

## Dynamic optimization in a multi-product firm

**Citation for published version (APA):**

Geldrop, van, J. H., & Withagen, C. A. A. M. (1987). *Dynamic optimization in a multi-product firm*. (Memorandum COSOR; Vol. 8718). Technische Universiteit Eindhoven.

**Document status and date:**

Published: 01/01/1987

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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EINDHOVEN UNIVERSITY OF TECHNOLOGY

COSOR-memorandum 87-18  
**Dynamic Optimization in a Multi-Product Firm**  
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March 1987

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## Dynamic Optimization in a Multi-Product Firm.

### Abstract

This paper considers the problem of optimal dynamic allocation of factors of production in a multiproduct firm. In particular we study a firm which produces by means of fixed assets an (intermediary) output that can be sold on the market or used within the firm to produce a second output which can be sold as a final product or can be added to the existing capital stock.

We show the existence of optimal trajectories, establish some stability properties and provide further characteristics.

### 1. Introduction

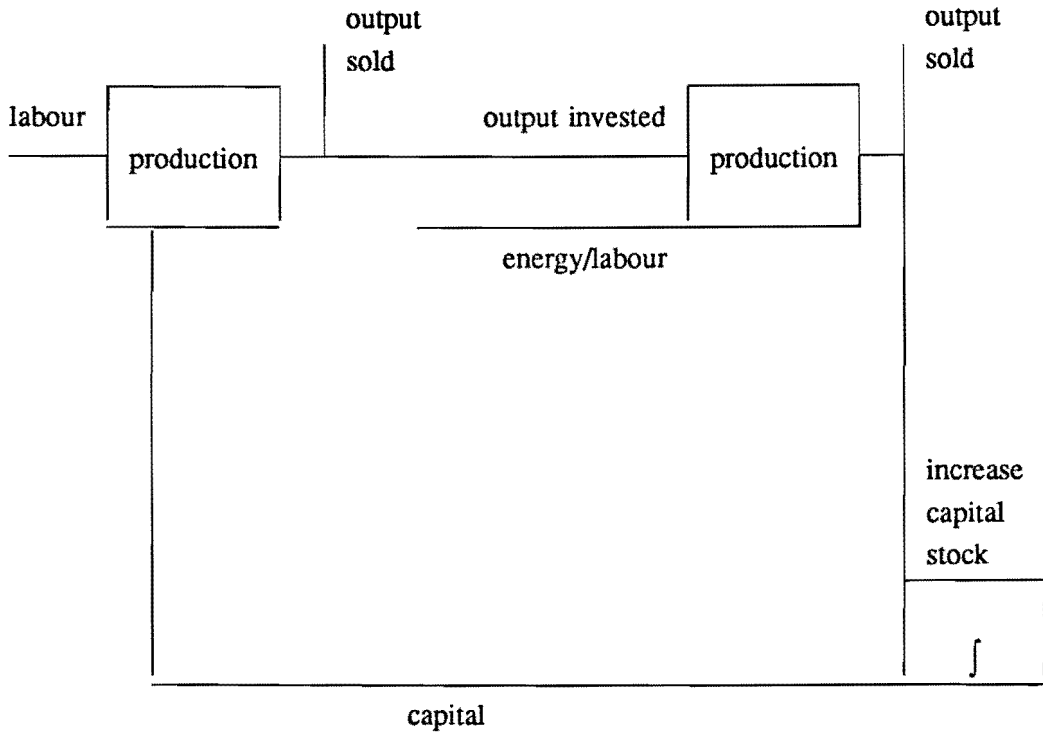
The phenomenon of multiproduct firms has been extensively studied in economics. To the early work one can attach the names of Pigou and Robinson. Hicks [1939] has given a thorough treatment of the case of only variable inputs. The importance of fixed inputs has been recognized by a.o. Dorfman [1951]. In his well-known book Ferguson [1969] argues that Pfouts [1968] was the first to construct a model "permitting switches of fixed inputs among various outputs". For more recent developments the standard reference is Fuss and McFadden [1978] who give a formal treatment of the issue. Characteristic for this type of studies is the fact that they are a-temporal. To our knowledge, extensions to a dynamic analysis can only be found in Debreu-type multi dimensional production sets with dated commodities.

These approaches however might not do justice to the many cases where one observes that the accumulation of "fixed" assets within the firms that produce them changes their production possibilities over time. This can be illustrated by variety of examples of which two will be presented below.

Consider first the problem of a chicken farmer. He has the disposal of a given amount of chickens. His labour or hired labour, together with food for the chickens, produce eggs which can be sold on the market. But the eggs can also be used in order to produce new chickens, again with the help of some input such as energy. The new chickens can be sold, for example as meat, or they can be added to the existing stock of chickens. Hence the farmer has to decide on the amounts of inputs he has to hire and on the amount to invest (eggs which are not sold). These decisions have to be taken within a dynamic context.

A similar argument applies to a steel mill which uses machinery and raw material to produce steel, that can be sold as such or be processed further within the firm in order to obtain new machinery to be sold or to be used within the firm.

Schematically the situation described above can graphically be drawn as follows



The aim of the present paper is to construct and analyse a model which captures the issues mentioned above and which takes into account several possible market structures.

The most important findings can be summarized as follows. If the initial stock of capital is small, a non-competitive firm will increase its stock of capital to a stationary value. Both final markets will be supplied if demand is sufficiently inelastic. When the initial stock of capital is large it will monotonically be decreased to the stationary value. Both final markets are supplied, at a decreasing rate.

The plan of the paper is as follows. Section 2 develops the model, discusses the assumptions and gives necessary and sufficient conditions for optimal trajectories. In section 3 we analyse the non-competitive firm and section 4 deals with the competitive case. The results are summarized in section 5 which also gives the conclusions.

## 2. The model and necessary conditions for optimality

The problem is to maximize

$$J := \int_0^{\infty} e^{-\rho t} \{U_1(C_1) + U_2(C_2) - q_1 V_1 - q_2 V_2\} dt , \quad (2.1)$$

subject to

$$C_1 + I = G(K, V_1) , \quad (2.2)$$

$$\dot{K} = F(I, V_2) - C_2, K(0) = K_0, \text{ given,} \quad (2.3)$$

$$C_1 \geq 0, C_2 \geq 0, V_1 \geq 0, V_2 \geq 0, I \geq 0. \quad (2.4)$$

Here  $K$  is the stock of the capital, which, together with some other input, the amount of which is denoted by  $V_1$ , yields an output according to production function  $G$ . Output is kept within the firm,  $I$ , or sold for final use,  $C_1$ . Investment  $I$ , together with the amount  $V_2$  of some other input, not necessarily equal to the input represented by  $V_1$ , produce a commodity that can be used to enlarge the existing stock of capital,  $\dot{K} := dK/dt$ , but also for selling on the market ( $C_2$ ). The non-negativity constraints with respect to  $V_1, V_2$  and  $I$  are self-evident. Those with respect to  $C_1$  and  $C_2$  need some comments. In principle one could argue that it might be advantageous in some circumstances to invest more than the firm produces itself or to add more to the existing stock of capital than is possible by means of own production of the relevant commodity. However, this is obviously ruled out a priori when the firm is a monopolist on both final product markets. And with less than perfect competition it is unlikely that an equilibrium would result. Finally, when full competition prevails then an optimal solution will exist only in very special circumstances, as is shown below.

