Almost invariance and non interacting control: a frequency domain analysis

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ALMOST INVARIANCE AND NON INTERACTING CONTROL: A FREQUENCY DOMAIN ANALYSIS

by

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ABSTRACT

In this paper we shall solve a number of feedback synthesis problems in the context of non interacting control or block diagonal decoupling for finite dimensional linear time invariant systems. We shall consider a plant that, apart from a control input and a measurement output, has a given number of exogenous input vectors and the same number of exogenous output vectors. The decoupling problem that will be studied here is to find dynamic compensators from the plant measurement output (which in this paper will be assumed to be the full plant state) to the plant control input in such a way that the following requirements are met: (i) the closed loop transfer matrix is block diagonal, (ii) the remaining diagonal blocks are stable with respect to an a priori given first stability set and (iii) the closed loop system is internally stable with respect to an a priori given second, in general larger, stability set. In addition, we will study the "almost" version of the above problem. In the latter the requirement of exact decoupling will be replaced by a requirement of approximate decoupling in the sense that the to-be-designed compensators should yield off-diagonal blocks in the closed loop transfer matrix that are arbitrarily small in $H^\infty$-norm. Necessary and sufficient conditions for the existence of such dynamic compensators will be formulated in terms of controlled invariant and almost controlled invariant subspaces.
1. Introduction

This paper deals with a number of feedback synthesis problems that appear in the context of non interacting control or (block) diagonal decoupling for finite-dimensional linear time-invariant systems. Over the past twenty-five or so years a considerable number of papers on this subject have appeared in control theory literature. For excellent overviews of the existing literature we refer to [7] or [5]. The set-up in the present paper will differ fundamentally from the one that is usually considered in the literature. We want to make clear from the outset that the purpose of this paper is not to present a new contribution to the "classical" problem of non interacting control as studied in the above references, but to formulate and resolve a number of new synthesis problems in the non interacting control context. These new synthesis problems are in principle independent of the existing problem formulations. The alternative point of view towards non interacting control as adopted in the present paper was initiated in [17], where also some preliminary results concerning the synthesis problems to be considered here can be found.

Following [17], we shall consider a plant that, apart from a control input and a measurement output (which in this paper will always be assumed to be the full state of the plant), has a given number of exogenous inputs and the same number of exogenous outputs. Basically, the problem of non interacting control that will be considered here is to design a dynamic feedback compensator from the measured plant output to the plant control input in such a way that the resulting closed loop system is block diagonal, with the sizes of the blocks compatible with the a priori given dimensions of the exogenous inputs and exogenous outputs. Stated differently: it is required to design an automatic feedback mechanism in such a way that in the closed loop system the existing interaction between the exogenous variables is eliminated and to make sure that these variables influence each other only one at a time. An illustration of this set-up is given in the figure below:

![Diagram of plant and compensator](image1)

![Diagram of closed loop system](image2)

The most important feature that distinguishes the above mentioned set-up from the "classical" one is that in this formulation the exogenous inputs are specified beforehand while in the classical case it is part of the problem to design these inputs. More precisely, the classical problem of non interacting control can be roughly stated as follows: given a plant with a control input, a measurement output and a given number of exogenous outputs, design exogenous input variables, a precompensator having these exogenous inputs as input variables and finally a compensator from the measured output to the plant control input such that the closed loop system as specified in the following diagram is block diagonal:

![Diagram of precompensator and plant](image3)

![Diagram of feedback compensator and closed loop system](image4)
Additionally, in order to avoid trivialities some typical requirements on output controllability or functional reproducability of the closed loop system are imposed. Requiring both the precompensator as well as the feedback compensator to be static then yields the so-called restricted decoupling problem, RDP [18], while allowing both compensators to be dynamic yields the extended decoupling problem EDP [18] (as explained in [5]).

