aligned}$$

for all $\varphi \in D(h_0)$ and $t \in \langle 0, 1 \rangle$ where $\widehat{\varphi}$ denotes the Fourier transform of φ . Therefore in the limit $t \rightarrow 0$ one has

$$h_0(\varphi) \geq 2^{-1} \int_{\mathbf{R}^d} d\xi |\widehat{\varphi}(\xi)|^2 \int_{\mathbf{R}^d} dx \rho(|x|^2) (\xi \cdot x)^2 = \mu \int_{\mathbf{R}^d} d\xi |\widehat{\varphi}(\xi)|^2 |\xi|^2$$

for all $\varphi \in D(h_0)$ with $\mu > 0$. Thus Condition II is satisfied.

The last statement of the theorem follows because Condition I implies that $h(\varphi) \geq \mu l(\varphi)$ for all $\varphi \in D(h) = D(l)$. Then h is closed, because of the complementary bound $h(\varphi) \leq \|C\| l(\varphi)$, and $h = h_0$ by the second statement of Proposition 2.1. \square

The argument that II \Rightarrow I in fact proves that the condition of strong ellipticity is equivalent to the seemingly weaker Gårding inequality.

Corollary 3.2 *Let H_0 be the viscosity operator with coefficients $C = (c_{ij})$. The following conditions are equivalent.*

- I. *There is a $\mu > 0$ such that $H_0 \geq \mu \Delta$ in the quadratic form sense on $L_2(\mathbf{R}^d)$.*
- II. *There are $\mu > 0$ and $\nu \geq 0$ such that $H_0 \geq \mu \Delta - \nu I$ in the quadratic form sense on $L_2(\mathbf{R}^d)$.*

Proof Obviously I \Rightarrow II but to establish the converse we evaluate the form condition $H_\varepsilon \geq \mu \Delta - \nu I$ with φ_k where $\varphi_k(x) = e^{ikx \cdot \xi} \varphi(x)$ and $\varphi \in C_c^\infty(\mathbf{R}^d)$. Then

$$\begin{aligned} \int_{\mathbf{R}^d} dx |\varphi(x)|^2 \left((\xi, C(x)\xi) + \varepsilon(\xi, \xi) \right) &= \lim_{k \rightarrow \infty} k^{-2} h_\varepsilon(\varphi_k) \\ &\geq \lim_{k \rightarrow \infty} k^{-2} (\mu l(\varphi_k) - \nu \|\varphi_k\|_2^2) = \int_{\mathbf{R}^d} dx |\varphi(x)|^2 \mu |\xi|^2. \end{aligned}$$

Therefore $C + \varepsilon I \geq \mu I$ almost everywhere, which gives $C \geq \mu I$. Then $H_0 \geq \mu \Delta$ by the implication I \Rightarrow II in the theorem. \square

The most surprising element of the theorem is the implication $IV \Rightarrow III$. The existence of small time lower bounds with the correct Euclidean geometry implies Gaussian upper and lower bounds on the semigroup kernel. This indicates that the key to the analysis in more general situations is the derivation of good lower bounds.

In fact the Carlen, Kusuoka and Stroock [CKS] argument shows that local lower bounds for one fixed t give a weaker form of ellipticity which indicates that H_0 is asymptotically strongly elliptic.

Corollary 3.3 *Let H_0 be the viscosity operator with coefficients $C = (c_{ij})$ and $K^{(0)}$ the distribution kernel of the contraction semigroup $S^{(0)}$ generated by H_0 . Further assume there exist $a, r, t > 0$ such that $K_t^{(0)}(x; y) \geq a$ for a.e. (x, y) with $|x - y| \leq r$.*

It follows that there exists a $\mu > 0$ such that

$$H_0 \geq \mu \Delta(I + \Delta)^{-1} \quad .$$

Proof One may suppose $t = 1$. Then repeating the calculation in the proof of the implication $IV \Rightarrow II$ in Theorem 3.1 with $t = 1$ gives

$$h_0(\varphi) \geq \int_{\mathbf{R}^d} d\xi |\widehat{\varphi}(\xi)|^2 \int_{\mathbf{R}^d} dx \rho(|x|^2) (1 - \cos \xi \cdot x) \quad .$$

