

# On the factorization of rational matrices depending on a parameter

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EINDHOVEN UNIVERSITY OF TECHNOLOGY

Faculty of Mathematics and Computing Science

Memorandum COSOR 87-06

On the factorization of rational  
matrices depending on a parameter

by

C. Praagman

Eindhoven, Netherlands

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# ON THE FACTORIZATION OF RATIONAL MATRICES DEPENDING ON A PARAMETER

C. Praagman  
November 1986

## Abstract

In 1961 Youla published his paper '*On the factorization of rational matrices*'. He proved that any proper rational parahermitian matrix, positive definite on the imaginary axis can be factorized as the product of a proper rational matrix, stable with respect to the closed right half plane, and its adjoint. In this paper I prove that for any positive definite, nonstrictly proper matrix this factorization can be given depending analytically on the original matrix, in a sufficiently small neighbourhood. This result is applied to the problem of metrizing the space of transfer matrices of linear systems, in accordance with Vidyasagar's *graph topology*.

Keywords: Graph topology, graph metric, spectral factorization, coprime factorizations, finite dimensional linear systems.

AMS subject classification: 47 A 68.  
47 D 15; 93 D 15; 93 D 25.

§ 1. Preliminaries on the Theorem of Youla

Let  $\mathcal{C}_+ = \{s \in \mathcal{C} \mid \text{Re } s > 0\}$ ,  $I^e = \{s \in \mathcal{C} \mid \text{Re } s = 0\} \cup \{\infty\}$  and  $\mathcal{C}_+^e = \mathcal{C}_+ \cup I^e$ . As usual let  $\mathbb{R}(s)$  be the field of real rational functions, and let  $S$  denote the subring of proper stable rational functions :

$$S = \mathbb{R}(s) \cap C(\mathcal{C}_+^e, \mathcal{C}).$$

Let  $gl_m S$  be the ring of square matrices of size  $m$  with entries in  $S$ , and  $Gl_m S$  the subgroup of invertible (in  $gl_m S$ ) elements. For any  $G \in \mathbb{R}(s)^{n \times m}$  define  $G^*$  by  $G^*(s) = G'(-s)$ . Then for all  $n, g_n$  defined by  $g_n(G) = G^*G$  maps  $S^{n \times m}$  into

$$L_m = \{G \in \mathbb{R}(s)^{m \times m} \cap C(I^e, \mathcal{C}^{m \times m}) \mid G^* = G\},$$

the space of parahermitian matrices. Denote by  $L_m^+$  the (open) subset of matrices positive definite on  $I^e$ . Finally let  $O(m)$  be the orthogonal group:  $O(m) = \{U \in Gl_m(\mathbb{R}) \mid U^t U = I\}$ .

With these notations a special case of the theorem of Youla ([8] theorem 2), reads :

Theorem 1.

$g_m$  maps  $Gl_m S$  onto  $L_m^+$ , and for any  $G \in L^+$ ,  $g_m^{-1}(G) = RO(m)$ , where  $g_m(R) = G$ .

For all  $n, m$  define a norm on  $\mathbb{R}(s)^{n \times m} \cap C(I^e, \mathcal{C}^{n \times m})$  by

$$\|H\| = \max_{s \in I^e} \|H(s)\|_1,$$

where  $\|\cdot\|_1$  is the usual operator norm for linear mappings between the Euclidean spaces  $\mathcal{C}^m$  and  $\mathcal{C}^n$ . With this norm  $L_m$  and  $gl_m S$  become normed spaces. In the appendix I have gathered some results on the induced topology and on the completion of these spaces.

The main result of this paper is that the solution to  $g_m(R) = G$  can be chosen such that if  $G$  depends in a certain regular way on a parameter, the same is true for  $R$ . As usual, for  $k = \{0, 1, 2, \dots, \infty, \omega\}$  let  $C^k$  denote the  $k$ -times continuously differentiable, resp. real analytic functions.

Theorem 2.

