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EQUIDISTANT CODES WITH DISTANCE 12

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Let C be an equidistant binary code with m words, pairwise at distance $2k$. It is known that if C is not trivial then $m \leq k^2 + k + 2$. Furthermore equality is possible if and only if a projective plane of order k exists. This settles the problem of determining the maximal m for $k < 10$ with the exception of $k = 6$. In this paper we show that if $k = 6$ then $m \leq 32$, and we give an example of a code with $m = 32$. To settle the next unknown case, i.e. $k = 10$, one would first have to know whether a projective plane of order 10 exists.

1. Introduction

Consider a binary code with m words of length n such that any two distinct words have Hamming distance $2k$. Such a code is *equidistant* and will be called an $(m, 2k, n)$-code. Let the m by n matrix C have the words of the code as its rows. If every column of C has m or $m - 1$ equal entries, then the code is called *trivial.* It is well known that if k is the order of a projective plane, then there exist nontrivial $(k^2 + k + 2, 2k, k^2 + 2k)$-codes (see [3, 11, 12]). It was shown by Deza [3] that $m \leq k^2 + k + 2$ for every nontrivial $(m, 2k, n)$-code. Van Lint proved that a nontrivial $(k^2 + k + 2, 2k, n)$-code exists only if there exists a projective plane of order k and n is sufficiently large (cf. [11, 12]). Therefore we know that a nontrivial $(m, 12, n)$-code must have cardinality $m \leq 43$. In this paper we shall prove

Theorem 1.1. *A nontrivial equidistant* $(m, 12, n)$*-code has cardinality* $m \leq 32$.

Example 1.2 below furnishes an example of a nontrivial equidistant code of distance 12 with 32 words.

After completion and submission of this work, the authors learned that the theorem has also been proven independently by McCarthy, Mullin and Vanstone in a long string of papers, the two most important of which are [6] and [7]. The basic idea of their proof, which was completed at approximately the same time as that presented here, is the same as that of this paper. However, in place of the transversal search of our Lemma 2.4 and reference to Nandi [8], their proof relies on a result of De Witte [2], a computer search of Schellenberg [10], and much more analysis of particular cases. The proof of McCarthy, Mullin and Vanstone is phrased in the language of (r, λ)-systems (cf. [7]).

Notation. Here and in the rest of the paper we use the following symbols for matrices which occur frequently.

I_p is the identity of size p;
$O_{p,q}$ is the p by q matrix with all entries 0;
$J_{p,q}$ is the p by q matrix with all entries 1;
$E^{(i)}_{p,q}$ is the p by q matrix with 1's in column i and 0's elesewhere;
$F^{(i)}_{p,q}$ is the p by q matrix with 1's in row i and 0's elsewhere.

Example 1.2. Let P be the incidence matrix of a projective plane of order 5.

$$C := \begin{pmatrix} O_{1,5} & O_{1,31} & O_{1,31} \\ J_{31,5} & I_{31} & P \end{pmatrix}$$

is the matrix of a nontrivial (32, 12, 67)-code.

2. Some lemmas

In this section we shall derive a number of necessary conditions for the existence of a nontrivial $(m, 12, n)$-code with $m > 32$. We now let C be a nontrivial $(33, 12, n)$-code and form the *matrix of the code*, which we shall also call C, whose rows are precisely the words of the code. We consider our code to be a subset of an n-dimensional vector space V over GF(2). In such a space Hamming distance is a translation invariant, so the code

$$C + \boldsymbol{x} := \{\boldsymbol{c} + \boldsymbol{x} : \boldsymbol{c} \in C\} \quad \text{for } \boldsymbol{x} \in V, \tag{2.1}$$

is also a $(33, 12, n)$-code. Of course, in terms of the matrix C this just corresponds to complementing the columns corresponding to the 1's of $\boldsymbol{x}$. Of particular interest are the codes $C + \boldsymbol{c}$ where $\boldsymbol{c} \in C$. In this case the zero vector $\mathbf{0}$ is in the code $C + \boldsymbol{c}$, hence all nonzero code words must have weight 12 and mutual inner products 6.

Lemma 2.1. *If c is a column sum of the matrix of a nontrivial $(33, 12, n)$-code, then $c \leq 7$ or $c \geq 26$.*

Proof. This is a consequence of (Deza [3, Lemma 3.1]).