As far as the maximand is concerned,  $\rho (>0)$  is the given discount rate,  $U_1$  and  $U_2$  are the proceeds from selling (possibly linear in  $C_1$  and  $C_2$ ), whereas  $q_1$  and  $q_2$  are the costs per unit of input. So  $J$  is total discounted profit. The maximization takes place with respect to  $C_1, C_2, V_1, V_2$  and  $I$ .

Some mathematical preliminaries are in order now.

It is required that  $(C_1(t), C_2(t), V_1(t), V_2(t), I(t))$  is piece-wise continuous on  $[0, \infty)$  and that the state-variable  $K(t)$  is continuous on  $[0, \infty)$  and piece-wise differentiable on  $[0, \infty)$ . Moreover the following assumptions will be made.

$A_{1a}$  holds in section 3 and in the sequel of this section,  $A_{1b}$  holds in section 4 only.

$A_{1a}$  For  $i = 1, 2, U_i$  is defined for  $C_i \geq 0$ .  $U_i$  is continuous and twice continuously differentiable on the open interval  $(0, \infty)$ ,  $U_i'(C_i) > 0, U_i''(C_i) < 0, \lim_{C_i \rightarrow \infty} U_i'(C_i) = 0$  and  $\lim_{C_i \rightarrow 0} U_i'(C_i) = \infty$ .

$A_{1b}$  For  $i = 1, 2, U_i(C_i) = p_i C_i$  where  $p_i$  is a given positive constant.

$A_2$   $F$  and  $G$  are defined, continuous, strictly concave and linearly homogeneous on  $R_+^2$ .  $F$  and  $G$  are twice continuously differentiable on  $R_{++}^2$ . The partial derivatives are positive.  $F(I, 0) = F(0, V_2) = G(K, 0) = G(0, V_1) = 0$ .

In view of the linear homogeneity we define  $z := V_1/K, x := V_2/I$  and

$$g(z) := G(1, z); \quad \phi(z) := g(z) - z g'(z) \text{ for } z > 0,$$

$$f(x) := F(1, x); \quad \psi(x) := f(x) - x f'(x) \text{ for } x > 0.$$

Then

$$g(0) = 0 ; g'(z) > 0 \text{ and } g''(z) < 0 \text{ for } z > 0 ,$$

$$f(0) = 0 ; f'(x) > 0 \text{ and } f''(x) < 0 \text{ for } x > 0 .$$

It is furthermore assumed that the following holds:

$$A_3 \quad \text{a) } \lim_{z \rightarrow \infty} g(z) = \infty ; \lim_{z \rightarrow \infty} g'(z) = 0 ; \lim_{z \rightarrow 0} g'(z) = \infty$$

$$\text{b) } \lim_{z \rightarrow 0} \phi(z) = 0 ; \lim_{z \rightarrow \infty} \phi(z) = \infty ;$$

$$\text{c) } \lim_{x \rightarrow \infty} f(x) = \infty ; \lim_{x \rightarrow \infty} f'(x) = 0 ; \lim_{x \rightarrow 0} f'(x) = \infty$$

$$\text{d) } \lim_{x \rightarrow 0} \psi(x) = 0 ; \lim_{x \rightarrow \infty} \psi(x) = \infty .$$

$A_3$  states the well-known Inada conditions.

$B(t) := (K(t), C_1(t), C_2(t), V_1(t), V_2(t), I(t))$  which satisfies (2.2) - (2.4) and fulfils the continuity requirements given above, is called feasible.  $B(t)$  is called optimal when it is feasible and maximizes  $J$ .

Without proof we state

Theorem 2.1 If  $A_1 - A_3$  hold, there exists a linear function  $M(K_0)$  such that  $J(B) \leq M(K_0)$  for all feasible  $B$ . □

We next apply the Pontryagin maximum principle. Define the Hamiltonian

$$H(B(t), \lambda(t), t) := e^{-\rho t} [U_1(C_1) + U_2(C_2) - q_1 V_1 - q_2 V_2] + \lambda (F(I, V_2) - C_2)$$

and the Lagrangean

$$L(B(t), \lambda(t), \mu(t), t) := H(B(t), \lambda(t), t) + \mu(G(K, V_1) - C_1 - I) .$$

Theorem 2.2 Let  $B(t)$  be optimal. Suppose that  $K(t) > 0, V_1(t) > 0$  for all  $t \geq 0$ . Then there exist  $\lambda(t)$ , continuous with piece-wise continuous derivative and  $\mu(t)$ , piece-wise continuous, possibly discontinuous at points of discontinuity of  $B(t)$ , such that

$$e^{-\rho t} U_1'(C_1) - \mu = 0 , \tag{2.5}$$

$$e^{-\rho t} U_2'(C_2) - \lambda = 0 , \tag{2.6}$$

$$\mu G_v = q_1 e^{-\rho t} , \tag{2.7}$$

$$\dot{\lambda} = -\mu G_k , \tag{2.8}$$

$$0 = \lambda F(I, V_2) - q_2 e^{-\rho t} V_2 - \mu I \geq \lambda F(\hat{I}, \hat{V}_2) - q_2 e^{-\rho t} \hat{V}_2 - \mu \hat{I}$$

for all  $\hat{I} \geq 0, \hat{V}_2 \geq 0$ . (2.9)

Proof

This is evident. □

The next theorem deals with sufficiency.

**Theorem 2.3** Suppose there exist a feasible  $B(t)$  and  $\mu(t)$  such that (2.5) - (2.9) are satisfied. If  $\lim_{t \rightarrow \infty} \lambda(t) K(t) = 0$  then  $B(t)$  is optimal.

**Proof**

The proof is straightforward and exploits the concavity of the functions involved. □

Bearing in mind that  $\mu > 0$  and  $\lambda > 0$  (see (2.5) and (2.6)), we introduce

$$p := q_2 / e^{\rho t} \lambda ; \quad r := \mu / \lambda .$$

After some manipulation, (2.5) - (2.9) can be rewritten as

$$U_1' - q_2 \frac{r}{p} = 0 , \tag{2.10}$$

$$U_2' - q_2 \frac{1}{p} = 0 , \tag{2.11}$$

$$g'(z) = \frac{q_1}{q_2} \frac{p}{r} , \tag{2.12}$$

$$\dot{p} / p = r \phi(z) - \rho , \tag{2.13}$$

$$0 = F(I, V_2) - p V_2 - r I \geq F(\hat{I}, \hat{V}_2) - p \hat{V}_2 - r \hat{I}$$

for all  $\hat{I} \geq 0, \hat{V}_2 \geq 0$ . (2.14)

Condition (2.14) will play an important role in the subsequent analysis. It requires shadow profits to be maximal which implies, in view of the linear homogeneity of  $F$ , that in order for  $I$  and  $V_2$  to be positive,  $r$  and  $p$  should be related as follows :  $p = f'(x)$  and  $r = \psi(x)$ . Hence  $r$  and  $p$  are on a curve in  $(r, p)$  space. Elimination of  $x$  gives  $r = \alpha(p)$  with  $d\alpha/dp < 0$  and  $f(x) = \alpha(p) - p d\alpha/dp$ . The curve  $r = \alpha(p)$  is called the factor-price frontier. Clearly  $r < \alpha(p)$  is not allowed in view of (2.14). If  $r > \alpha(p)$  then  $I = V_2 = 0$ . A typical representation is given in figure 2.1.

