Incomplete sums of multiplicative functions. I

KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
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MATHEMATICS

INCOMPLETE SUMS OF MULTIPLICATIVE FUNCTIONS. I

BY

N. G. DE BRUIJN AND J. H. VAN LINT

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1. *Introduction*

Let λ be a multiplicative function (i.e. a function defined on the positive integers and satisfying $\lambda(1)=1$, $\lambda(mn)=\lambda(m)\,\lambda(n)$ whenever m and n are relatively prime). Let $P(n)$ denote the largest prime factor of n. We shall be interested in

$$(1.1)\qquad \Lambda(x, y) = \sum_{n \leqslant x,\, P(n) \leqslant y} \lambda(n).$$

The case $\lambda(n) \equiv 1$ is particularly important; in that case we write $\Psi(x, y)$ instead of $\Lambda(x, y)$. This $\Psi(x, y)$ was studied by several authors (see [1], also for further references). A main result is that if u is fixed, $u>0$, then we have

$$(1.2)\qquad \Psi(y^u, y) \sim y^u \varrho(u) \qquad (y \to \infty),$$

where $\varrho(u)$ depends on u only. It is defined by $\varrho(u)=1$ $(0 \leqslant u \leqslant 1)$, and by the differential-difference equation $u\varrho'(u) = -\varrho(u-1)$ $(u>1)$.

If $\lambda(n)$ depends monotonically on n it is possible to apply (1.2) to (1.1) by the method of summation by parts. Actually it was shown by J. H. VAN LINT and H. E. RICHERT [2] that with $\lambda(n) \equiv n^{-1}$ we have

$$(1.3)\qquad \Lambda(y^u, y) \sim d(u) \log y \qquad (y \to \infty),$$

and even

$$(1.4)\qquad \Lambda(y^u, y) = d(u) \log y + O(1)$$

uniformly for $0<u<\log y$, where $d(u) = \int_0^u \varrho(v)\,dv$.

In the present paper we shall not assume a condition on $\lambda(n)$ as strong as monotonicity, and still we obtain results of the type (1.3).

Our main assumptions will be that $\lambda(n) \geqslant 0$ for all n, and that there is a constant $b \geqslant 0$ such that for all $u \geqslant 1$

$$(1.5)\qquad \sum_{y<p\leqslant y^u} \lambda(p) \to b \log u \qquad (y \to \infty).$$

(Except for section 2, the letter p is reserved for prime numbers.) We shall explain two entirely different methods for the study of $\Lambda(x, y)$. The one in Part I of this paper depends on Karamata's method, i.e. applying polynomial approximation to a step-function in order to derive asymptotic information about a sequence from the behaviour of certain moments. The method will seem rather artifical since we do not try to give any heuristic derivation of the special functions θ_b. That will be much clearer in Part II, where the differential-difference equation for that function θ_b arises quite naturally from functional equations satisfied by the sum $\Lambda(y^u, y)$. The results of Part II will contain generalizations that we cannot attack by the method of Part I, but on the other hand we cannot avoid to make extra assumptions on the sum $\sum_{n\leqslant y}\lambda(n)$. In Part I, these need not be postulated, but it will be proved that they follow from (1.5).

The main results of Part I are: If $b\geqslant 0$, $\lambda(n)\geqslant 0$, if (1.5) holds and if the (very weak) conditions (4.2) and (4.3) are satisfied, then we have

$$\Lambda(y^u, y) \sim \theta_b(u) \sum_{n\leqslant y} \lambda(n) \qquad (y\to\infty) \tag{1.6}$$

(uniformly for $0<\delta\leqslant u<\infty$). Here θ_b is the function explained in sec. 2; note that $\theta_1(u)$ is identical to $d(u)$ of (1.4). Furthermore, if we put

$$\sum_{n\leqslant y} \lambda(n) = (\log y)^b\, L\,(\log y), \tag{1.7}$$

then L is a slowly oscillating function (i.e. $L(ct)/L(t)\to 1$ for each fixed $c>0$, if $t\to\infty$). And finally we mention (3.7), i.e.

$$\prod_{p\leqslant y}(1+\lambda(p)+\lambda(p^2)+\ldots) \sim \Gamma(1+b)\, e^{b\gamma} \sum_{n\leqslant y}\lambda(n) \tag{1.8}$$

where γ is Euler's constant. This formula includes well-known formulas as

$$\prod_{p\leqslant y}(1-p^{-1}) \sim e^{\gamma}\log y,$$

$$\prod_{p\leqslant y}(1+p^{-1}) \sim e^{\gamma}\cdot 6\pi^{-2}\log y$$

and produces results like

$$\prod_{p\leqslant y}(1+2p^{-1}) \sim 2e^{2\gamma}\sum_{n\leqslant y}{}^{*}\, d(n)n^{-1},$$

where $d(n)$ stands for the number of divisors of n, and the asterisk indicates that n takes squarefree values only.

2. *The functions θ_b*

Let b be a parameter, and assume $b\geqslant 0$. Let θ_b be the function defined for all real u, and satisfying the following conditions:

(i) $\theta_b(u)$ is continuous for $u\geqslant 0$,

(ii) $\theta_b(u)=0$ for $u<0$,

(iii) $\theta_b(u)=u^b$ for $0\leqslant u\leqslant 1$,

(iv) $u\theta_b'(u)=b\theta_b(u)-b\theta_b(u-1)$ for $u>1$.

Obviously θ_b is uniquely determined by these conditions: since we know $\theta_b(u-1)$ for $1<u\leqslant 2$ as well as $\theta_b(1)$, we obtain $u^{-b}\,\theta_b(u)$ on the interval $1<u\leqslant 2$ by a single integration; in the next step we can deal with the interval $2<u\leqslant 3$, and so on.

If $b=0$ we obtain $\theta_b(u)=1$ for all $u>0$. If $b>0$, we can show that θ_b is a strictly increasing function of u: Assuming there is a value $v\geqslant 1$ such that $\theta_b(v)=\theta_b(v-1)$, we also have a minimal $v_0\geqslant 1$ such that $\theta_b(v_0)=$ $=\theta_b(v_0-1)$. By the mean value theorem we have a $v_1(v_0-1<v_1<v_0)$ with $\theta_b'(v_1)=0$ (it should be noted that θ_b is also differentiable at $u=1$, with $\theta_b'(1)=b$). As $\theta_b'(u)\neq 0$ for $0<u\leqslant 1$, we infer $v_1>1$. By (iv) we obtain $\theta_b(v_1)=\theta_b(v_1-1)$, and this contradicts the minimality of v_0.

From the monotonocity of θ_b it follows that $\theta_b(u)>0$ for all $u>0$, and now (iv) shows that $\theta_b'(u)/\theta_b(u)<b/u$. Therefore $\theta_b(u)\leqslant u^b$ for all $u\geqslant 0$, and it follows that $\int_0^\infty e^{-pu}\,\theta_b(u)\,du$ converges for all $p>0$. We shall evaluate this integral.

Theorem 2.1. If $b\geqslant 0$, $p>0$ we have

$$\int_0^\infty e^{-pu}\,\theta_b(u)\,du = p^{-1-b}\,\Gamma(1+b)\,\exp\{-b\int_p^\infty e^{-q}q^{-1}\,dq\}.$$

Proof. The formula is trivial if $b=0$; henceforth we assume $b>0$. Obviously $u\theta_b'(u)=b\theta_b(u)-b\theta_b(u-1)$ for all $u>0$, whence, for $p>0$,

$$\int_0^\infty e^{-pu}\,u\theta_b'(u)\,du = b\int_0^\infty e^{-pu}\,\theta_b(u)\,du - b\int_0^\infty e^{-pu}\,\theta_b(u-1)\,du =$$
$$= b(1-e^{-p})\int_0^\infty e^{-pu}\,\theta_b(u)\,du.$$

Putting $H(p)=\int_0^\infty e^{-pu}\,\theta_b'(u)\,du$, we derive that $\int_0^\infty e^{-pu}\,\theta_b(u)\,du=p^{-1}H(p)$, and

$$H'(p) = -\int_0^\infty e^{-pu}\,u\theta_b'(u)\,du = b(-1+e^{-p})p^{-1}H(p).$$

Hence $H(p)=C\,p^{-b}\exp\{-b\int_p^\infty e^{-q}q^{-1}\,dq\}$ with C not depending on p.