But by the choice of ρ one can find a $\mu > 0$ such that

$$\int_{\mathbf{R}^d} dx \rho(|x|^2) (1 - \cos \xi \cdot x) \geq \mu (|\xi|^2 \wedge 1) \geq \mu |\xi|^2 (1 + |\xi|^2)^{-1}$$

for all $\xi \in \mathbf{R}^d$. Therefore $h_0(\varphi) \geq \mu l((I + \Delta)^{-1/2} \varphi)$ for all $\varphi \in D(h_0)$. \square

4 Subelliptic operators

Strong ellipticity corresponds to non-degeneracy of the operator H and we next discuss a class of operators with mild degeneracies. Let $\mu_0(x)$ denote the smallest eigenvalue of the matrix $C(x) = (c_{ij}(x))$. Then μ_0 is an L_∞ -function and (2) is equivalent to μ_0 being bounded away from zero. Now assume that μ_0 has an isolated zero. To be specific assume

$$\mu_0(x) = c_0 \left(\frac{|x|^2}{1 + |x|^2} \right)^\gamma \quad (9)$$

for some $c_0 > 0$ and $\gamma \in [0, 1]$. Since $\Delta \geq \sigma^2 |x|^{-2}$ for $d \geq 3$ with $\sigma = (d - 2)/2$ (see, for example, [Kat] Remark VI.4.9a and (VI.4.24), or [ReS] Lemma on page 169) it follows that $\mu_0 \geq c_0 (\sigma^2 (\sigma^2 I + \Delta)^{-1})^\gamma$, where μ_0 is viewed as a multiplication operator. Therefore the viscosity operator H_0 satisfies

$$H_0 \geq c_0 \Delta (\sigma^2 (\sigma^2 I + \Delta)^{-1})^\gamma = c_0 \Delta^{1-\gamma} (\sigma^2 \Delta (\sigma^2 I + \Delta)^{-1})^\gamma \quad . \quad (10)$$

Consequently, for each $\mu \in \langle 0, c_0 \sigma^{2\gamma} \rangle$ there is a $\nu > 0$ such that

$$H_0 \geq \mu \Delta^{1-\gamma} - \nu I \quad . \quad (11)$$

This condition is a global analogue of the local subellipticity condition analyzed by Feferman and Phong [FeP]. It corresponds to strong ellipticity condition if $\gamma = 0$ by Corollary 3.2.

It is of interest that the semigroup kernels associated with operators satisfying (11) satisfy Gaussian upper bounds albeit with geometric modifications.

Lemma 4.1 *Let $S^{(0)}$ denote the positive contractive semigroup generated by the viscosity operator H_0 . Assume that there are $\mu > 0$, $\nu \geq 0$ and $\gamma \in [0, 1)$ such that*

$$H_0 \geq \mu \Delta^{1-\gamma} - \nu I \quad . \quad (12)$$

Then there is a $c_\gamma > 0$, whose value is independent of μ and ν , such that

$$\|S_t^{(0)}\|_{1 \rightarrow \infty} \leq c_\gamma (\mu t)^{-d/(2(1-\gamma))} e^{\nu t} \quad (13)$$

and

$$\|S_t^{(\varepsilon)}\|_{1 \rightarrow \infty} \leq c_\gamma (\mu t)^{-d/(2(1-\gamma))} e^{\nu t} \quad (14)$$

for all $t > 0$ and $\varepsilon \in \langle 0, 1 \rangle$.

Proof The result follows by a slight variation of Nash's original arguments for strongly elliptic operators. The starting point is the observation that the fractional Nash inequality

$$\|\varphi\|_2^{2+4(1-\gamma)/d} \leq c \|\Delta^{(1-\gamma)/2} \varphi\|_2^2 \|\varphi\|_1^{4(1-\gamma)/d}$$

is valid for all $\varphi \in L_1(\mathbf{R}^d) \cap D(\Delta^{(1-\gamma)/2})$ and a suitable $c > 0$. This can be verified by the reasoning on page 169 of [Rob]. Therefore

$$\|\varphi\|_2^{2+4(1-\gamma)/d} \leq c \mu^{-1} \left(h_\varepsilon(\varphi) + \nu \|\varphi\|_2^2 \right) \|\varphi\|_1^{4(1-\gamma)/d} \quad (15)$$

uniformly for all $\varepsilon > 0$ and $\varphi \in L_1(\mathbf{R}^d) \cap D(h)$ by the subellipticity condition (12).