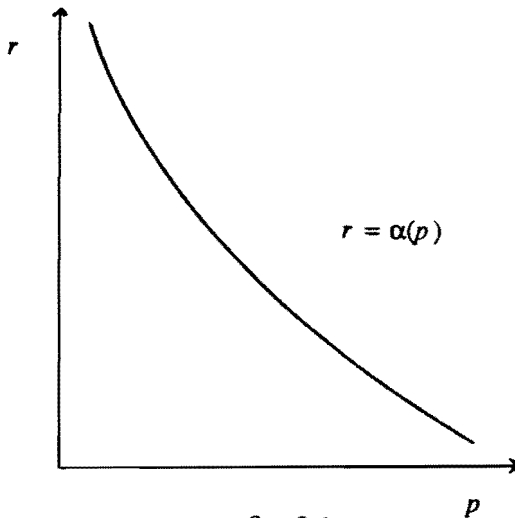


fig. 2.1.

### 3. The non-linear case, continued

Along an optimal trajectory there are essentially two possible regimes. One with  $I = V_2 = 0$ , the other with  $I > 0$  and  $V_2 > 0$ . What we wish to investigate is when each of these regimes will occur and whether switches from one regime to another are possible or not. To this end both regimes will be characterized.

Let  $p > 0$  and  $K > 0$  be given. If  $I > 0$  and  $V_2 > 0$  then the following system of equations holds. (Solutions are given by upper bars).

$$\bar{r} = \alpha(p), \quad (3.1)$$

$$U'_1(\bar{C}_1) = q_2 \frac{\alpha(p)}{p}, \quad (3.2)$$

$$U'_2(\bar{C}_2) = q_2 \frac{1}{p}, \quad (3.3)$$

$$g'(\bar{z}) = \frac{q_1}{q_2} \frac{p}{\alpha(p)}. \quad (3.4)$$

If  $I = V_2 = 0$  then the next should hold. (Solutions are given by lower bars).

$$\underline{C}_1 = K g(\underline{z}), \quad (3.5)$$

$$U'_1(\underline{C}_1) = q_2 \frac{r}{p}, \quad (3.6)$$

$$U'_2(\underline{C}_2) = q_2 \frac{1}{p}, \quad (3.7)$$

$$g'(\underline{z}) = \frac{q_1}{q_2} \frac{p}{r}. \quad (3.8)$$

By virtue of the assumption made  $\bar{r}(p)$ ,  $\bar{C}_1(p)$ ,  $\bar{C}_2(p)$ , and  $\bar{z}(p)$  are well-defined for any  $p > 0$ . It follows from (3.5), (3.6) and (3.8) that  $U'_1(K g(\underline{z})) g'(\underline{z}) = q_1$  so that  $\underline{z}(K)$  is well-defined. Then also  $\underline{C}_1(K) = K g(\underline{z}(K))$  is defined.  $\underline{C}_2(p)$  is found from (3.7) and finally  $r = p U'_1(\underline{C}_1)/q_2$ . Remark that  $\underline{C}_2 = \bar{C}_2$ . Remark also that  $\underline{z}(K)$  is decreasing,  $\underline{C}_1(K)$  is increasing,  $\bar{C}_1(p)$  and  $\bar{C}_2(p)$  are increasing and that  $\bar{z}(p)$  is decreasing.

Therefore given  $p > 0$  and  $K > 0$  both systems have a solution. This does however not imply that  $(p, K)$  leaves the regime in which the firm finds itself undetermined. On the contrary.

Define the open subsets

$$\bar{P} := \{ (p, K) >> 0 \mid K g(\bar{z}) > \bar{C}_1 \}.$$

$$\underline{P} := \{ (p, K) >> 0 \mid r > \alpha(p) \}.$$

So  $\bar{P}$  is the set of  $(p, K)$  constellations for which the solution of (3.1) - (3.4) indeed implies  $I > 0$  and  $V_2 > 0$ . And  $\underline{P}$  is the set of  $(p, K)$  constellations for which the solution of (3.5) - (3.8) implies  $r > \alpha(p)$  (and hence  $I = V_2 = 0$ ).

We now can prove



Proposition 3.1  $\bar{P}$  and  $\underline{P}$  do not intersect and for all  $(p, K) \gg 0$ ,  $(p, K) \in \bar{P}$  or  $(p, K) \in \underline{P}$  or  $(p, K) \in \partial \bar{P} \cap \partial \underline{P}$ , where  $\partial$  denotes the boundary.

Proof

Suppose  $(p, K) \in \bar{P} \cap \underline{P}$ . Since  $(p, K) \in \underline{P}$ ,  $r > \alpha(p)$  and  $g'(z) < \frac{q_1}{q_2} \frac{p}{\alpha(p)} = g'(\bar{z})$ . Hence  $z > \bar{z}$ . Also  $U_1'(\underline{C}_1) > q_2 \frac{\alpha(p)}{p}$ , whence  $\underline{C}_1 < \bar{C}_1$ . It follows that  $K g(z) = \underline{C}_1 < \bar{C}_1$  and that  $K g(z) > K g(\bar{z})$ . Hence  $\bar{C}_1 > K g(\bar{z})$  which contradicts  $(p, K) \in \bar{P}$ .

Now suppose  $(p, K) \notin \bar{P}$  and  $(p, K) \notin \underline{P}$ . Then  $K g(\bar{z}) \leq \bar{C}_1$  and  $r \leq \alpha(p)$ .

It follows from the latter inequality that  $U_1'(\bar{C}_1) \leq q_2 \frac{\alpha(p)}{p}$  and that  $g'(z) \geq \frac{q_1}{q_2} \frac{p}{\alpha(p)}$ .

Hence  $U_1'(\underline{C}_1) \leq U_1'(\bar{C}_1)$  and  $g'(z) \geq g'(\bar{z})$  implying that  $\underline{C}_1 \geq \bar{C}_1$  and  $z \leq \bar{z}$ . Since  $K g(\bar{z}) \leq \bar{C}_1$ ,  $K g(\bar{z}) \geq K g(z) = \underline{C}_1$ .

It follows that  $\bar{C}_1 = \underline{C}_1$ ,  $z = \bar{z}$ ,  $K g(\bar{z}) = \bar{C}_1$  and  $r = \alpha(p)$ . Hence  $(p, K)$  is on the boundary of  $\bar{P}$  and on the boundary of  $\underline{P}$ . □

Henceforth the common boundary of  $\bar{P}$  and  $\underline{P}$  is denoted by  $\Gamma$ . It is given by  $K = \underline{K}(p) = \bar{C}_1(p) / g(\bar{z}(p))$  and  $r = \alpha(p)$ . Remark that  $\underline{K}(p)$  is monotonically increasing. Furthermore, if  $p \rightarrow 0$  then  $\bar{C}_1(p) \rightarrow 0$  and  $g(\bar{z}(p)) \rightarrow \infty$ , whence  $\underline{K} \rightarrow 0$ . If  $p \rightarrow \infty$  then  $\underline{K} \rightarrow \infty$ . See figure 3.1 below.

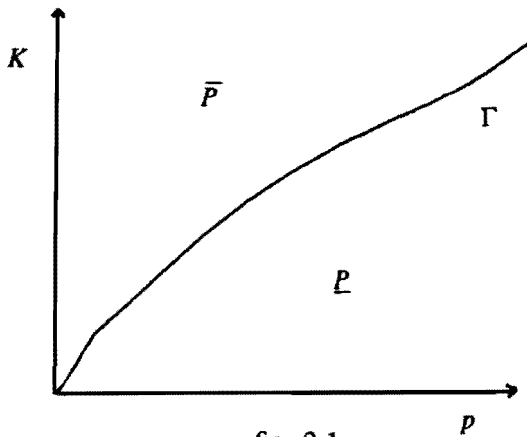


fig. 3.1.