The asymptotic behaviour of $H(p)$ for $p\to\infty$ is determined by $\theta_b'(u)$ in the neighbourhood of $u=0$. As $\int_0^\infty e^{-pu}\,bu^{b-1}\,du=p^{-b}\,\Gamma(1+b)$ we have $H(p)\sim p^{-b}\,\Gamma(1+b)$ for $p\to\infty$. It follows that $C=\Gamma(1+b)$, and the proof is complete.

Theorem 2.2. If $b\geqslant 0$ we have $\lim_{u\to\infty}\theta_b(u)=\Gamma(1+b)\,e^{b\gamma}$.

Proof. We now consider the behaviour of $\int_0^\infty e^{-pu}\,\theta_b(u)\,du$ for $p\downarrow 0$. Using the fact that

$$-\int_p^\infty e^{-q}\,q^{-1}\,dq-\log p$$

tends to Euler's constant γ if $p\downarrow 0$, we infer that

$$p\int_0^\infty e^{-pu}\,\theta_b(u)\,du \to \Gamma(1+b)\,e^{b\gamma}$$

if $p \downarrow 0$. Knowing that θ_b is monotonic, we successively deduce from this that $\theta_b(u)$ is bounded if $u \to \infty$, that it has a limit if $u \to \infty$, and that this limit equals $\Gamma(1+b)\, e^{b\gamma}$.

Remark. Theorem 2.1 can be generalized as follows: If $b > -1$, and if c is real, if θ is continuous for $u > 0$ and satisfies $\theta(u) = u^b$ $(0 < u \leqslant 1)$ and

$$u\theta'(u) = b\theta(u) - c\theta(u-1) \qquad (u \geqslant 1),$$

then we have, for all $p > 0$,

$$\int_0^\infty e^{-pu}\, \theta(u)\, du = p^{-1-b}\, \Gamma(1+b) \exp\{-c \int_p^\infty e^{-q}\, q^{-1}\, dq\}.$$

3. *Application of Karamata's method*

Assume that

$$\lambda(n) \geqslant 0 \qquad (n = 1, 2, 3, \ldots), \tag{3.1}$$

$$\lambda \text{ is a multiplicative function,} \tag{3.2}$$

that for each prime number p

$$1 + \lambda(p) + \lambda(p^2) + \ldots < \infty, \tag{3.3}$$

and put, for $y > 0$, $s \geqslant 0$,

$$f(y; s) = \prod_{p \leqslant y} (1 + \lambda(p)p^{-s} + \lambda(p^2)p^{-2s} + \ldots).$$

So obviously

$$f(y; s) = \sum_{P(d) \leqslant y} \lambda(d) d^{-s}. \tag{3.4}$$

Theorem 3.1. If (3.1), (3.2), (3.3) hold, if $b \geqslant 0$, $A > 0$, and if for each k $(k = 0, 1, 2, \ldots)$ we have

$$\lim_{y \to \infty} \log \frac{f(y;\, k/\log y)}{f(y;\, 0)} = b \int_0^1 \frac{e^{-ku} - 1}{u}\, du \tag{3.5}$$

(where uniformity with respect to k is not required), then we have

$$\Big(\sum_{P(d) \leqslant y,\, d \leqslant y^u} \lambda(d)\Big) \Big/ \Big(\sum_{n \leqslant y} \lambda(n)\Big) \to \theta_b(u) \tag{3.6}$$

if $y \to \infty$, uniformly for $A \leqslant u < \infty$. Furthermore we have

$$f(y;\, 0) \Big/ \Big(\sum_{n \leqslant y} \lambda(n)\Big) \to \Gamma(1+b)\, e^{b\gamma} \tag{3.7}$$

if $y \to \infty$, i.e. (1.8).

Proof. We shall use the notation of Stieltjes integrals. Note that $\theta_0(u) = 0 (u < 0)$, $\theta_0(u) = 1 (u \geqslant 0)$, whence $\int_{-\infty}^{\infty} f(u)\, d\theta_0(u) = f(0)$ for every continuous function f on $(-\infty, \infty)$. If $b > 0$, then $\int_{-\infty}^{\infty} f(u)\, d\theta_b(u) = \int_0^\infty f(u)\, d\theta_b(u) = \int_0^\infty f(u)\, \theta_b'(u)\, du$ if f is continuous.

If $k=1, 2, 3, \ldots$ we have

$$\int_0^1 (e^{-ku}-1)u^{-1}\,du = \int_0^k (e^{-u}-1)u^{-1}\,du = -\gamma - \log k - \int_k^\infty e^{-u}\,u^{-1}\,du,$$

whence, by theorem 2.1,

$$\exp\left\{b\int_0^1 \frac{e^{-ku}-1}{u}\,du\right\} = \frac{e^{-b\gamma}}{\Gamma(1+b)}\int_{-\infty}^{\infty} e^{-ku}\,d\theta_b(u),$$

and this remains true if $k=0$. If $b=0$ the formula is trivial, of course.

We abbreviate

$$e^{-b\gamma} f(y, 0)/\Gamma(1+b) = Z. \tag{3.8}$$

Using (3.4) and (3.5) we infer

$$Z^{-1} \sum_{P(d)\leqslant y} \lambda(d)\, d^{-k/\log y} \to \int_{-\infty}^{\infty} e^{-kv}\,d\theta_b(v)$$

for $k=0, 1, 2, \ldots, y\to\infty$. So if π is any polynomial, we have

$$Z^{-1} \sum_{P(d)\leqslant y} \lambda(d)\, \pi(e^{-\log d/\log y}) \to \int_{-\infty}^{\infty} \pi(e^{-v})\,d\theta_b(v). \tag{3.9}$$

Let, for $u>0$, the function χ_u be defined by

$$\chi_u(v) = \begin{cases} 1 \text{ if } v\leqslant u, \\ 0 \text{ if } v>u. \end{cases}$$

If b, ε and u are given ($b\geqslant 0$, $\varepsilon>0$, $u>0$) we can find polynomials π_1, π_2 such that

$$\pi_1(e^{-v}) \leqslant \chi_u(v) \leqslant \pi_2(e^{-v}) \qquad (0\leqslant v<\infty) \tag{3.10}$$

and

$$\int_{-\infty}^{\infty} \{\pi_2(e^{-v}) - \pi_1(e^{-v})\}\,d\theta_b(v) < \varepsilon. \tag{3.11}$$

It is easy to satisfy (3.11) if $b=0$, just by requiring that $\pi_1(1)=\pi_2(1)=1$. If $b>0$, we use the fact that $\theta_b'(u)$ is bounded on $0\leqslant u<\infty$ and that we can make

$$\int_0^\infty (\pi_2(e^{-v}) - \pi_1(e^{-v}))\,dv$$

as small as we please without violating (3.10).

It is easy to show that one and the same pair π_1, π_2 can be used for an open set of u's. That is, if b, ε, w are given ($b\geqslant 0$, $\varepsilon>0$, $w>0$), we can find π_1, π_2 and $\delta_w>0$ such that (3.10) and (3.11) hold for all u satisfying $w-\delta_w<u<w+\delta_w$.

Assuming that π_1, π_2, δ_w have been chosen that way, we put $v=\log d/\log y$ in (3.10), multiply by $\lambda(d)$, and take the sum over all integers $d\geqslant 1$.