For an equidistant $(33, 12, n)$-code as in Lemma 2.1, we call a column of the matrix *heavy* if its column sum c is at least 26 and *light* if $c \leq 7$.

Lemma 2.2. *If the nontrivial $(33, 12, n)$-code C exists, then for some r there exists a $(32, 12, r+1)$-code whose matrix C^* has*

(i) *all row sums* 7,
(ii) *all inner products of distinct rows* 1, *and*
(iii) *all column sums at most* 7.

Proof. We remark that any nontrivial $(m, 12, r+1)$-code which satisfies (i) must also satisfy (ii) and (iii), although here this will be apparent by our construction of C^*.

First suppose that D is the matrix of any $(33, 12, n)$-code which contains the zero vector, $\mathbf{0} \in D$. We claim that D has at most 7 heavy columns. Assume we can find 8 heavy columns in D. As the inner product of two nonzero rows of D is 6, there are at most 8 rows of D with 7 ones in our 8 fixed columns. If some row has 8 ones in those columns, then all other nonzero rows have at most 6 ones in them. In any case, the columns can contain at most

$$7 \times 8 + 6(33 - 9) = 200$$

ones. One the other hand, since the columns are heavy, they contain at least

$$8 \times 26 = 208$$

ones. This is a contradiction, hence as claimed D has at most 7 heavy columns. Now if necessary through replacing C by $C + \boldsymbol{c}$, for some $\boldsymbol{c} \in C$ (see (2.1)), we assume that $\mathbf{0} \in C$. In particular, by the previous comments C has at most 7 heavy columns.

Suppose now that there is a row $\boldsymbol{c}$ of C with $h \leq 5$ of its ones in heavy columns. Let k be the number of heavy columns of C without 1's in the row $\boldsymbol{c}$. Then $C + \boldsymbol{c}$ is a $(33, 12, n)$-code with $\mathbf{0} \in C + \boldsymbol{c}$ and having $12 - h + k$ heavy columns. Now $k \geq 0$ and $h \leq 5$, and since we must have $7 \geq 12 - h + k$, we find that $h = 5$ and $k = 0$. Thus C has 5 heavy columns and each nonzero row of C has ones in each of these columns. Hence, by permuting rows and columns, we can write

$$C = \begin{pmatrix} O_{1,5} & O_{1,n-5} \\ J_{32,5} & C^* \end{pmatrix}, \tag{2.2}$$

where, as desired, C^* is a $(32, 12, r+1)$-code with row sums 7 and column sums at most 7 for $r + 1 = n - 5$.

Thus we may assume that each nonzero row of C has 6 or 7 ones in heavy columns. In fact, if some row $\boldsymbol{c}$ has 7 ones in heavy columns, then $C + \boldsymbol{c}$ has only 5 heavy columns and $\mathbf{0} \in C + \boldsymbol{c}$. Thus C^* can be constructed from $C + \boldsymbol{c}$ as in (2.2). We may now assume that each nonzero row of C has 6 ones in heavy columns.

If there are only 6 heavy columns, then each nonzero row has a one in each of these columns. As nonzero inner products are 6 (or 12), all column sums are then from $\{0, 1, 32, 33\}$ and the code is trivial, against hypothesis. Thus C has 7 heavy columns and each nonzero row has 6 ones in these columns. Let $\boldsymbol{x}$ be the vector of V having weight 7, its 1's in the coordinate positions corresponding to the heavy columns of C. Then $C+\boldsymbol{x}$ is a $(33, 12, n)$-code with no heavy columns and all words of weight 7. Deleting any row from $C+\boldsymbol{x}$ gives a matrix C^* as required.

We remark that in the language of McCarthy, Mullin and Vanstone [6, 7] C^* is an incidence matrix for a (7, 1)-system. It should be noted that if C^* is any matrix as in Lemma 2.2, then the matrix given by (2.2) displays a $(33, 12, r+6)$-code.

We shall denote the column sums of C^* by $c_0, \ldots, c_r$ and let the number of columns of C^* with sum i be a_i, for $0 \leq i \leq 7$. We assume that all constantly zero columns of C^* have been deleted, so that $a_0 = 0$.