In our opinion both of the main problem formulations as stated above are useful in the context of non interacting control design. For some reason however the former one is highly neglected in control theory literature which, in our opinion, is rather surprising as its formulation appears to be a very natural one. In this paper we shall try to fill up this gap. We shall present an extensive treatment of the problem, including several stability issues. Moreover, the natural extension of the problem to the context of almost block diagonal decoupling will also be treated. In the latter problem the off-diagonal blocks are not required to be exactly identically equal to zero but can be made arbitrarily small in $H^\infty$-norm.

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Roughly speaking this paper is divided into two main parts, where the first part deals with the exact version of the non interacting control problem as sketched above and the second part with its almost version. The main contribution of the first part is a result that gives necessary and sufficient conditions for solvability of the (exact) non interacting control problem in a rather general formulation. Apart from block diagonal decoupling this formulation requires internal stability of the closed loop system with respect to a first stability set and at the same time input/output stability of the diagonal blocks with respect to a second, possibly smaller, stability set. The main contribution of the second part of the paper is a result that gives necessary and sufficient conditions for solvability of the 'almost' analogue of the above mentioned problem. If we take all stability sets involved to be equal to the entire complex plane, in both the exact as well as the almost version we reobtain the conditions found in [17] as special cases. (We do however note that in [17] no proof was given of the solvability conditions for the almost non interacting control problem).

The approach that will be adopted in this paper consists of a mixture of frequency domain concepts and concepts originating form the geometric approach to linear systems. An important role will be played by some typical controlled invariant and almost controlled invariant subspaces. These subspaces will be studied mainly in terms of their frequency domain characterizations, in particular in terms of $(\xi, \omega)$-representations. This concept was introduced in [4] and elaborated further in [8] and [11]. Typically in this paper, solvability conditions for the various synthesis problems will be given in terms of controlled invariant and almost controlled invariant subspaces, while the constructions of the actual dynamic compensators are based directly on the $(\xi, \omega)$-representation descriptions of these subspaces.

The paper is organized as follows. Sections 2 to 5 deal with the exact version of the non interacting control problem. In section 2 the main problem formulations are collected. In sections 3 and 4 some preliminary results with respect to these problems are derived and in section 5 the main results can be found. Sections 6 to 8 deal with the almost versions of the problems defined in section 2. These are stated in section 6. Section 7 gives some preliminary results on these problems and finally in section 8 the main results can be found. Some of the proofs are deferred to appendices A, B and C.

2. Non interacting control: problem formulation

Consider the finite-dimensional linear time-invariant system

$$
\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{k} G_i v_i(t) ,
$$

(2.1a)

$$
\sum_{i \in k} z_i(t) = D x(t) ,
$$

(2.1b)
with $x(t) \in \mathbb{R}^n =: X$ the state of the system, $u(t) \in \mathbb{R}^m =: U$ the control input, $v_i(t) \in \mathbb{R}^{q_i} =: V_i$ the $i$th exogenous input and $z_i(t) \in \mathbb{R}^{p_i} =: Z_i$ the $i$th exogenous output. $k$ is assumed to be an integer larger than 1 and the symbol $\mathbb{I}$ denotes the set \{1,2,...,k\}. In the above $A:X \rightarrow X, B:U \rightarrow X$ as well as $G_i:V_i \rightarrow X$ and $D_i:X \rightarrow Z_i$ are linear maps. As a standing assumption $B$ will be injective.

We shall be concerned with the design of dynamic compensators described by

$$
\begin{align*}
\dot{w}(t) &= Kw(t) + Lx(t) , \\
\Sigma_c \\
u(t) &= Mw(t) + Nx(t)
\end{align*}
$$

with $w(t) \in \mathbb{R}^l =: W$ the state of the compensator and $K: W \rightarrow W, L:X \rightarrow W, M:W \rightarrow U$ and $N:X \rightarrow U$ linear maps. The dimension $l$ of the state space $W$ will be denoted by $\dim \Sigma_c$. The feedback interconnection of $\Sigma$ with $\Sigma_c$ is a system with $(V_1, V_2, \ldots, V_k)$ as its input and $(Z_1, Z_2, \ldots, Z_k)$ as its output and is described by the equations