Noting that

$$\theta_b(u) = \int_{-\infty}^{\infty} \chi_u(v)\, d\theta_b(v) \leqslant \int_{-\infty}^{\infty} \pi_2(e^{-v})\, d\theta_b(v) < \theta_b(u) + \varepsilon,$$

we obtain from (3.9) that

$$\limsup_{y\to\infty} Z^{-1} \sum_{P(d)\leqslant y,\, d\leqslant y^u} \lambda(d) \leqslant \theta_b(u) + \varepsilon$$

uniformly for $w-\delta_w<u<w+\delta_w$.

A similar case of uniform convergence can be obtained for an infinite interval $[B, \infty)$. By virtue of the fact that $\int_0^\infty |\theta_b'(v)|\, dv$ converges it is easy to derive that, given $b\geqslant 0$, $\varepsilon>0$, we can find π_1, π_2, B such that (3.10) and (3.11) hold for $B\leqslant u<\infty$.

Now if $A>0$, we can cover the interval $[A, B]$ by a finite number of intervals $(w-\delta_w, w+\delta_w)$. Moreover we can do for the lim inf what we did for the lim sup. It results that

$$Z^{-1} \sum_{P(d)\leqslant y,\, d\leqslant y^u} \lambda(d) \to \theta_b(u) \tag{3.12}$$

if $y\to\infty$, uniformly for $A\leqslant u<\infty$.

If we specialize by taking $u=1$, and note that $\theta_b(1)=1$, we obtain

$$\sum_{n\leqslant y} \lambda(n) \sim Z \tag{3.13}$$

since every $n\leqslant y$ satisfies $P(n)\leqslant y$. From (3.12), (3.13), (3.8) the theorem follows at once.

Remark. We can interpret (3.6) also for $u=\infty$, and then it is nothing but (3.7). To this end we have to read $\theta_b(\infty)$ as $\lim_{u\to\infty}\theta_b(u)$, and to note that $d\leqslant y^u$ does not mean any restriction on d at all.

Finally $\sum_{P(d)\leqslant y} \lambda(d) = f(y;\, 0)$ according to (3.4).

Theorem 3.2. If the conditions of theorem 3.1 hold, then

$$\sum_{n\leqslant y} \lambda(n) = (\log y)^b\, L(\log y) \qquad (y>1), \tag{3.14}$$

where L is a slowly oscillating function, i.e.

$$L(ux)/L(x) \to 1 \qquad (x\to\infty)$$

uniformly on $A_1\leqslant u\leqslant A_2$, for every pair A_1, A_2 with $0<A_1<A_2$.

Proof. We take $0<A<1$, and we use that (3.6) holds uniformly on $A\leqslant u\leqslant 1$. As $\theta_b(u)=u^b$ for $0\leqslant u\leqslant 1$, and as for $0\leqslant u\leqslant 1$ the condition $P(d)\leqslant y$ is a mere consequence of $d\leqslant y^u$, (3.6) is translated into

$$\Big(\sum_{n\leqslant y^u} \lambda(n)\Big)\Big/\Big(\sum_{n\leqslant y} \lambda(n)\Big) \to u^b,$$

i.e. $L(u\log y)/L(\log y)\to 1$, uniformly on $A\leqslant u\leqslant 1$. So $L(ux)/L(x)\to 1$, uniformly on $A \leqslant u \leqslant 1$. It is a trivial consequence of this, that $L(vx)/L(x) \to 1$ uniformly on $1 \leqslant v \leqslant A^{-1}$.

4. *Sufficient conditions for* λ

It is our aim to replace condition (3.5) by a simpler set of conditions. We shall introduce one essential condition (4.1) and two extra conditions (4.2) and (4.3). The latter two are relatively weak and are quite easily replaced by still weaker ones, but we shall not take that trouble.

The essential condition is that there is a constant $b \geqslant 0$ and a constant $A > 1$ such that

$$\sum_{y < p \leqslant y^u} \lambda(p) \to b \log u \qquad (y \to \infty,\ 1 \leqslant u \leqslant A). \tag{4.1}$$

If this condition holds, and if $\lambda(p) \geqslant 0$ for all p, we can show that (4.1) holds uniformly on $1 \leqslant u \leqslant A$ for every finite A. The uniformity follows from the fact that the left-hand side of (4.1) depends monotonically on u, whereas the right-hand side is both monotonic and continuous. The possibility of extending (4.1) to arbitrary large finite u-intervals is given by the fact that if (4.1) holds for $u = a$, then it holds for $u = a^2$, since the interval $y < p \leqslant (y^a)^a$ splits into $y < p \leqslant y^a$ and $(y^a) < p \leqslant (y^a)^a$.

We shall also assume

$$\sum_p (\lambda(p))^2 < \infty \tag{4.2}$$

where the sum runs over all primes. Furthermore, if we put

$$2\lambda(p^2) + 3\lambda(p^3) + 4\lambda(p^4) + \ldots = \sigma(p),$$

we assume

$$\sum_p \sigma(p) < \infty. \tag{4.3}$$

Theorem 4.1. If (3.1), (4.2) and (4.3) hold, then we have, with the notation of theorem 3.1,

$$\lim_{y \to \infty} \left\{ \log \frac{f(y;\, s/\log y)}{f(y;\, 0)} - \sum_{p < y} \lambda(p)\, (p^{-s/\log y} - 1) \right\} = 0$$

for every $s \geqslant 0$.

Proof. Our assumptions imply $\lambda(p) = O(1)$, $\sigma(p) = O(1)$, and

$$\sum_{p < y} (\lambda(p) + \sigma(p))^2 \log p = o(\log y), \tag{4.4}$$

$$\sum_{p < y} \sigma(p) \log p = o(\log y). \tag{4.5}$$

Abbreviating $\lambda(p^\nu) = \lambda_\nu$, $2\lambda_2 + 3\lambda_3 + \ldots = \sigma$, $p^{-s/\log y} = w$, we have

$$(1 - w^2)\lambda_2 + (1 - w^3)\lambda_3 + \ldots \leqslant (1 - w)\sigma,$$

whence

$$\log \frac{1+\lambda_1 w+\lambda_2 w^2+\ldots}{1+\lambda_1+\lambda_2+\ldots} = \log\left\{1-\frac{(1-w)\lambda_1+(1-w^2)\lambda_2+\ldots}{1+\lambda_1+\lambda_2+\ldots}\right\} =$$

$$= -\frac{(1-w)\lambda_1+(1-w^2)\lambda_2+\ldots}{1+\lambda_1+\lambda_2+\ldots} + O\{(1-w)(\lambda_1+\sigma)^2\} =$$

$$= -\frac{(1-w)\lambda_1}{1+\lambda_1+\lambda_2+\ldots} + O\{(1-w)\sigma\} + O\{(1-w)(\lambda_1+\sigma)^2\} =$$

$$= -(1-w)\lambda_1 + O\{(1-w)\lambda_1\sigma\} + O\{(1-w)\sigma\} + O\{(1-w)(\lambda_1+\sigma)^2\}.$$

Taking the sum over all $p<y$, using $1-w=1-p^{-s/\log y}=O((\log p)(\log y)^{-1})$, and applying (4.4), (4.5), the theorem follows.