Setting $T_t^{(\varepsilon)} = e^{-\nu t} S_t^{(\varepsilon)}$ and using $\|S_t^{(\varepsilon)}\|_{1 \rightarrow 1} \leq 1$ one deduces that

$$\begin{aligned} \frac{d}{dt} \|T_t^{(\varepsilon)} \varphi\|_2^2 &= -2 \left(h_\varepsilon(T_t^{(\varepsilon)} \varphi) + \nu \|T_t^{(\varepsilon)} \varphi\|_2^2 \right) \\ &\leq -2 \mu c^{-1} \frac{\|T_t^{(\varepsilon)} \varphi\|_2^{2+4(1-\gamma)/d}}{\|T_t^{(\varepsilon)} \varphi\|_1^{4(1-\gamma)/d}} \leq -2 \mu c^{-1} e^{4\nu(1-\gamma)t/d} \frac{(\|T_t^{(\varepsilon)} \varphi\|_2^2)^{1+2(1-\gamma)/d}}{\|\varphi\|_1^{4(1-\gamma)/d}} \quad . \end{aligned}$$

Then, by integration,

$$e^{-\nu t} \|S_t^{(\varepsilon)} \varphi\|_2 = \|T_t^{(\varepsilon)} \varphi\|_2 \leq c_1 (\mu t)^{-d/(4(1-\gamma))} \|\varphi\|_1$$

for all $t > 0$, uniformly for $\varepsilon \in \langle 0, 1 \rangle$ with $c_1 = (cd/(4(1-\gamma)))^{d/(4(1-\gamma))}$. Therefore, by a limiting argument

$$\|S_t^{(0)}\|_{1 \rightarrow 2} \leq c_1 (\mu t)^{-d/(4(1-\gamma))} e^{\nu t}$$

for all $t > 0$. Then by duality and the semigroup property, one deduces that (13) is valid with $c_\gamma = c_1^2 2^{d/(2(1-\gamma))}$. The bounds (14) follow similarly. \square

In the proof of Lemma 4.1 we used the estimates

$$H_\varepsilon \geq \mu \Delta^{1-\gamma} - \nu I \quad (16)$$

uniformly for all $\varepsilon \in \langle 0, 1 \rangle$. Obviously (12) implies (16) for all $\varepsilon > 0$. But conversely, if (16) is valid uniformly for all $\varepsilon \in \langle 0, 1 \rangle$ then

$$((\lambda + \nu)I + H_\varepsilon)^{-1} \leq (\lambda I + \mu \Delta^{1-\gamma})^{-1}$$

for all $\lambda > 0$ and $\varepsilon \in \langle 0, 1 \rangle$. But H_ε converges to H_0 in the strong resolvent sense. So

$$((\lambda + \nu)I + H_0)^{-1} \leq (\lambda I + \mu\Delta^{1-\gamma})^{-1}$$

for all $\lambda > 0$ and (12) is valid.

It is an immediate consequence of the estimates of the lemma that $S^{(0)}$ has a bounded kernel $K^{(0)}$, which is automatically positive by Lemma 2.3, satisfying the bounds

$$0 \leq K_t^{(0)}(x; y) \leq c_\gamma (\mu t)^{-d/(2(1-\gamma))} e^{\nu t} \quad (x, y)\text{-a.e.} \quad (17)$$

for all $t > 0$. Then one can use Davies' perturbation method to extend the uniform bounds (17) to Gaussian upper bounds with respect to an appropriate distance.

One may associate with each positive-definite matrix C of coefficients $c_{ij} \in L_\infty(\mathbf{R}^d)$ a 'distance'

$$d_C(x; y) = \sup\{|\psi(x) - \psi(y)| : \psi \in C_c^\infty(\mathbf{R}^d), \psi \text{ real and}$$

$$\sum_{i,j=1}^d c_{ij}(z) (\partial_i \psi)(z) (\partial_j \psi)(z) \leq 1 \text{ for almost every } z\} .$$