$K_0$  is given. The question is therefore what will happen if the initial  $(p(0), K_0)$  is chosen in  $\bar{P}$  or  $\underline{P}$ . We shall first analyse  $\bar{P}$  in further detail. Suppose that  $(p(t), K(t)) \in \bar{P}$  for all  $t \geq t_0$  for some  $t_0 \geq 0$ . Then obviously  $r = \alpha(p)$  since otherwise  $(p(t), K(t)) \in \underline{P}$ . Hence the following system of differential equations should hold

$$\left. \begin{aligned} \dot{K} &= (g(\bar{z}(p)) K - \bar{C}_1(p)) f(x(p)) - \bar{C}_2(p) \\ \dot{p} &= (\alpha(p) \phi(\bar{z}(p)) - p) p \end{aligned} \right\} \quad (3.9)$$

with  $f(x(p)) = \alpha(p) - p \, d \alpha / d p$ . Remark that the second equation is autonomous in  $p$  and that  $p(t) \rightarrow p^*$  where  $p^*$  satisfies  $\alpha(p^*) \phi(\bar{z}(p^*)) = p$ .

Define

$$\bar{K}(p) = \frac{\bar{C}_1(p)}{g(\bar{z}(p))} + \frac{\bar{C}_2(p)}{g(\bar{z}(p))f(x(p))}, \quad (3.10)$$

where  $\bar{C}_1(p), \bar{C}_2(p), g(\bar{z}(p))$  are given by (3.2) - (3.4).  $(p, \bar{K}(p)) \in \bar{P}$  because  $\bar{K}(p)g(\bar{z}(p)) > \bar{C}_1$ . Therefore the curve representing  $\dot{K} = 0$  lies entirely in  $\bar{P}$ . Furthermore  $\bar{K}(p) \rightarrow 0$  as  $p \rightarrow 0$ , because then  $U'_1(\bar{C}_1) \rightarrow \infty, U'_2(\bar{C}_2) \rightarrow \infty, g'(\bar{z}(p)) \rightarrow 0, f(x) > \alpha$ , implying that  $\bar{C}_1 \rightarrow 0, \bar{C}_2 \rightarrow 0$  and  $g(\bar{z}) \rightarrow \infty$ .

We conclude that system (3.9) has a stationary point in  $\bar{P}$ . It will be denoted by  $(p^*, K^*)$ . See figure 3.2.

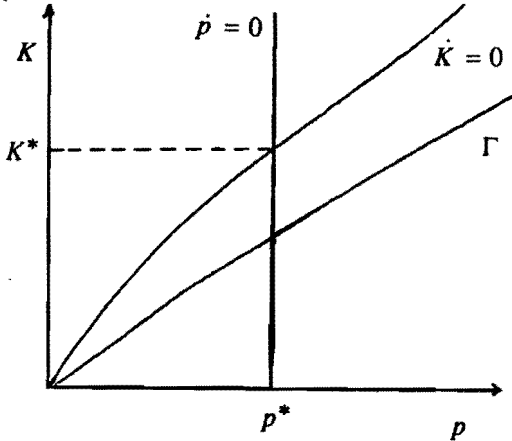


fig. 3.2.

It will be shown next that for  $K_0 \leq K^*$  there exists an optimal path, with  $I > 0$ , converging to  $(p^*, K^*)$ . For the sake of clarity the argument will be carried out in several steps. The first step is to prove that if  $(p(t), K(t))$  is optimal, and lies entirely in  $\bar{P}$  for all  $t$  greater than some  $t_0$ , it will converge to  $(p^*, K^*)$ . The second step is to show that there exists such a curve, at least for  $(p, K) \leq (p^*, K^*)$ . This curve "starts at"  $(0,0)$ .

Proposition 3.2

Let  $(p(t), K(t))$  be a curve satisfying the necessary conditions (2.2) - (2.9) and let  $(p(t), K(t)) \in \bar{P}$  for all  $t \geq t_0$ . Then  $(p(t), K(t)) \rightarrow (p^*, K^*)$  if  $(p(t), K(t))$  is optimal.

Proof

To abbreviate notation, write

$$\begin{aligned} \bar{g}(t) &:= g(\bar{z}(p(t))), \\ \bar{f}(t) &:= f(x(p(t))), \\ \bar{C}_1(t) &= \bar{C}_1(p(t)), \\ \bar{C}_2(t) &= \bar{C}_2(p(t)), \end{aligned}$$

where  $p(t)$  follows from the second part of 3.9 with a given  $p(0)$ . Hence

$$\dot{K} = \bar{g}(t)\bar{f}(t)K - \bar{C}_1(t)\bar{f}(t) - \bar{C}_2(t).$$

$$K(t) = e^{\int_0^t \bar{g}(\tau) \bar{f}(\tau) d\tau} \left( K(t_0) - \int_0^t e^{-\int_0^s \bar{g}(\tau) \bar{f}(\tau) d\tau} (\bar{C}_1(s) \bar{f}(s) + \bar{C}_2(s)) ds \right).$$

The second factor in this expression is monotonically decreasing and non-negative. Suppose it converges to a positive limit  $D$ .  $\rho = \alpha(p^*) \phi(z(p^*))$ . Hence

$$\rho = (f(x(p^*)) + p^* \frac{d\alpha}{dp}(p^*)) (g(\bar{z}(p^*)) - \bar{z}(p^*) g'(\bar{z}(p^*))).$$

Because  $d\alpha/dp < 0$ ,  $\rho < f(x(p^*)) g(\bar{z}(p^*))$ . There exists therefore  $\hat{t} > t_0$  such that for all  $t \geq \hat{t}$ ,  $\bar{g}(t) \bar{f}(t) > \rho + \epsilon$  for some  $\epsilon > 0$ . Then, for  $t \geq \hat{t} > t_0$

$$\begin{aligned} K(t) &> D e^{\int_0^t \bar{g}(\tau) \bar{f}(\tau) d\tau} - \int_0^t e^{\int_0^s \bar{g}(\tau) \bar{f}(\tau) d\tau} (\bar{C}_1(s) \bar{f}(s) + \bar{C}_2(s)) ds \\ &> D e^{\int_0^t \bar{g}(\tau) \bar{f}(\tau) d\tau} e^{(\rho + \epsilon)(t-t_0)} \\ e^{-\rho t} K(t) &> D e^{\int_0^t \bar{g}(\tau) \bar{f}(\tau) d\tau - \rho t - \epsilon t} * e^{\epsilon t}. \end{aligned}$$

Hence  $e^{-\rho t} K(t)/p(t) \rightarrow \infty$ . However, it is easily seen that for any trajectory satisfying the necessary conditions

$$J(T) = \frac{q_2}{p(0)} - \frac{q_2}{e^{\rho T} p(T)} K(T) + \int_0^T e^{-\rho t} \{U_1(C_1) - C_1 U_1'(C_1) + U_2(C_2) - C_2 U_2'(C_2)\} dt.$$

In the case at hand  $C_1$  and  $C_2$  are bounded because  $p$  is bounded, so that  $e^{-\rho t} K(t)/p(t) \rightarrow \infty$  is not optimal. The conclusion is that

$$K(t) = \int_t^\infty e^{-\int_t^s \bar{g}(\tau) \bar{f}(\tau) d\tau} (\bar{C}_1(s) \bar{f}(s) + \bar{C}_2(s)) ds.$$

It is easily seen that

$$\lim_{t \rightarrow \infty} K(t) = \frac{\bar{C}_1(p^*) f(x(p^*)) + \bar{C}_2(p^*)}{g(\bar{z}(p^*)) f(x(p^*))} = K^*. \quad \square$$

Elimination of  $t$  from the differential equations (3.9) gives the following relation between  $p$  and  $K$  along the stable path  $(p(t), K(t))$ :

$$K = \hat{K}(p) = \int_p^{p^*} e^{\int_p^w a(\pi) d\pi} b(w) dw,$$

where

$$a(p) := \frac{1}{p} \frac{g(\bar{z}(p)) f(x(p))}{\alpha(p) \phi(\bar{z}(p)) - \rho}$$

$$b(p) := \frac{1}{p} \frac{\bar{C}_1(p) f(x(p)) + \bar{C}_2(p)}{\alpha(p) \phi(\bar{z}(p)) - \rho}$$

Proposition 3.3.