Theorem 4.2. If (3.1) and (4.1) hold (with a constant $A>1$), and if $s \geqslant 0$, then

$$\lim_{y\to\infty} \sum_{p<y} \lambda(p)\,(p^{-s/\log y}-1) = b\int_0^1 (e^{-su}-1)u^{-1}\,du.$$

Proof. Putting

$$S_u = \sum_{y^u<p\leqslant y} \lambda(p) \qquad (0 \leqslant u \leqslant 1),$$

we can write our sum as a Stieltjes integral:

$$\sum_{p<y} \lambda(p)\,(p^{-s/\log y}-1) = -\int_0^{1+} (e^{-su}-1)\,d\,S_u.$$

As also

$$b\int_0^1 (e^{-su}-1)\,u^{-1}\,du = -\int_0^{1+} (e^{-su}-1)\,d\,(b\log u^{-1}),$$

it remains to show that

$$\lim_{y\to\infty} \int_0^{1+} (e^{-su}-1)\,d\,\{S_u - b\log u^{-1}\} = 0. \tag{4.6}$$

Noting that $e^{-su}-1=0$ at $u=0$, $S_u=\log u^{-1}=0$ at $u=1$, we integrate by parts, and we obtain for the integral

$$s\int_0^{1+} e^{-su}\,(S_u - b\log u^{-1})\,du. \tag{4.7}$$

For every separate u $(0<u\leqslant 1)$ we have $S_u - b\log u^{-1} \to 0$ as $y\to\infty$, by virtue of (4.1). So in order to establish that (4.7) tends to 0 we can apply Lebesgue's theorem on dominated convergence: it suffices to remark that

$$|S_u - b\log u^{-1}| < C(1+\log u^{-1}) \qquad (0<u\leqslant 1) \tag{4.8}$$

with C not depending on y. This can be derived as follows: By (4.1) there is a constant $C_1>0$ such that $\sum_{y<p\leqslant y^2}\lambda(p)<C_1$ for all y; hence

$$\sum_{y<p\leqslant y^4}\lambda(p)<2C_1,\quad \sum_{y<p\leqslant y^8}\lambda(p)<3C_1,\ \text{etc.}$$

It follows that for all $v\geqslant 1$, $y\geqslant 1$,

$$\sum_{y<p\leqslant y^v}\lambda(p)<C_1+C_1\,(\log v)/(\log 2).$$

Taking $v=u^{-1}$, (4.8) follows with $C=|b|+C_1/(\log 2)$. This completes the proof of the theorem.

5. *Applications*

We can, of course, combine theorems 3.1, 3.2, 4.1, 4.2:

Theorem 5.1. Let $b\geqslant 0$, and let λ be a non-negative multiplicative function, satisfying (4.1) (with some $A>1$), (4.2) and (4.3). Then we have (3.6), uniformly on every interval $\delta\leqslant u<\infty$, (if $\delta>0$), and moreover $(\log y)^{-b}\sum_{n\leqslant y}\lambda(n)$ is a slowly oscillating function of $\log y$. The limit case $u\to\infty$ (see (3.7)) gives (1.8).

Examples. 1. If λ satisfies $\lambda(p^2)=\lambda(p^3)=\ldots=0$ and $\lambda(p)\geqslant 0$ for all p, and if $p\lambda(p)\to 0$ when p runs through the primes, then the conditions of theorem 5.1 are satisfied with $b=0$. Therefore $\sum_{n\leqslant y}\lambda(n)$ is a slowly oscillating function of $\log y$.

2. Let φ be Euler's function, and $\lambda(n)=(\varphi(n))^{-1}$ if n is squarefree, $\lambda(n)=0$ otherwise. We have (4.1) with $b=1$, so by theorem 5.1

$$\sum^*_{P(d)\leqslant y,\,d\leqslant y^u}(\varphi(d))^{-1}\sim\theta_1(u)\sum^*_{n\leqslant y}(\varphi(n))^{-1}\tag{5.1}$$

for $0<u<\infty$ (the asterisk indicates that the summation index takes squarefree values only), and

$$\prod_{p\leqslant y}(1-p^{-1})^{-1}\sim e^{\gamma}\sum^*_{n\leqslant y}(\varphi(n))^{-1}.\tag{5.2}$$

The left-hand side of (5.2) is asymptotically $e^{\gamma}\log y$, so it follows that

$$(\log y)^{-1}\sum^*_{P(d)\leqslant y,\,d\leqslant y^u}(\varphi(d))^{-1}\to\theta_1(u).\tag{5.3}$$

Note that, for every $A>0$, (5.3) holds uniformly for $A\leqslant u<\infty$.

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KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
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N. G. DE BRUIJN AND J. H. VAN LINT

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1. *Introduction*

In part I we discussed the sum

$$\Lambda(x, y) = \sum_{n \leqslant x, P(n) \leqslant y} \lambda(n) \tag{1.1}$$

where $\lambda(n)$ is a multiplicative function and $P(n)$ denotes the largest prime factor of n. Our main assumptions were that $\lambda(n) \geqslant 0$ for all n and that for some $b \geqslant 0$ and all $u \geqslant 1$ we have (1.5) (see sec. 1.1 below). With some additional (very weak) conditions for $\lambda(n)$ we proved that

$$\Lambda(y^u, y) \sim \theta_b(u) \sum_{n \leqslant y} \lambda(n) \qquad (y \to \infty) \tag{1.2}$$

uniformly for $u \geqslant \delta > 0$. For the definition of $\theta_b(u)$, properties and further background material we refer to part I. We also proved that

$$\sum_{n \leqslant y} \lambda(n) = (\log y)^b \, L(\log y), \tag{1.3}$$

where L is a slowly oscillating continuous function.

In this paper we use a different method to discuss

$$\Lambda_a(x, y) = \sum_{n \leqslant x, P(n) \leqslant y} \lambda(n)\, n^a \tag{1.4}$$

with $a > 0$. Although it is exceptional in some respects, we might include $a = 0$ in our present discussion, but we shall not do this, because it would not produce results as strong as those obtained in part I. In this part II we shall not obtain a result of the type (1.3), but we shall take a formula of that type as one of our assumptions. (See **C** below.) Moreover we have to exclude the case $b = 0$.

As in part I we must impose some rather light conditions on λ in order to guarantee that prime powers p^i $(i > 1)$ have little influence. (See under **D** and **E**).

Finally we need an extra restriction on λ in the case that $0 < b \leqslant 1$ (see **E**).

1.1 Assumptions

A. The function λ is multiplicative (i.e. $\lambda(nm) = \lambda(n)\lambda(m)$ if m and n are co-prime positive integers), and $\lambda(n) \geqslant 0$ for all n.

The function L is continuous on $0<x<\infty$, and slowly oscillating. That is, $L(x)>0$ for all $x>0$, and for each fixed $q>0$ we have $L(qx)/L(x)\to 1$ if $x\to\infty$. It is a well-known consequence that this holds uniformly with respect to q in every interval $\delta\leqslant q\leqslant M$, provided that $0<\delta<M<\infty$ (see [4], [5]).

The numbers a and b satisfy $a>0$, $b>0$.

Throughout the paper, λ, L, a, b, are fixed. That is, numbers depending only on λ, L, a, b, are called constants, and none of our statements is intended to hold uniformly with respect to λ, L, a, b.

B. For every fixed $u>1$ we have

$$\lim_{y\to\infty}\sum_{y<p\leqslant y^u}\lambda(p)=b\log u, \tag{1.5}$$

where p runs through the primes.

C. For $y\to\infty$ we have

$$\sum_{n\leqslant y}\lambda(n)\,n^a\sim a^{-1}b\,y^a(\log y)^{b-1}L(\log y). \tag{1.6}$$

D. For every fixed $i\geqslant 2$ and every fixed $u>1$ we have

$$\lim_{y\to\infty}\sum_{y<p\leqslant y^u}\lambda(p^i)=0. \tag{1.7}$$

E. If $0<b\leqslant 1$ the following condition holds: for every i ($i=1, 2, 3, \ldots$) there is a constant C_i such that

$$\sum_{y<p\leqslant 2y}\lambda(p^i)<C_i/(\log y)\qquad(2<y<\infty). \tag{1.8}$$

1.2 Notations

For A_a see (1.4), for $P(n)$ see (1.1), for $\eta(u)$ see sec. 2. $\Phi(y, u)$ is an abbreviation:

$$\Phi(y,u)=y^{au}(\log y)^{b-1}L(\log y). \tag{1.9}$$

A phrase like $C=C(\delta)$ means: C may depend on δ, on λ, L, a, b, but not on any other parameters or functions.