Let  $R_0 \in Gl_m S$  and let  $l_k \in C^k(\Lambda, L_m^+)$ ,  $0 \in \Lambda \subset \mathbb{R}^n$  satisfy  $l_k(0) = g_m(R_0)$ . Then there exists an  $r_k \in C^k(\Lambda', Gl_m S)$ ,  $0 \in \Lambda' \subset \Lambda$  such that  $r_k(0) = R_0$  and  $g_m \circ r_k = l_k$ .

The proof of this theorem will be given in several steps. First I specialize to the case that  $R_0(\infty) = I$ , and  $l_k(\lambda)(\infty) = I$ . Let  $P_m$  be the set of positive definite real matrices of size  $m$ .

Lemma 1.

Let  $M \in Gl_m \mathbb{R}$ ; then there exists a unique  $W_M : P_m \rightarrow P_m$  such that  $g_m(W_M(K)M) = K$ , and  $W_M \in C^\omega(P_m, P_m)$ .

Proof. From Polderman [3] lemma 3.2 it follows that there exists a unique  $W_M : P_m \rightarrow P_m$

satisfying  $g_m(W_M(K)) = M^{*-1}KM^{-1}$ , and moreover that this  $W_M$  is analytic.

Now if one proves Theorem 2 in case  $R_0(\infty) = l_k(\lambda)(\infty) = I$ , then the general case follows by taking

$$r_k(\lambda) = \bar{r}_k(\lambda) W_M(l_k(\lambda)(\infty)) M ;$$

here  $M = R_0(\infty)$ , and  $\bar{r}_k(\lambda)$  is the solution in the special case where  $\bar{l}_k(\lambda) = M_\lambda^{*-1} l_k(\lambda) M_\lambda$ ,  $M_\lambda = W_I(l_k(\lambda)(\infty))$ .

§ 2. The proof of theorem 2

Let  $gl_m S_o = \{R \in gl_m S \mid R(\infty) = 0\}$  and  $L_m^o = \{G \in L_m \mid G(\infty) = 0\}$ , and define  $f : gl_m S_o \times L_m^o \rightarrow L_m^o$

$$f(R, G) = R^* + R + R^*R - G .$$

Note that  $f(R, G) = g_m(I+R) - (I+G)$ . Clearly  $f \circ \mu \in C^k(\Lambda, L_m^o)$  for any  $\mu \in C^k(\Lambda, gl_m S_o \times L_m^o)$

Theorem 3.

Let  $G_0 \in L_m^o$  satisfy  $I + G_0 \in L_m^+$ , and let  $l_k \in C^k(\Lambda, L_m^o)$  satisfy  $l_k(0) = G_0$ . There exists a unique map  $r_k : \Lambda \rightarrow Gl_m S_o$  such that  $f(r_k(\lambda), l_k(\lambda)) = 0$  and  $r_k \in C^k(\Lambda, Gl_m S_o)$

Proof. The uniqueness and existence of  $r_k$  are well known (Youla [8]), so the only thing to prove is the regularity conditions on  $r_k$ . Extend  $f$  to a map of  $gl_m \hat{S}_o \times \hat{L}_m^o \rightarrow \hat{L}_m^o$ , then  $\frac{\partial f}{\partial R}(r_k(0), l_k(0))$  defines a homeomorphism between  $gl_m \hat{S}_o$  and  $\hat{L}_m^o$  (theorem 7), so the implicit function theorem can be applied (Dieudonné [2], 10.2.1), to prove that on a small neighbourhood  $\Lambda'$  of zero there exists a function  $r'_k \in C^k(\Lambda', gl_m \hat{S}_o)$ , such that  $f(r'_k(\lambda), l_k(\lambda)) = 0$ .

Since  $f(r'_k(\lambda), l_k(\lambda)) = f(r_k(\lambda), l_k(\lambda))$  it follows that for all  $\lambda \in \Lambda'$

$$(I + r'_k(\lambda))^* (I + r'_k(\lambda)) = (I + r_k(\lambda))^* (I + r_k(\lambda))$$

and hence  $(I + r'_k(\lambda)) (I + r_k(\lambda))^{-1} = ((I + r_k(\lambda))(I + r'_k(\lambda))^{-1})^*$  is analytic in  $s$  on  $\mathcal{C} \cup \{\infty\}$ . So,  $I + r'_k(\lambda) = I + r_k(\lambda)$  implying that  $r_k = r'_k$ . Since this holds for arbitrary  $G_0$ ,  $r_k \in C^k(\Lambda, gl_m S_o)$ .