Lemma 2.3. *The following relations hold for the numbers c_i and a_i:*

$$\sum_{i=0}^{r} c_i = 224, \tag{2.3}$$

$$\sum_{i=0}^{r} \binom{c_i}{2} = 496, \tag{2.4}$$

$$\sum_{i=0}^{r} (c_i - 6)^2 = \sum_{k=1}^{7} (k-6)^2 a_k = -1436 + 36r. \tag{2.5}$$

Proof. (2.3) is obvious since each row of C^* has weight 7. (2.4) counts in two ways the sum of the inner products of all pairs of rows of C^*. (2.5) is a combination of (2.3) and (2.4).

Suppose now by deleting rows from C^* we obtain a submatrix D^* with d rows which contains two columns of weight 6 with the property that no row contains a 1 in each column. (These columns are *disjoint.*) We delete all constantly zero columns of D^* and permute rows and columns so that we may write

$$D^* = \begin{pmatrix} J_{6,1} & O_{6,1} & F^{(1)}_{6,6} & F^{(2)}_{6,6} & F^{(3)}_{6,6} & F^{(4)}_{6,6} & F^{(5)}_{6,6} & F^{(6)}_{6,6} & O_{12,s} \\ O_{6,1} & J_{6,1} & I_6 & I_6 & I_6 & I_6 & I_6 & I_6 & \\ & O_{d-12,2} & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & A' \end{pmatrix}, \tag{2.6}$$

where s is some integer with $s + 38 \leq r + 1$. Our aim is to give a concise description of the part of D^* denoted by T_i, for $1 \leq i \leq 6$, and from this to derive bounds for d (see Lemma 2.5).

We first comment that if D^* contains three pairwise disjoint columns of weight 6 (so that we may assume that A' has a column of weight 6), then it is not difficult to see that those 18 rows and their nonzero columns form an incidence matrix for a transversal design T(3, 6) (see [5, Chapter 15]). The remaining vectors of D^* are the characteristic vectors for parallel classes of the T(3, 6). It is well known (e.g. [5, Chapters 13 and 15]) that the transversal design T(3, 6) together with an arbitrary labelling of the three sets of size 6 (columns of weight 6) as "rows", "columns", or "entries" is equivalent to a Latin square of order 6. In this equivalence the parallel classes of the T(3, 6) correspond to the transversals of the Latin square. These observations motivate our construction.

We associate with D^* a matrix T of size $(d-12)\times 6$ with entries from $\{1, 2, 3, 4, 5, 6\}$. If D^* is as in (2.6) then we let $T_{k,q}$, for $1 \leq k \leq d-12$ and $1 \leq q \leq 6$, denote the position (i.e., 1 through 6) of the 1 of row $k+12$ of D^* which lies in T_q. Each row of T is then a permutation of 1 through 6, and no two rows agree in more than one column. Two rows of T have no column in which they agree if and only if the 1's in A' of the corresponding rows of D^* share a column. In particular, the property of not agreeing anywhere is an equivalence relation of the rows of T.

If A' contains a column of weight 6, then the rows of T corresponding to this column are the rows of the Latin square discussed above and the other rows of T are transversals of the square. Here column 0 provides "columns", column 1 "entries", and the column in A' "rows".

Suppose A' contains a column of weight 5, so that the corresponding rows of T form a 5×6 partial Latin square L'. L' completes in a unique way to a Latin square L. As all other rows of T meet each row of L' exactly once, they must in fact be transversals of L. The $d-17$ rows of T not in L' thus form a set of transversals of the Latin square L of order 6 such that each pair of transversal differ in at least five places.

Lemma 2.4. *A Latin square of order 6 has at most 12 transversals which are pairwise different in at least 5 positions.*

Proof. The proof is by inspection. From our discussion of transversal designs, we see that the property in question only depends on the isomorphism class of the underlying transversal design of the Latin square. From the 17 inequivalent Latin squares of order 6 (cf. [4]) it is easily checked that there are only 12 nonisomorphic T(3, 6). Of these, several clearly have no transversals at all, and it is easily found that only the two associated with the squares

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 1 & 4 & 5 \\ 3 & 6 & 2 & 5 & 1 & 4 \\ 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 4 & 1 & 6 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & 4 & 1 & 2 & 6 & 5 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{matrix}$$

produce more than 12 transversals (parallel classes).