If p is used as a summation index it is assumed to run through prime numbers only.

1.3 The main theorem

Let **A**, **B**, **C**, **D**, **E** hold. Let δ and M be constants, $0<\delta<M$. Then we have, if $y\to\infty$,

$$A_a(y^u,y)\sim a^{-1}b\,\eta(u)\,y^{au}(\log y)^{b-1}L(\log y), \tag{1.10}$$

uniformly for $\delta\leqslant u\leqslant M$ (for η see sec. 2).

1.4 Remarks

In the case $a=0$, $b \geqslant 0$ (part I) we had a similar, though simpler, result, viz.

$$(1.11) \qquad \Lambda(y^u, y) \sim \theta_b(u)\,(\log y)^b\, L(\log y).$$

Note that $\eta(u)=b^{-1}\,\theta_b{}'(u)$ (see sec. 2).

The constant $a^{-1}b$ in (1.6) and (1.10) is irrelevant, of course, since $a^{-1}b\,L$ is also a slowly oscillating function. We only introduced this factor in order to keep (1.6) in harmony with (1.3), as (1.3) can be obtained from (1.6) by a process of summation by parts.

In assumption **E** we require (1.8) only if $0<b\leqslant 1$. If $b>1$ we do not need this extra condition. It is not difficult to see from our proofs that (1.8) is not needed either if $b=1$, $L\equiv 1$, but we did not stress this fact in the form of a theorem.

2. *The function* η

For $u>0$ the function η is uniquely defined by the following set of conditions:

(i) $\eta(u)$ is continuous for $u>0$,
(ii) $\eta(u)=u^{b-1}$ for $0<u\leqslant 1$,
(iii) $u\eta'(u)=(b-1)\eta(u)-b\eta(u-1)$ for $u>1$.

This differential-difference equation can be written in the following integral form. If $\alpha\geqslant 1$, we have for $u\geqslant 1$

$$(2.1) \qquad \eta(u) = (u/\alpha)^{b-1}\,\eta(\alpha)-b\int_1^{u/\alpha}\eta(ux^{-1}-1)\,x^{b-2}\,dx.$$

The equivalence of (iii) and (2.1) is easily verified if we write (2.1) in the form

$$\int_\alpha^u \{(v^{-b}\,\eta(v))'+b\,v^{-b}\,\eta(v-1)\}\,dv = 0.$$

It is in the form (2.1) that the equation for η will arise in a natural way in our proof.

It is not difficult to derive from (i), (ii), (iii) that $\eta(u)=b^{-1}\,\theta_b{}'(u)$ if $u>0$, where θ_b is the function occurring in (1.2) (it is characterized by $\theta_b(u)=u^b$ $(0\leqslant u\leqslant 1)$, $u\,\theta_b{}'(u)=b\,\theta_b(u)-b\,\theta_b(u-1)$ $(u>1)$, θ_b continuous for $u\geqslant 0$).

3. *The functional equation for* $\Lambda_a(x, y)$

If $v>1$, $y>1$, then we have by (1.4),

$$(3.1) \qquad \Lambda_a(y^u, y^v)-\Lambda_a(y^u, y) = \sum_{n\leqslant y^u}{}'\,\lambda(n)\,n^a,$$

where the dash indicates that only those n are admitted whose largest prime factor p satisfies $y<p\leqslant y^v$. For such a prime factor we have $p^u>y^u$, whence p^i does not divide n if $i \geqslant u$. Therefore the right-hand side of (3.1) equals

$$\sum_{y<p\leqslant y^v} \sum_{1\leqslant i<u} \lambda(p^i)\, p^{ai} \sum_{m\leqslant y^u p^{-i},\, P(m)<p} \lambda(m)\, m^a,$$

whence

$$(3.2)\quad \Lambda_a(y^u, y^v) - \Lambda_a(y^u, y) = \sum_{1\leqslant i<u} \sum_{y<p\leqslant y^v} \lambda(p^i)\, p^{ai}\, \Lambda_a(y^u p^{-i}, p-1)$$

for all u, y, v with $u>0$, $y>1$, $v>1$.

In our proof of the main theorem it will turn out that the terms with $i>1$ are negligible.

4. *Some lemmas*

Our first lemma deals with uniform Riemann integrability. We consider a function $f_u(x)$ defined for $\xi \leqslant x \leqslant \eta$, depending on the parameter u $(\alpha \leqslant u \leqslant \beta)$. If we have a dissection of the interval $[\xi, \eta]$, given by

$$(4.1)\qquad \xi = x_0 < x_1 < \ldots < x_n = \eta,$$

then we define the lower step-function s_{1u} for $\xi < x \leqslant \eta$ by

$$s_{1u}(x) = \inf_{x_{i-1}<y\leqslant x_i} f_u(y) \qquad (x_{i-1} < x \leqslant x_i),$$

and the upper step-function s_{2u} similarly, with sup instead of inf.

We shall say that f_u is uniformly Riemann integrable over $\xi \leqslant x \leqslant \eta$ for $\alpha \leqslant u \leqslant \beta$, if $f_u(x)$ is bounded on that rectangle, and if, moreover, for every $\varepsilon>0$ there is a $\delta>0$ such that for every dissection of $[\xi, \eta]$ with maximal interval length less than δ and for all u in $[\alpha, \beta]$ we have

$$\int_\xi^\eta (s_{2u}(x) - s_{1u}(x))\, dx < \varepsilon.$$

The latter formula implies that the so-called upper and lower sums differ less than ε from the integral of f_u, uniformly with respect to u.

Lemma 1. Assume $0<\xi<\eta$, $0<\alpha<\beta$, $b \geqslant 0$. Let $\lambda(p)$ be defined and $\geqslant 0$ for all primes, and assume **B**. Let $f_u(x)$ be Riemann integrable over $[\xi, \eta]$, uniformly with respect to u $(\alpha \leqslant u \leqslant \beta)$. Put

$$\sum_{y^\xi<p\leqslant y^\eta} \lambda(p)\, f_u\left(\frac{\log p}{\log y}\right) = S[f_u].$$

Then we have

$$(4.2)\qquad \lim_{y\to\infty} S[f_u] = b \int_\xi^\eta f_u(x)\, x^{-1}\, dx,$$

uniformly with respect to u $(\alpha \leqslant u \leqslant \beta)$.

Proof. If the dissection (4.1) is fixed (not depending on u or y), we easily derive from **B** that for $y \to \infty$

$$\lim_{y\to\infty} S[s_{1u}] = b \int_{\xi}^{\eta} s_{1u}(x)\, x^{-1}\, dx, \tag{4.3}$$

uniformly with respect to u, since s_{1u} is uniformly bounded. Needless to say, we have a similar result for s_{2u}.

Let $\varepsilon > 0$ be given. By virtue of the uniform integrability we can take the dissection (4.1) such that

$$b \int_{\xi}^{\eta} (s_{2u}(x) - s_{1u}(x))\, x^{-1}\, dx < \tfrac{1}{2}\varepsilon$$

for all u simultaneously ($\alpha \leqslant u \leqslant \beta$). (Note that the factor x^{-1} is at most ξ^{-1}.) Next take y_0 such that for all $y > y_0$ the difference between $S[s_{1u}]$ and the right-hand side of (4.3) is less than $\varepsilon/4$, for all u simultaneously, and such that the analogous statement is true for the upper sum s_{2u}. As $\lambda(p) \geqslant 0$ for all p, we have

$$S[s_{1u}] \leqslant S[f_u] \leqslant S[s_{2u}],$$

and it follows that

$$\left| S[f_u] - b \int_{\xi}^{\eta} f_u(x)\, x^{-1}\, dx \right| < \varepsilon,$$

uniformly for $\alpha \leqslant u \leqslant \beta$. This proves the lemma.