1)  $\lim_{p \rightarrow 0} \hat{K}(p) = 0$

2)  $\hat{K}(p) > \underline{K}(p)$  for  $0 < p \leq p^*$  .

Proof

i) for  $0 < p < p^*$

$$a(p) \geq \frac{g(\bar{z}) f(x(p))}{p \alpha(p) g(\bar{z})} = \frac{f(x(p))}{\alpha(p) p}$$

Hence for  $0 < p_1 \leq p_2 < p^*$

$$\int_{p_1}^{p_2} a(p) dp \geq \int_{p_1}^{p_2} \frac{\alpha(p) - p \frac{d\alpha}{dp}}{p \alpha(p)} dp = \ln \frac{p_2}{p_1} \frac{\alpha(p_1)}{\alpha(p_2)}$$

Therefore

$$e^{-\int_{p_1}^{p_2} a(p) dp} \leq \frac{p_1}{p_2} \frac{\alpha(p_2)}{\alpha(p_1)}$$

ii) for  $0 < p < p^*$

$$b(p) = \frac{f(x(p))}{p \alpha(p)} \frac{\bar{C}_1 + \bar{C}_2 / f(x(p))}{\phi(\bar{z}) - \rho / \alpha(p)} = \frac{f(x(p))}{p \alpha(p)} R(p)$$

with  $R(p) \rightarrow 0$  as  $p \rightarrow 0$ , because  $\bar{C}_1 \rightarrow 0$ ,  $\bar{C}_2 \rightarrow 0$ ,  $\phi \rightarrow 0$ ,  $f(x(p)) \geq \alpha(p)$  and  $\alpha(p) \rightarrow \infty$ .

iii) with  $\bar{p}$  given ( $0 < \bar{p} < p^*$ ) and for  $0 < p < \bar{p}$

$$K(p) = \int_p^{\bar{p}} e^{-\int_p^w a(s) ds} b(w) dw + e^{-\int_p^{\bar{p}} a(s) ds} K(\bar{p})$$

iv) take some  $\epsilon > 0$ . Determine  $0 < \bar{p} < p^*$  such that

$$0 < R(p) < \frac{1}{2} \epsilon \text{ for } 0 < p \leq \bar{p}$$

and  $\bar{p} < \bar{p}$  such that

$$\frac{\alpha(\bar{p})}{\bar{p}} - \frac{p}{\alpha(p)} K(\bar{p}) < \frac{1}{2} \varepsilon \text{ for } 0 < p \leq \bar{p}.$$

v) for  $0 < p \leq \bar{p}$  we then have

$$\begin{aligned} 0 < K(p) &\leq \frac{1}{2} \varepsilon \int_p^{\bar{p}} e^{-\int_p^w a(s) ds} \frac{f(w)}{w \alpha(w)} dw + e^{-\int_p^{\bar{p}} a(s) ds} K(\bar{p}) \leq \\ &\leq \frac{1}{2} \varepsilon \int_p^{\bar{p}} \frac{p}{w} \frac{\alpha(w)}{\alpha(p)} \frac{f(w)}{w \alpha(w)} dw + \frac{p}{\alpha(p)} \frac{\alpha(\bar{p})}{\alpha(p)} K(\bar{p}) \leq \\ &\leq \frac{1}{2} \varepsilon \frac{p}{\alpha(p)} \int_p^{\bar{p}} \frac{\alpha(w) - w \frac{d\alpha}{dw}}{w^2} dw + \frac{1}{2} \varepsilon \leq \\ &\leq \frac{1}{2} \varepsilon \frac{p}{\alpha(p)} \left[ -\frac{\alpha(w)}{w} \right] \Big|_p^{\bar{p}} + \frac{1}{2} \varepsilon \leq \\ &\leq \frac{1}{2} \varepsilon \frac{p}{\alpha(p)} \left[ \frac{\alpha(p)}{p} - \frac{\alpha(\bar{p})}{\bar{p}} \right] + \frac{1}{2} \varepsilon < \varepsilon \end{aligned}$$

2) We will show that  $g(\bar{z}(p)) \hat{K}(p) / \bar{C}_1(p) > 1$ .

$$\begin{aligned} \hat{K}(p) &= \int_p^{p^*} e^{-\int_p^w a(\pi) d\pi} b(w) dw > \\ &> \int_p^{p^*} e^{-\int_p^w a(\pi) d\pi} \frac{1}{w} \frac{\bar{C}_1(w) f(x(w))}{\alpha(w) \phi(\bar{z}(w)) - \rho} dw \geq \\ &\geq \int_p^{p^*} e^{-\int_p^w a(\pi) d\pi} \frac{1}{w} \frac{\bar{C}_1(p) f(x(w))}{\alpha(w) \phi(\bar{z}(w)) - \rho} dw, \end{aligned}$$

since  $C_1$  increases as  $p$  increases. Recalling that  $\bar{z}(p)$  is decreasing, we find

$$\begin{aligned} \frac{g(\bar{z}(p)) \hat{K}(p)}{\bar{C}_1(p)} &> \int_p^{p^*} e^{-\int_p^w a(\pi) d\pi} a(w) dw = \\ &= -e^{-\int_p^w a(\pi) d\pi} \Big|_{w=p}^{w=p^*} = \end{aligned}$$

$$= 1 - e^{-\int_0^{p^*} a(\pi) d\pi} = 1 \quad \square$$

Proposition 3.4

$$\lim_{t \rightarrow \infty} \lambda(t) K(t) = 0$$

Proof

This is obvious □

The results of the previous propositions are summarized in

Theorem 3.1

For  $K_0 \leq K^*$  there exists an optimal solution. Along this solution

- 1)  $K(t)$  monotonically increases to  $K^*$ .
- 2)  $C_1(t)$  and  $C_2(t)$  monotonically increase. □

It is interesting to perform a sensitivity analysis with respect to  $\rho$ , the discount rate. The steady state  $p^*$  decreases as  $\rho$  increases. To see this observe that  $\alpha(p^*) \phi(\bar{z}(p^*)) = \rho$ ,  $d\alpha/dp < 0$  and  $d\phi/dp < 0$  since  $d\phi/d\bar{z} > 0$  ( $A_2$  and  $A_3$ ) and  $dz/dp < 0$ . So, if  $\rho$  increases the steady state rates of consumption become smaller. Furthermore the steady state capital stock becomes smaller as well because of the smaller rate of consumption  $C_1$  and the larger  $z$  (see (3.7)).

The case  $K(0) > K^*$  is less straightforward to analyse. This is so because the "stable branch"  $\hat{K}(p)$  will not lie in  $\bar{P}$  for all  $p > 0$  in general.