Remark.

In fact combining Lemma 1 and Theorem 3 one easily establishes that for any  $R \in Gl_m S$ , the map  $w_M : L^+ \rightarrow Gl_m S$  defined by

$$w_M(G) = \bar{F}(G) W_M(G) M$$

where

$$\bar{F}(G) = I + r(W_I((G(\infty))^{-1}) G W_I((G(\infty))^{-1}) - I),$$

$$W_M(G) = W_M(G(\infty)), \quad M = R(\infty),$$

satisfies : For any  $l_k \in C^k(\Lambda, L_m^+)$  the composition of  $w_M$  with  $l_k$ ,  $w_M \circ l_k \in C^k(\Lambda, Gl_m S)$ , and  $g_m \circ w_M(G) = G$ . Especially taking  $R \in g_m^{-1}(I)$  one gets a family of right inverses of  $g_m$  parameterized by the orthogonal group, giving rise to a foliation of  $Gl_m S$ , parameterized by the orthogonal group.

### § 3. The Normalized Graph Metric

In [6] and [7] Vidyasagar introduced the graph topology on transfermatrices of finite dimensional linear time invariant systems. In this set up a system is identified with its transfer matrix  $P \in \mathbb{R}(s)^{n \times m}$ . As is well known a  $P \in \mathbb{R}(s)^{n \times m}$  has a right coprime factorization (r.c.f.) over  $S$  (Vidyasagar [7], section 4.1).

#### Lemma 2.

Let  $P \in \mathbb{R}(s)^{n \times m}$ , then there exist  $N \in S^{n \times n}$  and  $D \in S^{m \times m}$  such that

- i)  $\det D \neq 0$ ,
- ii)  $XN + YD = I$ , for certain  $X \in S^{m \times n}$  and  $Y \in S^{m \times m}$ ,
- iii)  $P = ND^{-1}$ .

Note that the coprimeness of  $N, D$  can be expressed by the Bezout identity since  $S$  is a principal ideal domain.

Let  $R_{n,m}(S)$  be the subset of  $S^{(n+m) \times m}$  consisting of matrices  $M = \begin{bmatrix} N \\ D \end{bmatrix}$  such that  $N$  and  $D$  are right coprime, and  $\det D \neq 0$ .  $R_{n,m}(S)$  is equipped with the norm topology. Define  $p : R_{n,m}(S) \rightarrow \mathbb{R}(s)^{n \times m}$  by

$$p \begin{bmatrix} N \\ D \end{bmatrix} = ND^{-1}.$$

#### Definition.

The graph topology is the quotient topology under  $p$ .

#### Lemma 3.

Points are closed in the graph topology.

Proof. Let  $P = \bar{D}^{-1} \bar{N}$ , then  $p^{-1}(P) = \left\{ \begin{bmatrix} N \\ D \end{bmatrix} \in R_{n,m}(S) \mid (\bar{D}, -\bar{N}) \begin{bmatrix} N \\ D \end{bmatrix} = 0 \right\}$ .

In the recent literature several metrizations of this topology have been given. In [9] Zames and El Sakhary proposed the gap metric for  $\mathbb{R}(s)^{n \times m}$ , which induces the gap topology. Zhu proved in [10] that the gap topology and the graph topology coincide on  $\mathbb{R}(s)^{n \times m}$ . Vidyasagar himself introduced the graph metric in [6] and [7].