In fact it is not evident that this function is a distance since this would require it to be finite-valued. It does, however, follow that if $C = \kappa I > 0$ then $d_C(x; y) = \kappa^{-1/2}|x - y|$. Moreover, if $C_1 \geq C_2 \geq 0$ then $d_{C_1}(x; y) \leq d_{C_2}(x; y)$ for all $x, y \in \mathbf{R}^d$. Therefore uniform boundedness of the coefficients immediately implies the lower bounds

$$d_C(x; y) \geq \|C\|^{-1/2}|x - y|$$

for all $x, y \in \mathbf{R}^d$. Further if $C_\varepsilon = C + \varepsilon I$ with $\varepsilon > 0$ then

$$d_C(x; y) \geq d_{C_\varepsilon}(x; y) \quad (18)$$

for all $x, y \in \mathbf{R}^d$. Define

$$\overline{d}_C(x; y) = \lim_{\varepsilon \rightarrow 0} d_{C_\varepsilon}(x; y) = \sup_{\varepsilon > 0} d_{C_\varepsilon}(x; y)$$

for $x, y \in \mathbf{R}^d$. It then follows from (18) that

$$\overline{d}_C(x; y) \leq d_C(x; y)$$

for all $x, y \in \mathbf{R}^d$.

Proposition 4.2 *Assume that the viscosity operator H_0 satisfies the subellipticity condition (12). Then for all $\delta > 0$ there exists an $a > 0$ such that*

$$K_t^{(0)}(x; y) \leq a (\mu t)^{-d/(2(1-\gamma))} e^{\delta \nu t} e^{-\overline{d}_C(x; y)^2 ((4+\delta)t)^{-1}} \quad (x, y)\text{-a.e.} \quad (19)$$

uniformly for all $t > 0$. The value of a is independent of μ and ν .

Proof The result is established by Davies' perturbation method [Dav1] as elaborated by Fabes and Stroock [FaS] applied to the strongly elliptic approximants H_ε . The use of the approximants is necessary to avoid domain problems in the argument. The important features are the uniform bounds (14) on $\|S_t^{(\varepsilon)}\|_{1 \rightarrow \infty}$ and the positive-definiteness of C . The arguments lead to bounds

$$K_t^{(\varepsilon)}(x; y) \leq a (\mu t)^{-d/(2(1-\gamma))} e^{\delta \nu t} e^{-d_{C_\varepsilon}(x; y)^2 / ((4+\delta)t)^{-1}}, \quad (20)$$

with a independent of ε , μ and ν . Next one has the following convergence result.

Lemma 4.3 *If the viscosity operator H_0 satisfies the subellipticity condition (12) then the kernels $K_t^{(\varepsilon)}$ converge in the weak* sense on $L_\infty(\mathbf{R}^d \times \mathbf{R}^d)$, as $\varepsilon \rightarrow 0$, to the kernel $K^{(0)}$ of the semigroup generated by the viscosity operator H_0 .*

Proof If $\varepsilon \in \langle 0, 1 \rangle$ then $\|C_\varepsilon\| \leq \|C\| + 1$. Therefore $d_{C_\varepsilon}(x; y) \geq (1 + \|C\|)^{-1/2} |x - y|$ and it follows from (20) that

$$K_t^{(\varepsilon)}(x; y) \leq a (\mu t)^{-d/(2(1-\gamma))} e^{\nu t} e^{-b|x-y|^2 t^{-1}} \quad (21)$$

for all $t > 0$, $x, y \in \mathbf{R}^d$ and $\varepsilon \in \langle 0, 1 \rangle$. The convergence of the $K^{(\varepsilon)}$ follows from these uniform upper bounds and the L_2 -convergence of $S^{(\varepsilon)}$ to $S^{(0)}$ (see, for example, [EIR] Proposition 2.2). \square

Finally the kernel bounds (19) follow from the bounds (20) on $K^{(\varepsilon)}$ by taking the limit $\varepsilon \rightarrow 0$. We omit further details. \square

Note that if for some $x, y \in \mathbf{R}^d$ one has $\overline{d_C}(x; y) = \infty$ then $K_t^{(0)}(x; y) = 0$. Further the foregoing arguments give a large t bound on the kernel but with a factor $t^{-d/(2(1-\gamma))} e^{\varepsilon \nu t}$ which does not reflect the expected asymptotic behaviour, even if $\nu = 0$.

The Gaussian upper bounds give information on lower bounds by a variation of standard arguments for strongly elliptic operators.