$$
\begin{align*}
\dot{x}_c(t) &= A_c x_c(t) + \sum_{i=1}^{k} G_{i,c} v_i(t) , \\
z_i(t) &= D_{i,c} x_c(t) , \quad i \in \mathbb{I}
\end{align*}
$$

where we have denoted

$$
x_c = \begin{bmatrix} x \\ w \end{bmatrix}, \quad A_c = \begin{bmatrix} A & BN \\ L & BM \end{bmatrix}, \quad G_{i,c} = \begin{bmatrix} G_i \\ 0 \end{bmatrix}, \quad D_{i,c} = \begin{bmatrix} D_i \\ 0 \end{bmatrix}.
$$

We shall denote by $T$ the transfer matrix of the closed loop system (2.3). $T$ is equal to the composite matrix $(T_{ij})$, where

$$
T_{ij}(s) = D_{j,c}(Js - A_c)^{-1} G_{i,c} , \quad i,j \in \mathbb{I}
$$

represents the transfer matrix between the $i$th input $v_i$ and the $j$th output $z_j$. In [17] the following problem was introduced:

**PROBLEM I (Non interacting control).** Problem I is said to be solvable if there exists a compensator $\Sigma_c$ such that $T_{ij} = 0$ for all $i,j \in \mathbb{I}$ with $i \neq j$.

If a compensator $\Sigma_c$ is such that $T_{ij} = 0$ for all $i \neq j$ then it will be said to achieve non interaction. In that case the resulting closed loop transfer matrix is block diagonal:

$$
T = \text{blockdiag} \left( T_{11}, \ldots, T_{kk} \right).
$$

An important issue here will be stability. In the sequel a subset $\mathcal{C}_g$ of $\mathcal{C}$ will be called symmetric if $\mathcal{C}_g \cap \mathcal{R} \neq \emptyset$ and if it satisfies $\lambda \in \mathcal{C}_g \iff \overline{\lambda} \in \mathcal{C}_g$. A rational matrix will be called $\mathcal{C}_g$-stable (or $g$-stable) if its poles lie in $\mathcal{C}_g$. If, apart from non interaction, we require input/output stability of the closed loop transfer matrix from $(v_1, v_2, \ldots, v_k)$ to $(z_1, z_2, \ldots, z_k)$ we arrive at the following problem:

**PROBLEM II (Non interacting control with i/o-stability).** Given a symmetric subset $\mathcal{C}_g$ of $\mathcal{C}$, problem II is said to be solvable if there exists a compensator $\Sigma_c$ that achieves non interaction such that $T_{ii}$ is $g$-stable for all $i \in \mathbb{I}$.

A different stability issue is that of internal stability of the closed loop system. Of course, if we succeed in finding a dynamic compensator that achieves non interaction with i/o-
stability, due to the presence of uncontrollable and unobservable modes this does in general not mean that the closed loop system is internally stable (in the sense that $\sigma(A_x) \subseteq \mathcal{C}_g$).

PROBLEM III (Non interacting control with internal stability). Given a symmetric subset $\mathcal{C}_g$ of $\mathcal{C}$, problem III is said to be solvable if there exists a compensator $\Sigma_c$ that achieves non interaction such that $\sigma(A_x) \subseteq \mathcal{C}_g$.

In this paper, as it should, input/output stability and internal stability will be treated as two different requirements. Correspondingly we will specify two stability sets $\mathcal{C}_f$ and $\mathcal{C}_s$. Combining the two notions we shall require that the decoupled system is i/o-stable with respect to the stability set $\mathcal{C}_f$ and internally stable with respect to the stability set $\mathcal{C}_s$. Typically, this corresponds to requiring a fast response of the to-be-controlled output variables $z_1, z_2, \ldots, z_k$ and allowing a slower response of the internal part of the system (see also [4J or [10]). Formalizing this we arrive at the following version of the non interacting control problem:

PROBLEM IV (Non interacting control with i/o and internal stability). Given two symmetric subsets $\mathcal{C}_f \subseteq \mathcal{C}_g$ of $\mathcal{C}$, problem IV is said to be solvable if there exists a compensator $\Sigma_c$ that achieves non interaction such that $\mathcal{T}_{ii}$ is $f$-stable for all $i \in k$ and $\sigma(A_x) \subseteq \mathcal{C}_s$.