Lemma 2. Assume **A**, **B**, **E**. Let the number β satisfy $0 < \beta < b$ if $0 < b \leqslant 1$, and $1 < \beta < b$ if $b > 1$. Put $\gamma = \beta$ in the first case, $\gamma = \beta - 1$ in the second case. (So always $\gamma > 0$.) Then there is a positive constant $C = C(\beta)$ such that for all $y > 1$ and for all ε $(0 < \varepsilon \leqslant \frac{1}{2})$ we have

$$\sum_{y^{1-\varepsilon} < p \leqslant \frac{1}{2}y} \lambda(p) \left(\frac{\log (y/p)}{\log y} \right)^{\beta - 1} < C\, \varepsilon^{\gamma}. \tag{4.4}$$

Proof. a) If $1 < \beta < b$, the terms are at most $\lambda(p)\, \varepsilon^{\beta-1}$, so for $0 < \varepsilon \leqslant \frac{1}{2}$ (4.4) follows from the fact that

$$\sum_{y^{\frac{1}{2}} < p \leqslant y} \lambda(p)$$

is bounded (by **B** it has a finite limit).

b) Assume $0 < \beta < b \leqslant 1$, $y > 1$, $0 < \varepsilon \leqslant \frac{1}{2}$, and let N be the smallest integer such that $2^N \geqslant y^{\varepsilon}$. We enlarge the sum in (4.4) by replacing the interval $y^{1-\varepsilon} < p \leqslant \frac{1}{2}y$ by $2^{-N}y < p \leqslant \frac{1}{2}y$. Next we split this one into the intervals $2^{-k}y < p \leqslant 2^{-k+1}y$ $(k = 2, \ldots, N)$. On each one of these intervals we have

$$(\log (y/p))^{\beta-1} < (k \log 2)^{\beta-1},$$

whence, by **E**,

$$\sum_{y^{1-\varepsilon} < p \leqslant \frac{1}{2}y} \lambda(p)\, (\log (y/p))^{\beta-1} < \sum_{k=2}^{N} (k \log 2)^{\beta-1} C_1 / (\log (2^{-N} y)). \tag{4.5}$$

If y is large enough we have $2^{-N}y > y^{\frac{1}{2}}$. As $\beta > 0$ we have $\sum_{k=2}^{N} k^{\beta-1} = O(N^{\beta})$. Finally $(N-1)\log 2 < \varepsilon \log y$, by the definition of N. It follows that the right-hand side of (4.5) is less than a constant times $\varepsilon^{\beta}(\log y)^{\beta-1}$, and (4.4) follows.

Lemma 3. Let L be a continuous slowly oscillating function defined for $x \geqslant \frac{1}{2}$. Then for any $\delta > 0$ there exists a positive number $C = C(\delta)$ such that for all x_1, x_2 with $\frac{1}{2} \leqslant x_1 \leqslant x_2$ we have

$$|L(x_1)/L(x_2)| < C(\delta)\,(x_2/x_1)^{\delta}. \tag{4.6}$$

For a proof we refer to [5], [6].

Our main theorem will be proved in sec. 5 by induction. The first step of this induction is the following lemma.

Lemma 4. Assume **A**, **B**, **C**, **E**. Let M be any number > 1. Then as $y \to \infty$ we have, uniformly for $1 < u \leqslant M$,

$$\left\{\begin{aligned} &\sum_{y < p \leqslant y^u} \lambda(p)\,p^a \sum_{n \leqslant y^u/p} \lambda(n)\,n^a = \\ &\qquad = \Phi(y, u)\,\{a^{-1} b^2 \int_1^u (u-x)^{b-1} x^{-1}\,dx + o(1)\}. \end{aligned}\right. \tag{4.7}$$

Proof. We fix a number β satisfying the conditions mentioned in lemma 2, and we take $\delta = b - \beta$, so $\delta > 0$. With this δ we apply lemma 3. If $2 \leqslant y^u/p < y$ we can take $x_1 = y^u/p$, $x_2 = y$, whence

$$\left(\frac{\log(y^u/p)}{\log y}\right)^{b-1} \frac{L(\log(y^u/p))}{L(\log y)} < C(\delta)\left(\frac{\log(y^u/p)}{\log y}\right)^{\beta-1}. \tag{4.8}$$

It follows by **C** that if $2 \leqslant y^u/p < y$, we have the following rough estimate: there is a constant C with

$$\sum_{n \leqslant y^u/p} \lambda(n)\,n^a < C\,p^{-a}\,\Phi(y, u)\left(\frac{\log(y^u/p)}{\log y}\right)^{\beta-1}. \tag{4.9}$$

If $1 \leqslant y^u/p < 2$ this estimate is not efficient; in that case we just use that the left-hand side of (4.9) equals unity.

The total contribution to the left-hand side of (4.7) produced by those p for which both $y < p \leqslant y^u$ and $1 \leqslant y^u/p < 2$ hold, is relatively small. This contribution is at most

$$\sum_{\frac{1}{2}y^u < p \leqslant y^u} \lambda(p)\,p^a,$$

and by **E** this is less than $C_1\,y^{au}/\log y$ if $u \geqslant 1$, $y > 2$. By lemma 3 we have $(L(\log y))^{-1} = o((\log y)^b)$, since b is positive. It follows that the contribution of the p with $y < p \leqslant y^u$, $1 \leqslant y^u/p < 2$ is $o(\Phi(y, u))$, uniformly with respect to u.

Next choose an ε, $0 < \varepsilon < M^{-1}$, and consider the total contribution of those p for which both $y<p\leqslant y^u$ and $y^{(1-\varepsilon)u}<p\leqslant\frac{1}{2}y^u$ hold. For these terms we use (4.9), producing at most

$$C\Phi(y,u)\sum_{y^{(1-\varepsilon)u}<p\leqslant\frac{1}{2}y^u}\lambda(p)\left(\frac{\log(y^u/p)}{\log y}\right)^{\beta-1},$$

and this is at most $C\,\varepsilon^\gamma\,\Phi(y,u)$ according to lemma 2, with a new constant $C=C(\beta)$.

Finally we take the terms for which simultaneously

$$y<p\leqslant y^u,\quad p\leqslant y^{(1-\varepsilon)u}. \tag{4.10}$$

We remark that C now gives

$$\sum_{n\leqslant y^u/p}\lambda(n)\,n^a\sim a^{-1}\,b\,y^{au}\,p^{-a}\,(\log(y^u/p))^{b-1}\,L(\log y), \tag{4.11}$$

if $y\to\infty$, uniformly with respect to p and u (p restricted by (4.10), u by $1<u\leqslant M$). Note that $L(\log y)\sim L(\log(y^u/p))$, since (by (4.10) and $1<u\leqslant M$)

$$\varepsilon\log y<\log(y^u/p)<M\log y.$$

It does not do any harm to replace in (4.7) the expression on the left-hand side of (4.11) by the one on the right-hand side of (4.11). We then obtain as the contribution of the terms restricted by (4.10):

$$a^{-1}\,b\,\Phi(y,u)\sum_{y<p\leqslant y^M}\lambda(p)\,f_u\left(\frac{\log p}{\log y}\right), \tag{4.12}$$

where $f_u(x)$ is defined for $1\leqslant x\leqslant M$, $1\leqslant u\leqslant M$ by

$$f_u(x)=(u-x)^{b-1}\qquad\text{if } 1<x\leqslant(1-\varepsilon)\,u,$$
$$f_u(x)=0\qquad\text{if } \quad x>(1-\varepsilon)\,u.$$

(Note that for $1<u<(1-\varepsilon)^{-1}$ we have $f_u(x)=0$ for all x, and, accordingly, the sum (4.12) is empty in that case.)