Define  $g_{n,m} : R_{n,m}(S) \rightarrow L_m^+$  by  $g_{n,m}(M) = M^*M$ , and let  $A_{n,m}(S) = g_{n,m}^{-1}(I)$ .  $A_{n,m}(S)$  is called the set of normalized right coprime factorizations (n.r.c.f.). Let  $a : A_{n,m}(S) \rightarrow \mathbb{R}(s)^{n \times m}$  be the restriction of  $p$  then  $a$  is onto (Vidyasagar [7], section 7.3). Vidyasagar defines :

$$\bar{d}(P_1, P_2) = \min(\|A_1 - A_2 U\| \mid U \in gl_m S, \|U\| \leq 1)$$

$$d(P_1, P_2) = \max(\bar{d}(P_1, P_2), \bar{d}(P_2, P_1)) .$$

Here  $A_i$  is an arbitrary but fixed n.r.c.f. of  $P_i : A_i \in a^{-1}(P_i)$ . The condition  $\|U\| \leq 1$ , imposed to satisfy the triangle inequality, is not very elegant, and also leads to computational problems, not only from a practical point of view, but already in theory.

Therefore I propose as a third possibility the following metric, the *normalized graph metric* :

$$d_n(P_1, P_2) = \min(\|A_1 - A_2\| \mid A_i \in a^{-1}(P_i)) .$$

By straightforward calculation one verifies that  $d_n$  defines a metric on  $\mathcal{R}(s)^{nzm}$ . The essential point is of course the following theorem :

Theorem 4.

*The normalized graph metric metrizes the graph topology.*

Proof. Let  $V$  be an open subset of  $R_{n,n}(S)$ ,  $M \in V$  with  $p(M) = P$ , then  $p(V)$  is an open neighbourhood of  $P$  in the graph topology.

There exists an  $\epsilon > 0$  such that  $B(M, \epsilon) = \{M' \in R_{n,n}(S) \mid \|M - M'\| < \epsilon\} \subset V$ . Then  $B_n(P, \epsilon) = \{P' \in \mathcal{R}(s)^{nzm} \mid d_n(P, P') < \epsilon\} \subset p(B(M, \epsilon))$ . Hence  $p(V)$  is open in the topology induced by the normalized graph metric. On the other hand let  $V'$  be an open subset of  $\mathcal{R}(s)^{nzm}$  in the induced topology, and let  $P \in V'$ . There exists an  $\epsilon > 0$  such that  $B_n(P, \epsilon) \subset V'$ . Let  $A \in a^{-1}(P)$ , and let  $P_1 \in p(B(A, \delta))$  and let  $M \in B(A, \delta) \cap p^{-1}(P_1)$ . Let  $R = (w_I \circ g_{n,m}(M))^{-1}$ , then  $MR \in a^{-1}(P_1)$  and

$$\begin{aligned} \|A - MR\| &\leq \|A - AR\| + \|AR - MR\| \leq \\ &\leq \|A\| \|I - R\| + \|A - M\| \|R\| . \end{aligned}$$

Since  $w_I \circ g_{n,m}$  is continuous and  $w_I \circ g_{n,m}(A) = I$ , it is clear that if  $\delta$  is small enough, say  $\delta < \delta_0$ , then  $\|I - R\| < \frac{1}{2}\epsilon$ , and  $\|R\| < 2$ . So if  $\delta < \max(\delta_0, \frac{1}{4}\epsilon)$ , then

$$\|A - MR\| \leq \frac{1}{2}\epsilon + \frac{1}{4}\epsilon \cdot 2 = \epsilon .$$

Hence  $P_1 \in B_n(P, \epsilon)$ , so  $p(B(A, \delta)) \subset V'$ , and hence  $V'$  is open in the graph topology.

For computational purposes the normalized graph metric seems more promising in view of the following lemma.

Lemma 4.

Let  $P_1, P_2 \in \mathcal{R}(s)^{nzm}$ , and  $A_1 \in a^{-1}(P_1), A_2 \in a^{-1}(P_2)$ . Then

$$d_n(P_1, P_2) = \min(\|A_1 - A_2 U\| \mid U \in O(m)) .$$

Proof.  $a^{-1}(P_i) = \{A_i U \mid U \in O(m)\}$  (Vidyasagar [7], section 7.3). So



$$\begin{aligned} \inf (\|A_1' - A_2'\| \mid A_i' \in a^{-1}(P_i)) &= \inf (\|A_1 U_1 - A_2 U_2\| \mid U_i \in O(m)) \\ &= \inf (\|A_1 - A_2 U_2 U_1^{-1}\| \mid U_i \in O(m)) \end{aligned}$$

Since  $O(m)$  is a compact group, this proves the lemma.