If it did then the problem would be solved. But there are conditions under which it does not. The consequence is that there are values of  $K(0)$  to which, along an optimal path, one should associate  $p(0)$  such that  $(p(0), K(0)) \in \underline{P}$ . Then the question is : are there appropriate starting points in  $\underline{P}$  such that eventually  $(p^*, K^*)$  is approached? The answer is in the affirmative as will be shown below. We shall proceed as follows. First we prove that, under mild conditions, the graph of  $\hat{K}(p)$  intersects  $\Gamma$  at least once. Let the first point of intersection be denoted by  $(\hat{p}, \hat{K}(\hat{p})) = (\hat{p}, \hat{K})$ . Then we consider a set of differential equations in  $\underline{P}$ , taking  $(\hat{p}, \hat{K})$  as an endpoint and prove that for any  $K(0) > \hat{K}$  there exists  $p(0)$  with  $(p(0), K(0)) \in \underline{P}$  such that  $(\hat{p}, \hat{K})$  is reached within finite time.

Proposition 3.5

Suppose that the elasticities of marginal utility  $(\eta_i(C_i))$  are bounded and that

$$\forall \varepsilon > 0 \exists \partial > 0 \left[ \frac{U'_i(C_i)}{U'_j(C_j)} < \varepsilon \Rightarrow \frac{C_i}{C_j} > \partial \right], i \neq j.$$

Then there is some  $p > p^*$  such that  $(p, \hat{K}(p)) \in \Gamma$ .

Proof

$$a) \quad \frac{U_1''(\bar{C}_1)\bar{C}_1}{U_1'(\bar{C}_1)} \frac{1}{\bar{C}_1} \frac{d\bar{C}_1}{dp} = \frac{(\frac{d\alpha}{dp} p - \alpha(p))/p^2}{\alpha(p)/p}.$$

Hence

$$\frac{d\bar{C}_1}{dp} = -\frac{\bar{C}_1}{\eta_1} \frac{f(x)}{p\alpha(p)}.$$

$$b) \quad \frac{U_2''(\bar{C}_2)\bar{C}_2}{U_2'(\bar{C}_2)} \frac{1}{\bar{C}_2} \frac{d\bar{C}_2}{dp} = -1/p$$

Hence

$$\frac{d\bar{C}_2}{dp} = -\frac{\bar{C}_2}{\eta_2 p}$$

$$c) \quad \frac{d(\rho - \alpha\phi)}{dp} = g(\bar{z}) \frac{f(x) - \alpha(p)}{p} + \frac{q_1}{q_2} \bar{z} \geq 0.$$

$\hat{K}(p)$  is the solution of  $d\hat{K}/dp = a(p)K - b(p)$ . (See the proof of proposition 3.3). Hence  $d\hat{K}/dp \leq -b(p)$  if  $p > p^*$ . Since

$$\frac{d(\rho - \alpha\phi)}{dp} > 0 \text{ for } p > p^*$$

there exists  $M_1 > 0$  such that

$$\frac{d\hat{K}}{dp} \leq \frac{M_1}{p} \frac{\bar{C}_1 f + \bar{C}_2}{\rho} = \frac{M_1}{p} \frac{1}{p} \left\{ -\eta_1 \rho \alpha \frac{d\bar{C}_1}{dp} - \eta_2 p \frac{d\bar{C}_2}{dp} \right\}$$

It has been assumed that  $\eta_1$  and  $\eta_2$  are bounded. Therefore there exists  $M > 0$  such that

$$\frac{d\hat{K}}{dp} \leq M \left[ \frac{d\bar{C}_1}{dp} + \frac{d\bar{C}_2}{dp} \right] \text{ for } p \geq p(0) > p^*.$$

It follows that

$$\hat{K}(p) - \hat{K}(p(0)) \leq M(\bar{C}_1(p) + \bar{C}_2(p)) - M(\bar{C}_1(p(0)) + \bar{C}_2(p(0))).$$

$$\frac{\hat{K}(p)}{\bar{C}_1(p)} \leq \frac{\hat{K}(p(0))}{\bar{C}_1(p)} + M + M \frac{\bar{C}_2(p)}{\bar{C}_1(p)} - M \frac{\bar{C}_1(p(0)) + \bar{C}_2(p(0))}{\bar{C}_1(p)}$$

Hence  $\hat{K}(p)/\bar{C}_1(p)$  is bounded because  $\bar{C}_1(p)$  is increasing and  $\bar{C}_2/\bar{C}_1$  is bounded. Hence

$$\lim_{p \rightarrow \infty} \frac{g(\bar{z})\hat{K}(p)}{\bar{C}_1(p)} = 0 < 1.$$

□

The situation is depicted in figure 3.3

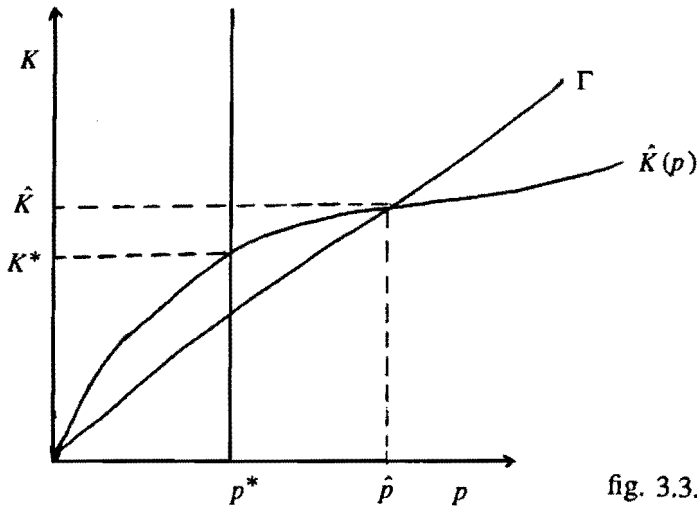


fig. 3.3.

Now consider the following system of differential equations in  $\underline{P}$  (going backwards).

$$\left. \begin{aligned} \dot{K} &= \underline{C}_2(p), K(0) = \hat{K}, \\ \dot{p} &= p \left( \rho - \frac{1}{q_2} p U'_1(\underline{C}_1) \phi(\underline{z}(K)) \right), p(0) = \hat{p} \end{aligned} \right\} \quad (3.12)$$

Let  $(p(t), \tilde{K}(t))$  be a solution of this system. Then clearly  $\tilde{K}(t)$  increasing.

It follows that along a solution  $U'_1(\underline{C}_1) \phi(\underline{z}(K))$  is decreasing (see  $A_3$ ). Whether or not  $p$  is increasing depends on the sign of  $\rho - \frac{1}{q_2} p U' \phi$ . Let us therefore consider the set  $\Delta$  of points  $(p, K)$  for which  $p = \rho q_2 / U'_1(\underline{C}_1) \phi(\underline{z}(K))$ .

**Proposition 3.6**

If  $K = \bar{C}_1(p) / g(\bar{z}(p))$  and  $p = \rho q_2 / U'_1(\underline{C}_1) \phi(\underline{z})$  then  $p = p^*$ .