The second one has 24 transversals forming two sets of 12 having the required property. The first one has 32 transversals but no subset of more than 8 of these has the required property. The Lemma has been easily proven by hand. As a check an inspection was carried out by a Burroughs B6700 computer in 15 seconds.

We thus have

Lemma 2.5. *Suppose D^* in (2.6) has a column of weight 5 in A'. Then $d \leqslant 29$.*

Proof. We have constructed above from such a D^* a Latin square L of order 6 with a set of $d-17$ transversals of L which pairwise differ in at least 5 places. Hence by Lemma 2.4, $d-17 \leqslant 12$.

We now present a standard representation of C^* which will be essential in the remainder of the paper. By permuting columns of C^* we may choose any column to be column 0, the *leading column* of C^*. This column has weight $c := c_0$ and with suitable permutations of rows and columns we have

$$C = \begin{pmatrix} J_{c,1} & F^{(1)}_{c,6} & F^{(2)}_{c,6} & \cdots & F^{(c)}_{c,6} & O_{c,r-6c} \\ O_{32-c,1} & B_1 & B_2 & \cdots & B_c & A \end{pmatrix}. \tag{2.7}$$

Here each row of each B_j has exactly one 1. Of special interest are the cases $c=6$ and $c=7$. Assuming $c=7$, there are seven B_j. Hence A contains no 1's and so does not exist. Thus if $a_7 \neq 0$, then taking $c=7$ we find $r=42$. When $c=6$, each row of A contains exactly one 1.

Remember that above we called two columns of C^* disjoint if no row had a 1 in each column. If such a (unique) row does exist we shall say that the two columns *intersect* (or *hit*).

Lemma 2.6. (i) *In* (2.7), $c = c_1 = c_2 = 7$ *is impossible.*

(ii) *In* (2.7) *with* $c=6$, A *does not contain two columns of weight* 6.

(iii) *In* (2.7) *with* $c=6$, A *does not contain a column of weight* 6 *and a column of weight* 5.

Proof. We delete rows 1 and 2 from the possible matrix of (i), delete row 1 from that of (ii), and leave undisturbed the matrix of (iii). We now have three possible submatrices of C^* with at least 30 rows and containing three pairwise disjoint columns with weights 6, 6 and 5. By suitable row and column permutations, each of these matrices may be arranged as D^* in (2.6) with a column of weight 5 in A' and $d \geqslant 30$. This contradicts Lemma 2.5.

3. Nonexistence of a nontrivial $(33, 12, n)$-code with $r \leqslant 42$

In this section we investigate the possibility of a nontrivial $(33, 12, n)$-code C. We analyse the corresponding 32 by $r+1$ matrix C^* given by Lemma 2.2 and prove $r > 42$.

Lemma 3.1. *If $r \leqslant 42$ then we may assume $c_0 = 7$.*

Proof. Let the ones of row 1 be in the first 7 columns of C^*. Then $\sum_{i=0}^{6} c_i = 38$, and we may assume that $c_0 \geqslant 38/7 > 5$. Hence c_0 is either 6 or 7. Suppose $c_0 = 6$. Then in (2.7), A has at most 6 columns. We add $42 - r$ columns of zeros to C^*, then a new row with 1's in columns 0 and 37 through 42. The new row has weight 7 and inner product 1 with all rows of C^*. Now by discarding a row of C^* not intersecting column 0, we find a new matrix C^* with $c_0 = 7$ as desired.

Lemma 3.2. *If $r \leqslant 42$ we may assume $c = c_1 = c_2 = 7$ in* (2.7).

Proof. By Lemma 3.1 we may assume $c_0 = 7$ so that $r = 42$ and C^* has the form (2.7) with $c = 7$. We note that it is enough to show that $c_1 \geqslant c_2 \geqslant 6$. For in that case without disturbing column 0 we may proceed as in Lemma 3.1 to construct an appropriate matrix C^* with the desired property. As before, we first take $c_1 \geqslant \frac{1}{6}(38-7)$. That is, $c_1 \geqslant 6$. In particular we remark that any row of C^* has at least two 1's in columns of weight 6 or more.