Remark.

If one compares the graph topology with the gap topology the definition of the normalized graph metric is analogous to the definition of the gap metric. For computational purposes the gap is more suited than the graph metric. Therefore one might introduce the following measure for the distance between plants ; fix  $A_i \in a^{-1}(P_i)$  and define

$$\begin{aligned} \vec{d}_g(P_1, P_2) &= \inf (\|A_1 - M_2\| \mid M_2 \in p^{-1}(P_2)) \\ d_g(P_1, P_2) &= \max (\vec{d}_g(P_1, P_2), \vec{d}_g(P_2, P_1)) . \end{aligned}$$

The calculation of these numbers is treated in Vidyasagar [7] chapter 6.

A second application of theorem 2 would be the 'normalization' of the metric defined by ZHU [11] for the linear distributed systems without poles on the imaginary axis, having a Bezoutian factorization.

**Appendix on Function Spaces**

In § 2 use was made of the completions of the spaces  $gl_m S_o$  and  $L_m^o$ . In this appendix I have gathered some lemma's and theorems needed for the calculation of the completions.

Theorem 5 (Mergelyan).

Let  $f$  be a complex valued function, continuous on  $\{ |s| \leq 1 \}$  and analytic on  $\{ |s| < 1 \}$ . Then for all  $\epsilon > 0$  there exists a  $p \in \mathcal{C}[s]$ , such that

$$\sup_{|s| \leq 1} | f(s) - p(s) | < \epsilon .$$

Proof. See Rudin [5], theorem 20.5.

Corollary.

Let  $S^c = \mathcal{C}(s) \cap C(\mathcal{C}_+^e, \mathcal{C})$ , with the sup norm. Then  $S^c$  is dense in  $O^c = C^\omega(\mathcal{C}_+, \mathcal{C}) \cap C(\mathcal{C}_+^e, \mathcal{C})$ .

Proof. Let  $f \in O^c$ , then  $\hat{f}$  defined by  $\hat{f}(s) = f\left(\frac{1+s}{1-s}\right)$  is analytic inside the unit disc, and continuous on the closed disc. Fix  $\epsilon > 0$ , then there exists a polynomial  $\hat{p}$  such that  $|\hat{p}(s) - \hat{f}(s)| < \epsilon$  for all  $|s| \leq 1$ . Define the rational function  $p$  by  $p(s) = \hat{p}\left(\frac{s-1}{s+1}\right)$ , then

$$p \in S^c, \text{ and for all } s \in \mathcal{C}_+^e : | f(s) - p(s) | = \left| \hat{f}\left(\frac{s-1}{s+1}\right) - \hat{p}\left(\frac{s-1}{s+1}\right) \right| < \epsilon .$$

Proposition.

$$\hat{S}_o = \{ f \in O^c \mid f = \bar{f}, f(\infty) = 0 \} .$$

Proof. Let  $\{p_n\} \subset S_o$  be a Cauchy sequence. Then the  $p_n$  converge uniformly on  $\mathcal{C}_+^e$ , so  $\lim_{n \rightarrow \infty} p_n \in O^c$ . Clearly  $\lim_{n \rightarrow \infty} p_n(\infty) = 0$ , and  $\lim_{n \rightarrow \infty} p_n(x)$  is real if  $x$  is real. So now let  $f \in O^c$  with  $f(\mathbb{R}_+) \subset \mathbb{R}$ , and  $f(\infty) = 0$ . There exists  $p_n \in S^c$  such that  $p_n \rightarrow f$  uniformly on  $\mathcal{C}_+^e$ . Hence  $\bar{p}_n \rightarrow \bar{f} = f$ , hence  $\frac{1}{2}(\bar{p}_n + p_n) \rightarrow f$ , and  $\frac{1}{2}(\bar{p}_n + p_n) \in S$ . Taking  $q_n = \frac{1}{2}(\bar{p}_n + p_n) - \frac{1}{2}(\bar{p}_n + p_n)(\infty)$  one has  $q_n \in S_o$  and  $q_n \rightarrow f$ .