Corollary 4.4 *Assume that the viscosity operator H_0 satisfies the subellipticity condition (12). Let $r, t > 0$. Then there is an $a' > 0$ such that*

$$(\varphi, S_t^{(0)} \varphi) \geq a' \|\varphi\|_1^2 \quad (22)$$

for all positive $\varphi \in L_1(\mathbf{R}^d) \cap L_2(\mathbf{R}^d)$ with $\text{diam}(\text{supp } \varphi) \leq r$. Hence if $K_t^{(0)}$ is continuous at $(x, x) \in \mathbf{R}^d \times \mathbf{R}^d$ then

$$K_t^{(0)}(x; x) \geq a' .$$

The value of a' depends on H_0 only through the parameters μ, ν, γ and $\|C\|$.

Proof First we prove the analogue of (22) for the approximants $S^{(\varepsilon)}$.

Since $S^{(\varepsilon)}$ is self-adjoint it follows that $(\varphi, S_t^{(\varepsilon)} \varphi) = \|S_{t/2}^{(\varepsilon)} \varphi\|_2^2 \geq 0$. Hence

$$|(\varphi, S_t^{(\varepsilon)} \psi)|^2 \leq (\varphi, S_t^{(\varepsilon)} \varphi) (\psi, S_t^{(\varepsilon)} \psi) \quad (23)$$

for all $\varphi, \psi \in L_2(\mathbf{R}^d)$. Next let $x_0 \in \mathbf{R}^d$ and φ be a positive integrable function with support in the Euclidean ball $B_E(x_0; r) = \{y \in \mathbf{R}^d : |y - x_0| < r\}$. Further let ψ be the

characteristic function of the ball $B_E(x_0; R)$ with $R > r$. We evaluate (23) with this choice of φ and ψ .

Using positivity and contractivity of $S^{(\varepsilon)}$ on $L_\infty(\mathbf{R}^d)$ one deduces that

$$(\psi, S_t^{(\varepsilon)}\psi) \leq V_E(R)$$

where $V_E(R)$ is the volume of $B_E(x_0; R)$. But as H_ε is strongly elliptic $S_t^{(\varepsilon)}\mathbf{1} = \mathbf{1}$ and

$$(\varphi, S_t^{(\varepsilon)}\psi) = (\varphi, \mathbf{1}) - (\varphi, S_t^{(\varepsilon)}(\mathbf{1} - \psi)) \geq \|\varphi\|_1 \left(1 - \sup_{x \in B_E(x_0; r)} \int_{\{y: |y-x_0| \geq R\}} dy K_t^{(\varepsilon)}(x; y)\right) .$$

Then since $K_t^{(\varepsilon)}$ satisfies the bounds (21) one can choose R sufficiently large that

$$(\varphi, S_t^{(\varepsilon)}\psi) \geq 2^{-1}\|\varphi\|_1$$

uniformly for $\varepsilon \in \langle 0, 1 \rangle$ and $x \in \mathbf{R}^d$. Therefore substituting these last two estimates in (23) one deduces that

$$(\varphi, S_t^{(\varepsilon)}\varphi) \geq (4V_E(R))^{-1} \|\varphi\|_1^2$$

uniformly for $\varepsilon \in \langle 0, 1 \rangle$. Since $S_t^{(\varepsilon)}$ converges strongly to $S_t^{(0)}$ it follows that (22) is valid with $a' = (4V_E(R))^{-1}$. The value of R is dictated by the Gaussian bounds (21) and hence depends on H_0 only through the parameters μ, ν, γ and $\|C\|$.

Finally suppose $K_t^{(0)}$ is continuous at a diagonal point, which we may take to be $(0, 0)$. Then for $\lambda > 0$ replace φ in (22) by φ_λ where $\varphi_\lambda(x) = \lambda^{-d}\varphi(\lambda^{-1}x)$. It follows that $\|\varphi_\lambda\|_1 = \|\varphi\|_1$. Moreover,

$$\lim_{\lambda \rightarrow 0} (\varphi_\lambda, S_t^{(0)}\varphi_\lambda) = \lim_{\lambda \rightarrow 0} \int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy \varphi(x) \varphi(y) K_t^{(0)}(\lambda x; \lambda y) = \|\varphi\|_1^2 K_t^{(0)}(0; 0) .$$