We note that we can assume $c_2 \geqslant \frac{1}{5}(38-14)$. As we are done if $c_2 \geqslant 6$, this allows us to take $c_2 = 5$. As column 2 is incident with 5 rows, the above remark ensures the existence of at least 10 distinct columns with weight at least 6. Now C^* has the form (2.7) with $c = 7$, so at least one of the matrices B_i has two columns with weight 5 or more. We may now renumber columns so that $c_1 \geqslant c_2 \geqslant 6$ as desired.

Lemma 3.3. $r > 42$.

Proof. If $r \leqslant 42$ then Lemma 3.2 contradicts Lemma 2.6(i).

4. Nonexistence of a nontrivial (33, 12, n)-code

In this section we show the nonexistence of a nontrivial (33, 12, n)-code C. We consider the associated matrix C^* of Lemma 2.2 which, in view of Lemma 3.3, has $r \geqslant 43$.

Lemma 4.1. $a_7 = 0$.

Proof. Referring to (2.7) we see that $c = 7$ implies $r = 42$.

Lemma 4.2. *The column sums corresponding to the 1's of a row of C^* are of one of the following types:*

$$\begin{aligned}&\text{(a) } \{6,6,6,6,6,6,2\}, \quad \text{(b) } \{6,6,6,6,6,5,3\}, \quad \text{(c) } \{6,6,6,6,6,4,4\},\\ &\text{(d) } \{6,6,6,6,5,5,4\}, \quad \text{(e) } \{6,6,6,5,5,5,5\}.\end{aligned} \tag{4.1}$$

Proof. The column sums corresponding to the 1's of any given row of C^* form a partition of 38 into 7 parts from 1 through 6. The result follows.

In particular $a_6 \neq 0$. Letting $c = 6$ in (2.7), we define $t := r - 36$, the number of columns in A.

Lemma 4.3. (i) *Any column of weight* 2 *hits only columns of weight* 6.
(ii) *Any column of weight* 6 *which hits a column of weight* 2 *is disjoint from a unique other column of weight* 6, *which hits the same column of weight* 2.

Proof. (i) This follows from Lemma 4.2.
(ii) If a column of weight 6 hits a column of weight 2, we take the column of weight 6 as the leading column in (2.7). The column of weight 2 hits a column of A, which must have weight 6 by part (i). This column is unique by Lemma 2.6(ii).

Lemma 4.4. $a_2 > 0$.

Proof. We have seen in Lemma 4.2 that $a_6 > 0$, i.e. C^* can be represented as in (2.7) with $c = 6$. Suppose $a_2 = 0$. Since A has 26 1's we must have $t = 7$ or $t = 8$. By (2.5) we must have $\sum_{i=0}^{r}(c_i - 6)^2 = 36t - 140$. If $a_2 = 0$ then the maximal value of $\sum_{i=0}^{36}(c_i - 6)^2$ is 60, which is achieved if each B_i in (2.7) is of type (4.1)(b). The contribution of the columns of A to $\sum_{i=0}^{r}(c_i - 6)^2$ is at most 42 if $t = 7$ (realized by $\{3,3,3,3,4,5,5\}$) resp. 64 if $t = 8$ (realized by $\{3,3,3,3,3,3,3,5\}$). In both cases we have $\sum_{i=0}^{r}(c_i - 6)^2 < 36t - 140$.

Lemma 4.5. $a_3 = 0$.

Proof. Suppose $a_3 \neq 0$. By the previous lemma, $a_2 > 0$. A row with a 1 in a column of weight 2 and a row with a 1 in a column of weight 3 have inner product 1. This means that there is a column of weight 6 which hits both the column of weight 2 and the column of weight 3. Take this as the leading column in (2.7) and let the column of weight 2 be the next column. By Lemma 4.3 there is a column of weight 6 in A. The column of weight 3 hits two columns of A which by Lemma 4.2 both have weight at least 5. But this is impossible by Lemma 2.6(ii) and (iii).