Now lemma 1 provides the asymptotic behaviour of (4.12). It results that the left-hand side of (4.7) is

$$\Phi(y,u)\,\{a^{-1}\,b^2\int_1^M f_u(x)\,x^{-1}\,dx+R\}, \tag{4.13}$$

where $\limsup_{y\to\infty}|R|\leqslant C\,\varepsilon^\gamma$, uniformly with respect to u ($1<u\leqslant M$). As finally

$$\lim_{\varepsilon\to 0}\int_1^M f_u(x)\,x^{-1}\,dx=\int_1^u(u-x)^{b-1}\,x^{-1}\,dx,$$

uniformly with respect to u ($1<u\leqslant M$), the lemma follows.

Lemma 5. Assume **A**, **C**, **D**, **E**. Let i be a fixed integer >1 and let M be any number >1. Then we have

$$\sum_{y<p\leqslant y^u}\lambda(p^i)\,p^{ai}\sum_{n\leqslant y^u/p^i}\lambda(n)\,n^a=o(\Phi(y,u)) \tag{4.14}$$

uniformly for $1<u\leqslant M$.

Proof. We shall use the letter q as a summation index running through all numbers p^i (i fixed, p prime).

The inner sum in (4.14) is certainly zero if $y^u/p^i<1$, so the left-hand side of (4.14) equals

$$\sum_{y^i<q\leqslant y^u} \lambda(q)\, q^a \sum_{n\leqslant y^u/q} \lambda(n)\, n^a, \tag{4.15}$$

(so this is zero for $u<i$).

Next we remark that if ξ, η, α, β, f_u satisfy the conditions of lemma 1, then

$$\lim_{y\to\infty} \sum_{y^\xi<q\leqslant y^\eta} \lambda(q)\, f_u\left(\frac{\log q}{\log y}\right) = 0, \tag{4.16}$$

uniformly for $\alpha\leqslant u\leqslant\beta$. The fact that the q are not prime is of no concern in the proof of that lemma: the lemma can still be used to show that our assumption **D**, i.e.

$$\lim_{y\to\infty} \sum_{y<q\leqslant y^u} \lambda(q) = 0$$

(for every fixed $u>1$) leads to (4.16). (This means specializing b in lemma 1 to $b=0$, but this is not the same b we have in our present lemma 5: the b occurring in assumption **C** is positive according to **A**.)

A further preparatory remark is that lemma 2 and its proof remain true if we replace p by q, provided that $\sum_{y^{\frac{1}{2}}<q\leqslant y} \lambda(q)$ is bounded, and this is certainly the case because it has limit 0, by **D**.

We can now prove lemma 5 by repetition of the proof of lemma 4, replacing p's by q's. There are two minor differences:

(i) The summation in (4.15) runs from y^i onward instead of from y onward. This gives no trouble, we can first show that the sum with $y<q\leqslant y^u$ is $o(\Phi(y,u))$, and then remark that (4.15) is even less.

(ii) In (4.13) we have to replace $a^{-1}b\int_1^M f_u(x)\,x^{-1}\,dx$ by zero.

5. *The main theorem*

We shall now prove the theorem announced in sec. 1.3.

If $0<\delta<M\leqslant 1$, the result is a direct consequence of **C**, since $\eta(u)=u^{b-1}$ $(0<b\leqslant 1)$ and since

$$\Lambda_a(y^u, y) = \sum_{n\leqslant y^u} \lambda(n)\, n^a \qquad (0<u\leqslant 1). \tag{5.1}$$

It has to be noted that $L(\log y^u)/L(\log y)\to 1$ uniformly for $\delta\leqslant u\leqslant M$.

Next we prove the theorem for $0<\delta<M$, $1<M\leqslant 2$. By (3.2) we have, if $1<u\leqslant 2$

$$\Lambda_a(y^u, y) = \Lambda_a(y^u, y^u) - \sum_{y<p\leqslant y^u} \lambda(p)\, p^a \sum_{n\leqslant y^u/p} \lambda(n)\, n^a,$$

since the terms with $i>1$ do not give a contribution here ($p>y$ implies

$p^2 > y^u$). Applying **C** to $\Lambda_a(y^u, y^u)$ (see (5.1)) and then lemma 4, to the double sum, we obtain

$$\Lambda_a(y^u, y)/\Phi(y^u, y) = a^{-1}\, b\, \{u^{b-1} - b \int_1^u (u-x)^{b-1}\, x^{-1}\, dx + o(1)\},$$

uniformly for $1 \leqslant u \leqslant 2$. Since (2.1) (with $\alpha = 1$) gives

$$u^{b-1} - b \int_1^u (u-x)^{b-1}\, x^{-1}\, dx = \eta(u) \qquad (1 < u \leqslant 2),$$

we have now proved the theorem for $M \leqslant 2$.

We proceed by induction. Assuming that the theorem has been proved for a certain $M \geqslant 2$, we show that it is correct for M replaced by $M' = M + \frac{1}{2}$, i.e. we show that (1.10) holds uniformly for $M < u \leqslant M + \frac{1}{2}$.

We apply (3.2) with $v = \frac{1}{2}u$;

$$(5.2)\quad \Lambda_a(y^u, y^{u/2}) - \Lambda_a(y^u, y) = \sum_{1 \leqslant i < u}\ \sum_{y < p \leqslant y^{u/2}} \lambda(p^i)\, p^{ai}\, \Lambda_a(y^u/p^i, p-1).$$

We have

$$\Lambda_a(y^u/p^i, p-1) \leqslant \sum_{n \leqslant y^u/p^i} \lambda(n)\, n^a,$$

and so, by lemma 5, the contribution of each fixed $i > 1$ to the right-hand side is $o(\Phi(y, u))$, uniformly for $1 \leqslant u \leqslant M + \frac{1}{2}$. We have to consider at most $M - \frac{1}{2}$ different values of i, so their total contribution is $o(\Phi(y, u))$, and we can restrict ourselves to the remaining terms with $i = 1$.

For the values of u and p under consideration ($M < u \leqslant M + \frac{1}{2}, y < p \leqslant y^{u/2}$) we have

$$\tfrac{1}{2} < \frac{\log (y^u/p)}{\log (p-1)} \leqslant (u-1)\frac{\log p}{\log (p-1)} \leqslant (M - \tfrac{1}{2})\frac{\log p}{\log (p-1)} < M$$

for all y exceeding a certain constant $C = C(M)$. Hence we may apply the induction hypothesis:

$$\Lambda_a(y^u/p, p-1) =$$

$$= \{1 + o(1)\}\, a^{-1}\, b\, \eta\left(\frac{\log (y^u/p)}{\log (p-1)}\right) y^{au}\, p^{-a} (\log (p-1))^{b-1}\, L(\log (p-1)) =$$

$$= \{1 + o(1)\}\, a^{-1}\, b\, \eta\left(u\frac{\log y}{\log p} - 1\right) \frac{\Phi(y, u)}{p^a} \left(\frac{\log p}{\log y}\right)^{b-1},$$

uniformly for $M < u \leqslant M + \frac{1}{2}$. (Note that η is uniformly continuous and positive on $[\frac{1}{2}, M]$; moreover $\log (p-1)/\log y$ lies between $\frac{1}{2}$ and $\frac{1}{2}M + \frac{1}{4}$, whence $L(\log (p-1))$ may be replaced by $L(\log y)$.