Therefore $K_t^{(0)}(0; 0) \geq a'$. □

Remark 4.5 If the kernel $K_t^{(0)}$ is a continuous function over $\mathbf{R}^d \times \mathbf{R}^d$ it follows from the corollary that it is strictly positive on the diagonal, i.e.,

$$\inf_{x \in \mathbf{R}^d} K_t^{(0)}(x; x) \geq a' > 0 .$$

If, however, if $K_t^{(0)}$ is uniformly continuous then one has a stronger off-diagonal property. Explicitly, if

$$\lim_{x \rightarrow 0} \|L(x)K_t^{(0)} - K_t^{(0)}\|_\infty = 0$$

where $(L(z)K_t^{(0)})(x; y) = K_t^{(0)}(x - z; y)$ then it follows from Corollary 4.4 that there are $a', r > 0$ such that

$$K_t^{(0)}(x; y) \geq a' > 0$$

for all $x, y \in \mathbf{R}^d$ with $|x - y| < r$. Uniform continuity of the kernel in the first variable is of course equivalent to uniform continuity in the second variable, by symmetry, and separate uniform continuity is equivalent to joint uniform continuity.

The upper bound (17) on the kernel gives an estimate $t^{-d/(2(1-\gamma))}$ on the singularity of the kernel as $t \rightarrow 0$. Explicitly, $\|K_t^{(0)}\|_\infty \leq a e^\nu (\mu t)^{-d/(2(1-\gamma))}$ for all $t \in \langle 0, 1 \rangle$. This is an optimal estimate for the local singularity for the class of subelliptic operators under consideration. If $C = \mu_0 I$ with μ_0 given by (9) and if $d \geq 3$ then the kernel has this singularity. This can be established by a local variation of the reasoning used to deduce Corollary 4.4.

First, by the rescaling $x \rightarrow t^{1/(2(1-\gamma))}x$ and the simultaneous replacement $\varepsilon = \delta t^{\gamma/(1-\gamma)}$ one finds

$$K_t^{(\varepsilon)}(x; y) = t^{-d/(2(1-\gamma))} K_1^{(t, \delta)}(t^{-1/(2(1-\gamma))}x; t^{-1/(2(1-\gamma))}y)$$

where $K^{(t, \delta)}$ is the semigroup kernel of the operator $H_{t, \delta} = H_t + \delta \Delta$ with $H_t = -\nabla c^{(t)} \nabla$ and $c^{(t)}(x) = c_0 \left(|x|^2 (1 + |x|^2 t^{1/(1-\gamma)})^{-1} \right)^\gamma$. Note that $H_{t, \delta}$ is the strongly elliptic approximant to the operator H_t . Taking the weak* limit $\delta \rightarrow 0$ of the last identity one obtains the identity

$$K_t^{(0)}(x; y) = t^{-d/(2(1-\gamma))} K_1^{(t, 0)}(t^{-1/(2(1-\gamma))}x; t^{-1/(2(1-\gamma))}y)$$

for all $t \in \langle 0, 1 \rangle$.

Secondly, $H_{t, \delta} \geq H_{1, \delta} \geq H_0$ and since H_0 satisfies the subellipticity condition (12) one has bounds $H_{t, \delta} \geq \mu \Delta^{1-\gamma} - \nu I$ uniformly for $t, \delta \in \langle 0, 1 \rangle$. Therefore, by Corollary 4.4, one deduces that for all $r > 0$ there is an $a' > 0$ such that

$$(\varphi, S_1^{(t, 0)} \varphi) \geq a' \|\varphi\|_1^2$$

for all positive integrable φ with support in $\{x : |x| < r\}$ uniformly for $t \in \langle 0, 1 \rangle$. Hence

$$\sup_{|x|, |y| < r} K_1^{(t, 0)}(x; y) \geq a'$$

uniformly for $t \in \langle 0, 1 \rangle$. But then by scaling one has

$$\begin{aligned} a' t^{-d/(2(1-\gamma))} &\leq t^{-d/(2(1-\gamma))} \sup_{|x|, |y| < r} K_1^{(t, 0)}(x; y) \\ &= \sup_{|x|, |y| < r} K_t^{(0)}(t^{1/(2(1-\gamma))}x; t^{1/(2(1-\gamma))}y) \leq \sup_{|x|, |y| < r} K_t^{(0)}(x; y) \end{aligned}$$

where the last inequality uses $t \in \langle 0, 1 \rangle$. In particular, it follows together with Lemma 4.3, that there are $a, a' > 0$ such that

$$a' t^{-d/(2(1-\gamma))} \leq \|K_t^{(0)}\|_\infty \leq a t^{-d/(2(1-\gamma))}$$

for all $t \in \langle 0, 1 \rangle$.