**Proof**

$$p = \frac{\rho q_2}{U'_1(\underline{C}_1(K)) \phi(\underline{z}(K))} = \frac{\rho p}{\underline{r}(p, K) \phi(\underline{z}(K))}$$

$$K = \frac{\bar{C}_1(p)}{g(\bar{z}(p))} \text{ and } \underline{r}(p, K) = \alpha(p).$$

Hence

$$\alpha(p) \phi(\underline{z}(K)) = \rho = \alpha(p^*) \phi(\underline{z}(\bar{p}^*)) \text{ (see (3.9)) .}$$

$$p > p^* \Leftrightarrow \alpha(p) < \alpha(p^*) \Leftrightarrow \phi(\underline{z}(K)) > \phi(\bar{z}(\bar{p}^*)) \Leftrightarrow$$

$$\underline{z}(K) > \bar{z}(\bar{p}^*) \Leftrightarrow g'(\underline{z}(K)) < g'(\bar{z}(\bar{p}^*)) \Leftrightarrow$$



$$\frac{q_1}{q_2} \frac{p}{r(p, K)} < \frac{q_1}{q_2} \frac{p^*}{\alpha(p^*)} \Leftrightarrow \frac{p}{\alpha(p)} < \frac{p^*}{\alpha(p^*)} \Leftrightarrow p < p^* \quad \square$$

Proposition 3.5 implies that for  $p = \hat{p}$ ,  $p < q_2 \rho / U'_1(C_1) \phi(z(K))$ . We conclude that a solution of (3.12) has increasing  $p$ . Furthermore  $\bar{C}_2$  increases (because  $p$  increases). Therefore  $K \rightarrow \infty$ . See figure 3.4. Remark that for solutions of 3.12 starting below  $\Delta$  there is a finite  $t$  such that  $K(t)$  is zero.

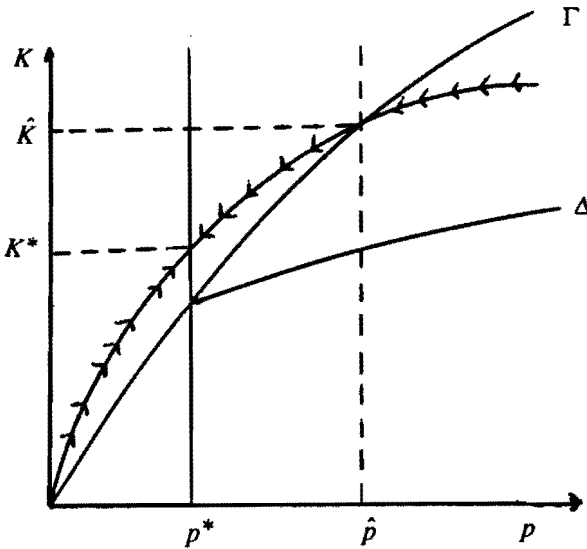


fig. 3.4.

There is one single problem that could arise, namely that the solution curve enters  $\bar{P}$  for  $(p, K) > (\hat{p}, \hat{K})$ . Indeed this possibility has not been excluded by the foregoing analysis. On the other hand no examples where this occurs have been found as yet. Obviously such a case would not pose any mathematical problem. But the main difficulty is to give an economic interpretation to such a phenomenon. Why should for some large  $K_0$  the firm start with not investing ( $I = 0$ ) and with still larger  $K_0$  start with investing ( $I > 0$ )? This remains an open question.

However, it is easy to provide conditions on the functions involved which are sufficient for the phenomenon described not to occur.

Let  $\eta_i$  be the elasticity of marginal utility with respect to  $C_i$ :  $\eta_i = -C_i U''_i(C_i) / U' C C_i$  and let  $\sigma$  be defined as  $\sigma := -z g''(z) / g'(z)$ . Straightforward calculations yield

$$\frac{\frac{dk}{dp}}{\frac{d\bar{C}_1 / g(\bar{z})}{dp}} = \frac{C_2(p)}{\bar{C}_1(p)} \frac{\alpha}{(\alpha - p \frac{d\alpha}{dp})} \frac{1}{\rho - \alpha\phi} \frac{g^2(\bar{z})}{\frac{1}{\eta_1} g(z) + \frac{1}{\sigma} z g'(\bar{z})}, \quad (3.13)$$

where  $dk / dp$  is taken along  $\Gamma$ .

Now suppose that  $\eta_1$  and  $\eta_2$  are constant and that  $\eta_1 \leq \eta_2 \leq 1$ . Suppose furthermore that  $\sigma$  is constant and smaller than unity and that  $\frac{p}{\alpha} \frac{d\alpha}{dp}$  is decreasing. Then the stable branch intersects with  $\Gamma$  and each factor of the right hand side of (3.13) is decreasing in  $p$ , the result we desire.

The central results for  $K_0 \geq K^*$  are summarized in

Theorem 3.2

For  $K_0 \geq K^*$  there exists an optimal solution. Along this solution

- 1)  $K(t)$  monotonically decreases
- 2)  $C_1(t)$  and  $C_2(t)$  monotonically decrease. □

4. The linear case

Without loss of generality  $q_2$  is put equal to unity. It is easily seen that necessary conditions for the linear case are

$$p_1 - \frac{r}{p} \leq 0, C_1(p_1 - \frac{r}{p}) = 0, \tag{4.1}$$

$$p_2 - \frac{1}{p} \leq 0, C_2(p_2 - \frac{1}{p}) = 0, \tag{4.2}$$

$$\frac{r}{p} g'(z) = q_1 \text{ as long as } K > 0, \tag{4.3}$$

$$\dot{p} / p = r\phi(z) - \rho \text{ as long as } K > 0, \tag{4.4}$$

$$0 = F(I, V_2) - p V_2 - r I \geq F(\hat{I}, \hat{V}_2) - p \hat{V}_2 - r \hat{I}, \forall (\hat{I}, \hat{V}_2) \geq 0. \tag{4.5}$$

At least for an initial interval of time  $K(t) > 0$  because of our continuity requirements with respect to  $K(t)$ . Below it will be pointed out that along an optimal trajectory the stock of capital will remain positive.

Much of the linear case can be analysed in  $(r, p)$  space.

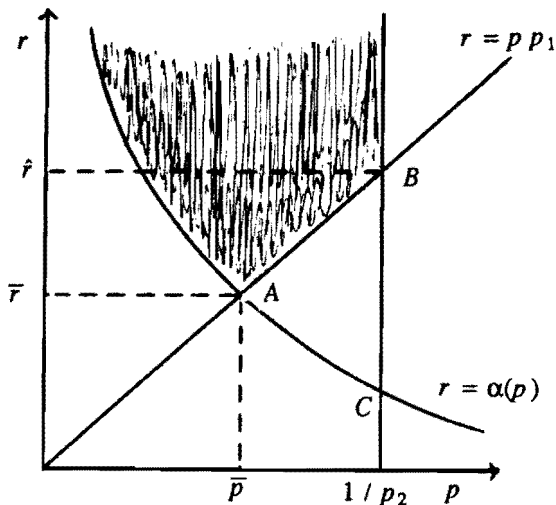


fig. 4.1.

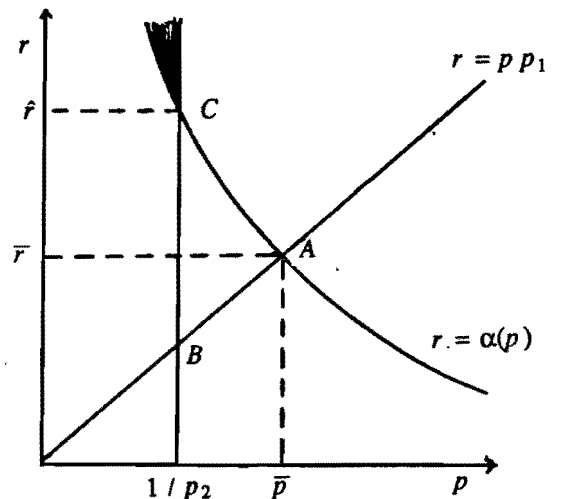


fig. 4.2.