As (1.10) has already been proved for $u = 2$ we have

$$\Lambda_a(y^u, y^{u/2}) \sim a^{-1}\, b\, \eta(2)\, (\tfrac{1}{2}u)^{b-1}\, \Phi(y, u),$$

uniformly for $M<u\leqslant M+\frac{1}{2}$. So it follows from (5.2) that

$$\Lambda_a(y^u, y)/\Phi(y, u) = a^{-1} b\, \eta(2)\, (\tfrac{1}{2}u)^{b-1} -$$
$$- a^{-1} b \sum_{y<p\leqslant y^{u/2}} \lambda(p) \left\{\eta\left(u\frac{\log y}{\log p}-1\right)+o(1)\right\}\left(\frac{\log y}{\log p}\right)^{b-1},$$

uniformly for $M<u\leqslant M+\frac{1}{2}$.

We now apply lemma 1 with $\xi=1$, $\eta=\frac{1}{2}M+\frac{1}{4}$, $\alpha=M$, $\beta=M+\frac{1}{2}$, and

$$f_u(x)=\begin{cases}\eta(ux^{-1}-1)\,x^{b-1} & \text{if } \ 1\leqslant x\leqslant \frac{1}{2}u,\\ 0 & \text{if } \ \frac{1}{2}u<x\leqslant\frac{1}{2}M+\frac{1}{4}.\end{cases}$$

This leads to

$$\Lambda_a(y^u, y)/\Phi(y, u) =$$
$$= a^{-1} b\, \eta(2)\, (\tfrac{1}{2}u)^{b-1} - a^{-1} b^2 \int_1^{\frac{1}{2}u} \eta(ux^{-1}-1)\, x^{b-2}\, dx + o(1),$$

uniformly for $M<u\leqslant M+\frac{1}{2}$. By (2.1) (with $\alpha=2$) the right-hand side is $a^{-1}b\,\eta(u)+o(1)$, and this completes the induction step.

6. *Applications*

6.1 If $\lambda(n)=n^{-1}$ for all n, and if $a=1$, $b=1$, the conditions of our theorem are satisfied, with $L\equiv 1$. The result is that if $\Psi(x, y)$ is the number of ntegers $\leqslant x$, free of prime factors $>y$, then $\Psi(y^u, y) \sim \eta(u)y^u$ (u fixed, $y\to\infty$). This result was first obtained by A. A. Buchstab [8], and extended to cases where $u\to\infty$ in [1].

6.2 In Part I ($=$[7]) we proved

$$\sum_{P(d)\leqslant y,\, d\leqslant y^u} \mu^2(d)\, (\varphi(d))^{-1} \sim \theta_1(u) \log y. \tag{6.1}$$

Inserting an extra factor d, we now obtain from our present theorem (see (1.10))

$$\sum_{P(d)\leqslant y,\, d\leqslant y^u} \mu^2(d)\, d(\varphi(d))^{-1} \sim \eta(u)\, y^u \tag{6.2}$$

if $u>0$ is fixed, $y\to\infty$. In this case we have $\lambda(n)=\mu^2(n)/\varphi(n)$, $a=1$, $b=1$, $L\equiv 1$. We omit a detailed verification of the conditions **A**, **B**, **C**, **D**, **E**; **A** and **D** are trivial, **B** and **E** depend on the fact that the expression

$$\textstyle\sum_{p<x} p^{-1} - \log\log x$$

has a limit if $x\to\infty$; for **C** we need

$$\sum_{n\leqslant y} \mu^2(n)\, n(\varphi(n))^{-1} \sim y.$$

The latter relation can be seen, for example, from the identity

$$\sum_1^\infty \mu^2(n)\, n(\varphi(n))^{-1}\, n^{-s} = \zeta(s) \prod_p \left(1+\frac{1}{p-1}\frac{1}{p^s}-\frac{p}{p-1}\frac{1}{p^{2s}}\right),$$

where the infinite product can be expanded into a Dirichlet series which converges absolutely for $s > \frac{1}{2}$ and has the value 1 at $s = 1$.

6.3 If we define the multiplicative function λ by $\lambda(n) = (n\, d(n))^{-1}$, where $d(n)$ stands for the number of divisors of n, then we have by [10]

$$\sum_{n \leqslant x} \lambda(n)\, n = \sum_{n \leqslant x} (d(n))^{-1} \sim c\, x (\log x)^{-\frac{1}{2}},$$

with a certain positive constant c. The function λ evidently satisfies conditions **A**, **B**, **C**, **D**, **E** with $a = 1$, $b = \frac{1}{2}$, $L \equiv c$. Therefore by (1.10) we have

$$\sum_{P(d) \leqslant y,\, d \leqslant y^u} \mu^2(n)\, (d(n))^{-1} \sim c\, \eta(u)\, y^u (\log y)^{-\frac{1}{2}}$$

where η is the function defined in sec. 2 with $b = \frac{1}{2}$.

6.4 Another example with $b = \frac{1}{2}$ is found by defining

(i) $\lambda(p^i) = 0$ if $i = 1, 3, 5, \ldots$; $p \equiv 3 \pmod 4$,
(ii) $\lambda(p^i) = p^{-i}$ otherwise,
(iii) λ multiplicative.

It is well-known that for $n \geqslant 1$ we have $n\lambda(n) = 1$ if n is the sum of two squares, $n\lambda(n) = 0$ otherwise. Thus we have in this case

$$\Lambda_1(y^u, y) = \sum_{n \leqslant y^u} \lambda(n)\, n = \sum_{n \leqslant y^u,\, P(n) \leqslant y}{}' \; 1,$$

where the dash indicates that n is omitted if n is not the sum of two squares.

For the partial sums we have

$$\sum_{n \leqslant x} \lambda(n)\, n \sim c\, x (\log x)^{-\frac{1}{2}},$$

where $c = \{2 \prod_{p \equiv 3 \pmod 4} (1 - p^{-2})\}^{-\frac{1}{2}}$ (cf. [9], § 176), and the verification of **A**, **B**, **C**, **D**, **E** (with $a = 1$, $b = \frac{1}{2}$, $L \equiv c$) is easy. So by (1.10) we have

$$\Lambda_1(y^u, y) \sim c\, \eta(u)\, y^u (\log y)^{-\frac{1}{2}}$$

with the same function η as in example 2.

6.5 In all previous examples the function L occurring in our theorem was constant. It is not difficult to construct an example where this is not the case. We define

(i) $p\lambda(p) = 1 + (\log \log p)^{-1}$ if $p > 2$,
(ii) $\lambda(p^i) = 0$ if $p = 2$ or $i \geqslant 2$,
(iii) λ multiplicative.

By a theorem of Wirsing [11] we now have

$$\sum_{n \leqslant x} \lambda(n)\, n \sim e^{-\gamma} x (\log x)^{-1} \prod_{2 < p \leqslant x} (1 + \lambda(p)) \sim 4\pi^{-2}\, x\, G(x),$$

where $G(x) = \prod_{2 < p \leqslant x} \{1 + (p+1)^{-1} (\log \log p)^{-1}\}$, and γ is Euler's constant.

In order to prove that G is a slowly oscillating function of $\log x$ we must show that

$$\lim_{x\to\infty} \prod_{x<p\leqslant x^c} \{1+(p+1)^{-1}(\log\log p)^{-1}\} = 1$$

for every $c>1$, and to show this it is sufficient to show that

$$\lim_{x\to\infty} \int_x^{x^c} \frac{1}{t\log\log t}\frac{dt}{\log t} = 0$$

for every $c>1$. (Here the prime number theorem is applied in the familiar way.) This is verified by straightforward calculation.

Also, it is easy to see that $L(x)\to\infty$ if $x\to\infty$.

Thus we have given an example of a multiplicative function λ, satisfying **A**, **B**, **C**, **D**, **E** with $a=1$, $b=0$ and L is a slowly oscillating function which is not a constant (not even asymptotically). We omit the simple verification of **A**, **B**, **C**, **D**, **E**.

Technological University, Eindhoven
Netherlands

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