If $K_t^{(0)}$ is continuous at $(0, 0)$ then $K_1^{(t, 0)}$ is continuous at $(0, 0)$ and $K_1^{(t, 0)}(0; 0) \geq a'$ uniformly for all $t \in \langle 0, 1 \rangle$ for some $a' > 0$ by Corollary 4.4. Then

$$a' t^{-d/(2(1-\gamma))} \leq K_t^{(0)}(0; 0) \leq a t^{-d/(2(1-\gamma))}$$

for all $t \in \langle 0, 1 \rangle$. The point $(0, 0)$ is, of course, special since the coefficients of H vanish at the origin. If $x \neq 0$ one would expect $K_t^{(0)}(x; x) \asymp t^{-d/2}$ for small t . Therefore any improvement in the upper bounds would require techniques which give a position dependent singularity.

One can also construct examples of second-order operators satisfying the subellipticity condition (12) for which the singularity of $t \mapsto \|K_t^{(0)}\|_\infty$, as $t \rightarrow 0$, is given by $t^{-D/2}$ with $D \in \langle d, d/(1-\gamma) \rangle$. Therefore the condition (12) does not determine the singularity of the kernel. Hence one cannot expect to obtain a characterization of the subellipticity condition by kernel bounds analogous to the statement of Theorem 3.1 in the strongly elliptic case. Nevertheless one can establish a weaker result in a similar direction.

Theorem 4.6 *Let H_0 be the viscosity operator with coefficients $C = (c_{ij})$ and $K^{(0)}$ the distribution kernel of the contraction semigroup $S^{(0)}$ generated by H_0 . Assume*

1. H_0 satisfies the subellipticity condition (12) for some $\gamma \in \langle 0, 1 \rangle$.
2. There are $a, r > 0$ such that

$$K_1^{(0)}(x; y) \geq a$$

for almost every $x, y \in \mathbf{R}^d$ with $|x - y| \leq r$.

Then for all $\delta > 0$ there exists an $a > 0$ such that

$$K_t^{(0)}(x; y) \leq a t^{-d/2} e^{-\bar{d}_C(x;y)^2((4+\delta)t)^{-1}} \quad (x, y)\text{-a.e.}$$

for all $t \geq 1$.

Remark 4.7 Note that the local lower bounds of Condition 2 follow from the subellipticity of Condition 1 if $K_t^{(0)}$ is uniformly continuous. This is a consequence of Remark 4.5.

Proof Again the proof is based on Nash's original arguments as elaborated by Carlen, Kusuoka and Stroock [CKS].

By Corollary 3.3 there exists a $\sigma > 0$ such that

$$h_0(\varphi) \geq \sigma \int_{\mathbf{R}^d} d\xi |\widehat{\varphi}(\xi)|^2 (|\xi|^2 \wedge 1) \quad (24)$$

for all $\varphi \in D(h)$ with $\widehat{\varphi}$ the Fourier transform of φ .

Assume that $\varphi \in D(h) \cap L_1$. It then follows by Fourier transformation, as in the proof of Corollary 4.9 in [CKS], that

$$\begin{aligned} \|\varphi\|_2^2 &= \int_{\{\xi: |\xi| \leq R\}} d\xi |\widehat{\varphi}(\xi)|^2 + \int_{\{\xi: |\xi| \geq R\}} d\xi |\widehat{\varphi}(\xi)|^2 \\ &\leq c R^d \|\varphi\|_1^2 + \int_{\{\xi: |\xi| \geq R\}} d\xi (R^{-2} |\xi|^2 \wedge 1) |\widehat{\varphi}(\xi)|^2 \\ &\leq c R^d \|\varphi\|_1^2 + R^{-2} \int_{\mathbf{R}^d} d\xi (|\xi|^2 \wedge 1) |\widehat{\varphi}(\xi)|^2 \\ &\leq c R^d \|\varphi\|_1^2 + R^{-2} \sigma^{-1} h_\varepsilon(\varphi) \end{aligned}$$

for all $R \in \langle 0, 1 \rangle$ and $\varepsilon > 0$ where the last inequality uses (24). Then the Nash inequality