Theorem 6.

Let  $f \in C(I^e, \mathcal{C})$ ; then there exists for all  $\epsilon > 0$  a  $p \in \mathcal{C}(s)$  such that  $|p(s) - f(s)| < \epsilon$  for all  $s \in I^e$ .

Proof.  $\mathcal{C}(s) \cap C(I^e, \mathcal{C})$  forms a complex algebra of complex valued functions on  $I^e$ , which vanishes nowhere, is selfadjoint, and separates points. Hence by the Stone-Weierstrass theorem (RUDIN [4] th. 7.33) one has that  $\mathcal{C}(s) \cap C(I^e, \mathcal{C})$  is dense in  $C(I^e, \mathcal{C})$ .

Proposition.

$$\hat{L}_m^o = \left\{ M \in C(I^e, \mathcal{C})^{m \times m} \mid M(\infty) = 0, \bar{M} = M = M^* \right\} .$$

Proof. The preceding theorem yields that  $\hat{L}_m^o \subset \{\dots\}$ , and on the other hand that if  $M \in C(I^e, \mathcal{C})^{m \times m}$ , then there exist  $G_n \in \mathcal{C}(s)^{m \times m}$  such that  $G_n \rightarrow M$ . First of all without loss of generality we may assume that  $G_n(\infty) = 0$ , and secondly  $G_n^* \rightarrow M, \bar{G}_n \rightarrow M$ , so  $\frac{1}{4}(\bar{G}_n + G_n + G_n^* + \bar{G}_n^*) \rightarrow M$ , and  $\bar{G}_n + G_n + \bar{G}_n^* + \bar{G}_n^* \in L_m^o$ .

Theorem 7.

Let  $R \in Gl_m S$ . The map  $h : gl_m \hat{S}_o \rightarrow \hat{L}_m^o$  defined by  $h(K) = R^* K + K^* R$  is a homeomorphism.

Proof. Clearly  $h$  is a bounded linear operator:  $\|h\| \leq 2\|R\|$ . Further if  $h(K) = 0$ , then  $R^* K = -K^* R$  or  $KR^{-1} = -R^{*-1} K^*$ . Hence  $KR^{-1}$  is analytic on  $\mathcal{C} \cup \{\infty\}$  and hence constant. Since  $(KR^{-1})(\infty) = 0$ , this implies that  $K = 0$ .

Now let  $\Phi \in \hat{L}_m^o$ , define  $K(s) = \frac{1}{2\pi i} R^{*-1}(s) \int_{I^e} \Phi(z) / (z-s) dz$  for  $\text{Re } s > 0$ .  $K$  is analytic on  $\mathcal{C}_+$  and for all  $s_0 \in I^e$  (Heins [2] XIV (5.3)):

$$K(s_0) = \lim_{s \rightarrow s_0} K(s) = R^{*-1}(s_0) \left[ \frac{1}{2\pi i} \int_{I^e} \Phi(z) / (z-s_0) dz + \frac{1}{2} \Phi(s_0) \right],$$

where  $\int$  denotes the Cauchy principal value. This map is continuous since  $\|K\| \leq \|R^{-1}\| \|\Phi\|$ .

Finally :

$$\begin{aligned} h(K)(s_0) &= R^*(s_0) K(s_0) + K^*(s_0) R(s_0) \\ &= \frac{1}{2\pi i} \int_{I^e} \Phi(z) / (z-s_0) dt + \frac{1}{2} \Phi(s_0) + \frac{1}{2\pi i} \int_{I^e} \Phi(z) / (z+s_0) dz + \frac{1}{2} \Phi^*(s_0) \\ &= \Phi(s_0). \end{aligned}$$

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