We shall now tabulate the remaining possibilities for the matrix C^* in the form (2.7) with $c = 6$. Since $a_2 > 0$ we may choose this leading column in such a way that there is a column of weight 6 in A. The possible contributions of the matrices B_i to the column sum catalogue are found from (4.1)(a), (c), (d) and (e) by leaving out one 6 corresponding to c_0. Besides the one column of weight 6, all other columns of $A =: A(t)$ have weight 2 or 4 (by Lemmas 2.6, 4.2 and 4.5). Since A contains exactly 26 rows, the number of columns of each weight in $A(t)$ is determined by the value of t $(7 \leqslant t \leqslant 11)$. A has $2t - 12$ columns of weight 2 and $11 - t$ columns of weight 4. This reasoning shows that C^* always has at least 2 columns of weight 2. Therefore we may even assume that the leading column is chosen in such a way that the first two rows of C^* are of type (4.1)(a). Among the matrices B_3 to B_6 let there be a α corresponding to (4.1)(a), β corresponding to (4.1)(c), γ corresponding to (4.1)(d), and $4-\alpha-\beta-\gamma$ corresponding to (4.1)(e). Table 1 shows the contributions to the catalogue of column sums of C^*.

Table 1

			B_3 to B_6				
	c_0	B_1, B_2	(4.1)(a)	(4.1)(c)	(4.1)(d)	(4.1)(e)	$A(t)$
a_2	0	2	α	0	0	0	$2t-12$
a_4	0	0	0	2β	γ	0	$11-t$
a_5	0	0	0	0	2γ	$4(4-\alpha-\beta-\gamma)$	0
a_6	1	10	5α	4β	3γ	$2(4-\alpha-\beta-\gamma)$	1

Lemma 4.6. (i) $t \neq 7$, 8, *or* 10.
(ii) *If* $t=9$ *then* $a_2=10$, $a_4=6$, $a_5=0$, $a_6=30$.
(iii) *If* $t=11$ *then* $a_2=16$, $a_4=a_5=0$, $a_6=32$.

Proof. (a) Let $t=7$. As observed in the proof of Lemma 4.4 the equation (2.5) now has the form $\sum_{k=2}^{6}(k-6)^2a_k=36t-140$. From Table 1 we substitute the values of a_2 to a_6 and solve the equation subject to the conditions $\alpha \geqslant 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$, $\alpha+\beta+\gamma \leqslant 4$. As a result we find that either $a_2=4$, $a_4=12$, $a_5=0$, $a_6=28$ or $a_2=5$, $a_4=6$, $a_5=8$, $a_6=25$. Once we know these numbers we may take any column of weight 6 as the leading column in (2.7). We claim that then A has a column of weight 6. If this were not so then A would have at most three columns of weight 2. Therefore there would be a column of weight 2 which hits the leading column, and by Lemma 4.3 this implies that A has a column of weight 6. Thus each column of weight 6 has a unique partner of weight 6, disjoint from it. Hence a_6 is even, and the second possibility above is ruled out. Furthermore we now know that each weight 6 column hits precisely 2 columns of weight 2. Then $a_2=4$ implies $a_6=24$, a contradiction with the value obtained above. Hence $t=7$ is impossible.

(b) Let $t=8$. As in (a) we use Table 1 and (2.5) to calculate the possible values of a_k. Now $a_2=8$, $a_4=3$, $a_5=8$, $a_6=26$ or $a_2=7$, $a_4=9$, $a_5=0$, $a_6=29$. As A has 26 rows and 8 columns, and $a_2 \geqslant 7$, any column of weight 6 hits a column of weight 2. Hence, by Lemma 4.3 and Table 1 for any choice of leading column with $c=6$ in (2.7), A contains one column of length 6 and three of weight 4. The first solution can not occur as $a_4=3$ then implies that no column of weight 4 hits a column of weight 6. The second solution can not occur as every column of weight 6 has a unique disjoint partner, forcing a_6 to be even.

(c) Let $t = 10$. Now A has one column of weight 6 and one of weight 4. Any column corresponding to one of the matrices B_i with weight $\geqslant 4$ hits at least 3 columns of A, i.e. it hits at least one column of weight 2, and hence it must have weight 6. It follows that all the B_i are of type (4.1)(a). Hence $a_4 = 1$ and $a_2 = 14$. Therefore each column of weight 6 hits a column of weight 2, and so has a unique partner of weight 6. As $a_4 = 1$ this means that any column of weight 6 taken as leading column in (2.7) would result in the unique column of weight 4 being a column of A. This column of weight 4 hits no column of weight 6, an obvious falsehood.

Now (a), (b), (c) settle part (i) of the lemma.