Let  $(\bar{r}, \bar{p})$  be defined by  $\bar{r} = \alpha(\bar{p})$ ,  $\bar{r} = \bar{p}p_1$  and  $\hat{r}$  by  $\hat{r} = \alpha(1/p_2)$ . In  $(r, p)$ -space a distinction can be made between various regions

- if  $r/p > p_1$  then  $C_1 = 0$ , whereas  $r/p < p_1$  is not allowed.
- if  $p < 1/p_2$  then  $C_2 = 0$ , whereas  $p > 1/p_2$  is not allowed.
- if  $r > \alpha(p)$  then  $I = V_2 = 0$ , whereas  $r < \alpha(p)$  is not allowed.

So only the shaded regions in figures 4.1 and 4.2 should be taken into consideration as feasible  $(r, p)$  constellations. Moreover the interiors of these sets are not feasible because there  $C_1 = C_2 = I = V_2 = V_1 = 0$ . But since  $g'(0) = \infty$  profits are foregone which cannot be optimal. The half lines above  $B$  are excluded for the same reason.

We investigate the behaviour of  $p$  along  $r = \alpha(p)$ . If  $r = \alpha(p)$  then  $g'(z) \alpha(p) / p = q_1$ . Let  $z(p)$  denote the solution of this equation. Clearly  $z'(p) < 0$ . Bearing in mind that  $\phi'(z) > 0$ , we have

$$d \alpha(p) \phi(z(p)) / dp < 0 \quad (4.6)$$

Let  $\alpha(p^*) \phi(z(p^*)) = \rho$ . Then, along  $r = \alpha(p)$ ,  $p > p^* \Leftrightarrow \dot{p} < 0$ .

The behaviour of  $p$  along  $r = p p_1$  is also easily traced. Let  $g'(\hat{z}) = q_1 / p_1$ . It follows that  $p > \rho / p_1 \phi(\hat{z}) \Leftrightarrow \dot{p} > 0$ .

Now consider figure 4.1. Suppose there exists an interval of time where  $C_2(t) > 0$ . Then, according to (4.2)  $p = 1/p_2$ . If  $K(t) = 0$  then necessarily  $C_1(t) = I(t) = 0$ . Hence  $\dot{K}(t) = -C_2(t) < 0$ , a contradiction. Therefore, along the interval  $K(t) > 0$  and  $\frac{r}{p} g'(z) = q_1$  and  $r \phi(z) = \rho$ . Furthermore  $I(t) = V_2(t) = 0$  because, in figure 4.1,  $(r, p) > (\bar{r}, \bar{p})$ . Since  $K(t) > 0$  it is profitable to produce the  $C_1$ -commodity. Hence  $C_1(t) > 0$  along the interval, implying that  $\frac{r}{p} = p_1$  (4.1). We therefore end up with the following set of equations.

$$\left. \begin{aligned} p_1 g'(z) &= q_1 \\ \frac{p_1}{p_2} \phi(z) &= \rho \end{aligned} \right\} \quad (4.7)$$

It is highly unlikely that (4.7) has a solution. But if it does, then it is easily seen that an optimal trajectory is indeed characterized by  $I(t) = V_2(t) = 0$  for all  $t \geq 0$ .

But we may safely assume away the existence of a solution of (4.7). In that case  $C_2(t) = 0$  for all  $t \geq 0$ . Since  $K_0 > 0$  and  $\dot{K} = F(I, V_2) \geq 0$  we have  $K(t) > 0$  for all  $t \geq 0$  so that (4.3) and (4.4) hold along the entire optimal trajectory (if any).

Let  $\bar{z}$  denote the solution of (4.3) with  $p = \bar{p}$  and  $r = \bar{r}$ . It will turn out that the sign of  $\bar{r} \phi(\bar{z}) - \rho$  is crucial in the subsequent analysis.

If  $\bar{r} \phi(\bar{z}) > \rho$  then  $\dot{p} / p > 0$  for all  $p \geq \bar{p}$  and any trajectory along which  $p(t_1) \geq \bar{p}$  for some  $t_1$ , will necessarily enter a non-feasible region (namely north-east of  $B$ ). This implies that  $p(t) < \bar{p}$  for all  $t$ . But then  $C_1(t) = 0$  for all  $t$ , whereas  $K(t) > 0$  for all  $t$ . Hence no optimal path exists.

If  $\bar{r}\phi(\bar{z}) < \rho$  then any trajectory along which  $p(t) \leq \bar{p}$  for some  $t_1$  will necessarily have  $p(t) < \bar{p}$  for all  $t > t_1$ . But then for all  $t > t_1$   $C_1(t) = 0$ , whereas  $K(t) > 0$ , which cannot be optimal. Therefore  $p(t) > \bar{p}$  for all  $t \geq 0$ . The optimal solution is characterized by  $\dot{K}(t) = I(t) = V_2 = C_2(t) = \dot{C}_1(t) = 0$ .

Finally, if  $\bar{r}\phi(\bar{z}) = \rho$  then  $\dot{p} > 0$  for  $p > \bar{p}$ . So in this case  $p(t) \leq \bar{p}$  for all  $t$ . Now it is optimal to have  $p(t) = \bar{p}$  for all  $t \geq 0$  and the optimal trajectory looks as in the previous case.

A consideration of  $\bar{p} > 1/p_2$  is in order now. See figure 4.2.

Clearly  $C_1(t) = 0$  for all  $t$ . By the same reasoning as above the only feasible solution is found when  $\hat{r}\phi(\hat{z}) = \rho$ . In that case it is optimal to have  $\dot{K}(t) = 0$ .

Obviously the non-existence of an optimal solution is due to the linearity assumption. In fact, a very large discount rate forces the firm to get rid of the stock of capital as soon as possible.

A final remark concerns the non-existence of a solution when  $C_1$  and  $C_2$  are not restricted to be non-negative. Then one should have  $p = 1/p_2$  and  $r = p p_1$  which is a border case only.

## 5. Conclusion

The findings of the present paper can be summarized as follows.

In the case of less than full competition there exists a steady state to which the firm monotonically converges. Along the optimal trajectory both final markets are being supplied at a rate which is increasing over time when the stock of capital is increasing, and vice-versa. The steady state stock of capital is inversely related to the discount rate. Although the stock of capital is decreasing when it is large, this does not mean that the production process which provides the capital good is not carried out. On the contrary, when the firm is close to its steady state new capital goods are produced but instead of being added to the firm's capital stock they are sold on the market.

These results stand in sharp contrast to the competitive case. First of all existence of an optimal trajectory is not guaranteed. But if there exists an optimal solution it shows entirely different characteristics. It is optimal to keep the capital stock unchanged, whatever its initial value, to provide only one market, at a constant rate, and not to produce the commodity that could in principle be added to the capital stock.

## References

Dorfman, R., Application of linear programming to the theory of the firm, Berkely, University of California Press, 1951.

Ferguson, C.E., The neoclassical theory of production and distribution, Cambridge, Cambridge University Press, 1969.

Fuss, M. and McFadden, M., Production Economics: a Dual Approach to Theory and Applications, Amsterdam, North-Holland, 1978.

Hicks, J.R., Value and Capital, Oxford, Clarendon Press, 1939.

Pfouts, R.W., "The theory of cost and production in the multi-product firm", *Econometrica*, vol. 29, pp. 650 - 658, 1961.