$$\|\varphi\|_2^{2+4/d} \leq c' h_\varepsilon(\varphi) \|\varphi\|_1^{4/d} \quad (25)$$

follows for all $\varphi \in D(h_\varepsilon) \cap L_1$ with $h_\varepsilon(\varphi) \leq \|\varphi\|_1^2$ by setting $R = (h_\varepsilon(\varphi)/\|\varphi\|_1^2)^{1/(d+2)}$. The inequality is uniform for $\varepsilon \in \langle 0, 1 \rangle$. Note that (25) is analogous to the earlier Nash inequality (15) but with $\gamma = 0$ and $\nu = 0$. In addition there is the important restriction $h_\varepsilon(\varphi) \leq \|\varphi\|_1^2$.

Next it follows from the contractivity of $S^{(\varepsilon)}$ on L_1 that

$$\|S_t^{(\varepsilon)}\|_{1 \rightarrow \infty} \leq \|S_1^{(\varepsilon)}\|_{1 \rightarrow \infty} \quad (26)$$

for all $t \geq 1$. In particular $t \mapsto \|S_t^{(\varepsilon)}\|_{1 \rightarrow \infty}$ is uniformly bounded for $t \geq 1$. The conditions (25) and (26) correspond to the assumptions of Theorem 2.9 of [CKS]. Therefore the theorem establishes the large time estimates

$$\|S_t^{(\varepsilon)}\|_{1 \rightarrow \infty} \leq a t^{-d/2} \quad (27)$$

for all $t \geq 1$. These estimates are again uniform for $\varepsilon \in \langle 0, 1 \rangle$.

The estimates (27) convert to large time Gaussian bounds, with the distance associated with C_ε , by Davies perturbation theory as in Proposition 4.2, but with $\gamma = 0$ and $\nu = 0$. Specifically one deduces that for all $\delta > 0$ there exists an $a > 0$ such that

$$K_t^{(\varepsilon)}(x; y) \leq a t^{-d/2} e^{-dC_\varepsilon(x; y)^2((4+\delta)t)^{-1}}$$

uniformly for all $t \geq 1$, $x, y \in \mathbf{R}^d$ and $\varepsilon \in \langle 0, 1 \rangle$. Finally, taking the limit $\varepsilon \rightarrow 0$, one obtains the large time bounds on $K^{(0)}$. \square

An essential feature of the above analysis was the estimate (24) which follows from the local lower bound. This complements the subellipticity estimate (12) which can be reformulated as

$$h_0(\varphi) \geq \int_{\mathbf{R}^d} d\xi \left(\mu |\xi|^{2(1-\gamma)} - \nu \right) |\widehat{\varphi}(\xi)|^2 \quad .$$

Combination of the two conditions then gives

$$h_0(\varphi) \geq \int_{\mathbf{R}^d} d\xi \left(\sigma(|\xi|^2 \wedge 1) \right) \vee \left(\mu |\xi|^{2(1-\gamma)} - \nu \right) |\widehat{\varphi}(\xi)|^2 \quad .$$

Thus if f is any positive function on $\langle 0, \infty \rangle$ such that $f(x) \asymp \left(\sigma(x \wedge 1) \right) \vee \left(\mu x^{1-\gamma} - \nu \right)$ one then has

$$h_0(\varphi) \geq c \int_{\mathbf{R}^d} d\xi f(|\xi|^2) |\widehat{\varphi}(\xi)|^2 = c(\varphi, f(\Delta)\varphi)$$

for all $\varphi \in D(f(\Delta))$. In particular one can choose $f(x) = \mu' x (1+x)^{-\gamma}$ with $\mu' > 0$. The positivity bound (24) is a restriction on the lower part of the spectrum and is reflected by the fact that $f(x) \asymp x$ as $x \rightarrow 0$. This behaviour effectively means that H_0 is asymptotically strongly elliptic. Subellipticity is, however, a restriction on the upper part of the spectrum reflected by $f(x) \asymp x^{1-\gamma}$ as $x \rightarrow \infty$. This special choice of f gives the operator inequality

$$H_0 \geq \mu' \Delta(I + \Delta)^{-\gamma}$$

which incorporates both the local and global behaviour. It is analogous to the bound (10) in the example at the beginning of the section.

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