(d) Suppose $t = 9$. Further suppose that some column corresponding to the B_i-part of C^* has weight at least 5. Such a column would hit 4 columns of A, while A has only 3 columns with weight not 2. Thus the column must have weight 6. So $a_5 = 0$, i.e. $\gamma = 0$, $\beta = 4 - \alpha$. By substituting the values of a_2, a_4, a_6 in (2.5) we find $\alpha = 2$ from which part (ii) follows.

(e) Suppose $t = 11$. Any column corresponding to the B_i-part of C^* with weight more than 2 hits at least two columns of $A(t)$ and so at least one column of weight 2. Therefore the column has weight 6. Hence all columns have weight 2 or 6 and each B_i is of type (4.1)(a), This gives (iii).

The lemmas proved in this section show that the matrix C^* in (2.7) must have either

$$t = 9, \quad a_2 = 10, \quad a_4 = 6, \quad a_6 = 30, \tag{4.3}$$

or

$$t = 11, \quad a_2 = 16, \quad a_6 = 32, \tag{4.4}$$

where we have left out the a_i which are 0. We reorder the columns of C^* by partitioning according to column sums:

$$C^* = \left(\begin{array}{c|cccccc|c}
 & 1 & 0 & 0 & \cdots & & 0 & \\
D_1 & 1 & 0 & 0 & \cdots & & 0 & 0 \\
 & 0 & 1 & 0 & \cdots & & 0 & \\
 & 0 & 1 & 0 & \cdots & & 0 & \\
 & & & & \ddots & & & \\
 & 0 & \cdots & \cdots & \cdots & 0 & 1 & \\
 & 0 & \cdots & \cdots & \cdots & 0 & 1 & \\ \hline
 & & & & & & & \\
D_2 & & & 0 & & & & *
\end{array}\right),$$

where for $t = 11$ the starred, 0-sections and D_2 are empty. Referring to Lemma 4.6(ii) and (iii), for $t = 9$ the matrix D_1 is a 20 by 30 matrix, for $t = 11$ it is 32 by 32.

$D = \binom{D_1}{D_2}$ is composed of $\frac{1}{2}a_6$ pairs of columns of weight 6. D_1 has row sums 6. In our proof of Lemma 4.6(ii) we found that $t = 9$ gave $\alpha = 2$. Thus for $t = 9$, D_1 has column sums $\alpha + 2 = 4$, and for $t = 11$, D_1 has column sum 6.

We now index the columns of D_1 by $\{\pm 1, \pm 2, \ldots, \pm\frac{1}{2}a_6\}$, where columns indexed by $+a$ and $-a$ correspond to disjoint columns of weight 6 in D. We interpret the rows of D_1 as the blocks of a configuration with elements ± 1, $\pm 2, \ldots, \pm\frac{1}{2}a_6$ (i.e. we ignore the 0's in the rows of D_1). This configuration, which we shall also denote by D_1, has $2a_2$ blocks, each with 6 elements. We consider the associated configuration

$$D_1^* := \{\{|a|, |b|, |c|, |d|, |e|, |f|\}: \{a, b, c, d, e, f\} \in D_1\}.$$

Note that by Lemma 4.3 the two rows of D_1 which are incident with a given column of weight 2 give rise to the same block of D_1^*. Therefore the configuration D_1^* has a_2 blocks of size 6 from a set of $\frac{1}{2}a_6$ points, each point occurring 4 times if $t = 9$, resp. 6 times if $t = 11$. Also, we claim that any two blocks of D_1^* intersect twice. For if P and Q are blocks of D_1, then $P \cap Q = \{p\}$ and $P \cap -Q = \{q\}$ where $q \neq \pm p$, since the rows $+Q$ and $-Q$ are disjoint as are the columns $+p$ and $-p$. Hence the blocks $|P|$ and $|Q|$ of D^* intersect in the points $|p|$ and $|q|$. Thus D_1^* is the dual of a block design with $\lambda = 2$. The parameters of the block design are $(10, 15, 6, 4, 2)$ for $t = 9$ and $(16, 16, 6, 6, 2)$ for $t = 11$. All such (nonisomorphic) designs are known.

We consider the problem of reconstructing D_1 from D_1^*. Let E^* be the dual of one of the appropriate designs. We wish to attach signs to all indices in order to form a configuration E with the following properties:

$$\begin{aligned} &\text{(i) if } P \text{ is a block of } E, \text{ so is } -P; \\ &\text{(ii) if } P \text{ and } Q \text{ are blocks of } E \text{ and } Q \neq \pm P \text{ then } |P \cap Q| = 1. \end{aligned} \tag{4.5}$$

Before we proceed to show that the two configurations D_1 do not exist, we make two remarks.

(i) For $t = 11$, the configuration D_1 and its relation to $(16, 16, 6, 6, 2)$-designs has been discussed by Payne [9], who proves nonexistence in this case.

However,

(ii) The nonexistence of D_1 for $t = 9$ will imply the nonexistence for $t = 11$ since the designs $(10, 15, 6, 4, 2)$ are residuals of the designs $(16, 16, 6, 6, 2)$ (cf. [5, p. 103]). For if we delete from the matrix D_1 for $t = 11$ two disjoint columns and all rows having 1's in these columns, the resulting configuration is a suitable D_1 for $t = 9$.

We shall now show that for $t = 9$ the configuration D_1 does not exist. We observe that if $|P \cap Q| = 1$, then $|P \cap -Q| = 1$ also. Hence we only need check (4.5)(ii) for one block from each pair $\{Q, -Q\}$. We use the three nonisomorphic designs $(10, 15, 6, 4, 2)$ given by Nandi [8]. For each of these the dual E^* contains the blocks

$$\{1,2,3,4,8,9\}, \quad \{1,2,5,6,10,11\}, \quad \{1,3,5,7,12,13\}, \quad \{1,4,6,7,14,15\}.$$

In attaching signs to these we may without loss of generality take the signed blocks to be

$$\{1,2,3,4,8,9\}, \quad \{1,-2,5,6,10,11\}, \quad \{1,-3,-5,7,12,13\},$$
$$\{1,-4,-6,-7,14,15\},$$

where the minus signs are forced by (4.5)(ii).

For the designs denoted by α_{11} and α_3 in [8] the dual E^* also contains the blocks

$$\{2,3,10,12,14,15\}, \quad \{2,7,8,11,13,14\}, \quad \{4,5,9,11,12,14\}.$$

We can attach signs to these in accordance with the previous 4 blocks and (4.5)(ii), finding the three pairs of possible blocks

$$\{2,-3,10,-12,\pm 14,\mp 15\}, \quad \{2,-8,11,\pm 7,\mp 13,\pm 14\},$$
$$\{4,-9,14,\pm 5,\mp 11,\pm 12\},$$

where exactly one block from each pair must be in D_1. But the possible choices of blocks are found to be inconsistent with (4.5)(ii). We sketch a proof. Assume $P=\{2,-3,10,-12,14,-15\}\in D_1$. Then $P\cap\{2,-8,11,7,-13,14\}=\{2,14\}$, so $Q=\{2,-8,11,-7,13,-14\}\in D_1$. But neither of the two possible blocks $\{4,-9,14,\pm 5,\mp 11,\pm 12\}$ has intersection of size 1 with both P and Q. Hence $P\notin D_1$, and $P'=\{2,-3,10,-12,-14,15\}\in D_1$. In a similar manner this leads to a contradiction.

For the design denoted by β_{21} in [8] we have, in the dual, the blocks

$$\{2,7,8,10,12,14\}, \quad \{2,7,9,11,13,15\}, \quad \{3,6,8,11,13,14\}$$

which give the possible block pairs

$$\{2,-8,10,\pm 7,\mp 12,\pm 14\}, \quad \{2,-9,11,\pm 7,\mp 13,\pm 15\},$$
$$\{3,-8,13,\pm 6,\mp 11,\pm 14\}.$$

As above, the possible block pairs are mutually inconsistent.

As we have now shown that the C^* of Lemma 2.2 does not exist, the proof of the main Theorem 1.1 is complete.

As a corollary to our theorem, we resolve a question of Bose and Shrikhande [1] (see that paper for definitions).

Corollary 4.7. *A regular two component pairwise balanced design with parameters* $[35, \binom{7;5}{3;4}, 1]$ *does not exist.*

Such a configuration would give rise to a nontrivial (36, 12, 48)-code. This corollary has also been proven by DeWitte [2], using methods similar to those of Bose and Shrikhande [1]. The proof of the main theorem due to McCarthy, Mullin, and Vanstone [6, 7] relies on DeWitte's proof of the corollary.

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