It is our purpose to establish necessary and sufficient conditions for the solvability of the above four problems that can be checked constructively. Clearly, once we have established these for problem IV we are done, since problems II and III may be obtained as special cases of problem IV by taking $\mathcal{C}_f = \mathcal{C}_g$, $\mathcal{C}_s = \mathcal{C}$ and $\mathcal{C}_f = \mathcal{C}_s = \mathcal{C}_g$ respectively. Obviously, problem I requires only $\mathcal{C}_f = \mathcal{C}_s = \mathcal{C}$. Thus, instead of considering each problem separately, we shall concentrate on deriving conditions for solvability of IV.

3. Some geometric concepts

Given a system $(\mathcal{A}, \mathcal{B})$ with state space $X$ and a subspace $K$ of $X$ we shall denote by $V^*(K)$ the supremal controlled invariant subspace contained in $K$. If $\mathcal{C}_g$ is a symmetric subset of $\mathcal{C}$ then $V_g^*(K)$ will denote the supremal stabilizability subspace in $K$, [4]. If instead of one we specify two stability sets $\mathcal{C}_f$ and $\mathcal{C}_s$ then $V_f^*(K)$ and $V_s^*(K)$ will denote the supremal stabilizability subspace with respect to $\mathcal{C}_f$ and $\mathcal{C}_s$ respectively. The system $(\mathcal{A}, \mathcal{B})$ will be called $g$-stabilizable ($s$-stabilizable) if it is stabilizable with respect to $\mathcal{C}_g$ ($\mathcal{C}_s$). A similar terminology will be used in the context of detectability.

If $\xi$ and $\omega$ are an $n$-vector and an $m$-vector of real rational functions respectively and if $x_0 \in X$ then the expression

$$x_0 = (Is - A)\xi(s) - B\omega(s) \quad (3.1)$$

will be called a $(\xi, \omega)$-representation of $x_0$. A $(\xi, \omega)$-representation will be called regular if both $\xi$ and $\omega$ are strictly proper. Given a symmetric subset $\mathcal{C}_g$ of $\mathcal{C}$ a $(\xi, \omega)$-representation will be called g-stable if $\xi$ is $g$-stable. Assuming that $B$ is injective this implies that also $\omega$ is $g$-stable. The notion of $(\xi, \omega)$-representation can be used to give frequency domain characterizations of the various controlled invariant and almost controlled invariant subspaces appearing in the literature on the geometric approach to linear systems (see e.g. [4], [8] or [11]). In particular, if $\mathcal{C}_g$ is a symmetric subset of $\mathcal{C}$ and if $K$ is a subspace of $X$ such that $K = \text{ker} \mathcal{H}$ then we have

$$V_g^*(K) = \{x_0 \in X_0 \mid x_0 \text{ has a regular } g \text{-stable } (\xi, \omega)\text{-representation with} \quad (3.2)$$

$$\mathcal{H}\xi = 0\}.$$
In our considerations on non interacting control with i/o and internal stability an important role will be played by controlled invariant subspaces that, instead of on one subspace $K$ and one stability set $E$, rather depend on a pair of subspaces and a pair of stability sets. In the following, let $K_1$ and $K_2$ be subspaces of $X$ such that $K_2 \subseteq K_1$. Let $H_1$ and $H_2$ be linear maps such that $K_i = \ker H_i$. Let $C_f \subseteq C_g$ be symmetric subsets of $C$. The following definition slightly generalizes [4, def. 4.2]:

**DEFINITION 3.1**

$$V_{f,s}(K_1, K_2) := \{ x_0 \in X \mid x_0 \text{ has a regular } s\text{-stable } (\xi, \omega)\text{-representation with } H_1 \xi = 0 \text{ and } H_2 \xi f\text{-stable} \}.$$  

Clearly, this defines a linear subspace of $X$ which, by (3.2), is contained in $V^*_s(K_1)$, the supremal $s$-stabilizability subspace in $K_1$. Using the fact that $K_2 \subseteq K_1$ it is also clear that the above defined subspace contains $V^*_s(K_2)$, the supremal $f$-stabilizability subspace in $K_2$. We note that if in the above we take $H_1 = 0$ and $H_2 = H$ then $V_{f,s}(K_1, K_2) = V_{f,s}(X, \ker H)$, the subspace of all $x_0 \in X$ having a regular $s$-stable $(\xi, \omega)$-representation with $H_1 \xi f$-stable. The latter coincides with the subspace $S_{12}$ as studied in [4, p. 707]. In fact, by slightly adapting the proof of [4, th 4.3], we arrive at the following representation of $V_{f,s}(K_1, K_2)$ (see also [12, th. 4.5]):

**THEOREM 3.2**

$$V_{f,s}(K_1, K_2) = V^*_s(K_1) + V^*_s(K_2).$$

In the considerations on non interacting control to come the following observation will turn out to be instrumental:

**LEMMA 3.3** Let $E$ be a linear subspace of $X$ and let $E$ be a linear map such that $\text{im } E = E$. Then $(A, B)$ is $s$-stabilizable and $E \subseteq V_{f,s}(K_1, K_2)$ if and only if there exist strictly proper $s$-stable real rational matrices $X$ and $U$ such that $(Is - A) X(s) - BU(s) = I$, $H_1 XE = 0$ and $H_2 XE f$-stable.

**PROOF** $(\Rightarrow)$ Without loss of generality, assume that $E$ is injective. Let $E : V \to X$, with $V = \mathbb{R}^q$. By applying the definition of $V_{f,s}$ to the vectors $Ee_i$, where $e_i$ is the $i$th column of the $q \times q$ identity matrix, we find strictly proper $s$-stable real rational matrices $X_1$ and $U_1$ such that $E = (Is - A) X_1(s) - BU_1(s)$, $H_1 X_1 = 0$ and $H_2 X_1 f$-stable. Let $E'$ be a linear map such that $(E')'$ is bijective. Since $(A, B)$ is $s$-stabilizable, there are strictly proper $s$-stable real rational matrices $X_2$ and $U_2$ such that $E' = (Is - A) X_2(s) - BU_2(s)$ (see [4, cor. 2.20]. Now define $X := (X_1 X_2)(E')^{-1}$ and $U := (U_1 U_2)(E')^{-1}$. Then $(Is - A) X(s) - BU(s) = I$. Moreover, $H_1 XE = H_1 X_1 = 0$, $H_2 XE = H_2 X_1$ is $f$-stable and $X$ and $U$ are $s$-stable and strictly proper. $(\Leftarrow)$ Let $Ev \in E$. By taking $\xi := XEv$ and $\omega := UEv$ we obtain a $(\xi, \omega)$-representation of $Ev$ with the properties required in DEF. 3.1. The fact that $(A, B)$ is $s$-stabilizable follows from [4, cor. 2.20].

Note that by taking $C_f = C_g$ and $K_1 = K = \ker H$ in the above lemma we find that $(A, B)$ is $s$-stabilizable and $E \subseteq V^*_s(K)$ if and only if there exist strictly proper $s$-stable real rational matrices $X$ and $U$ such that $(Is - A) X(s) - BU(s) = I$ and $HXE = 0$. Finally note that we do not need TH. 3.2 in the proof of LEMMA 3.3. The importance of TH. 3.2 is that it enables us to calculate explicitly the subspace $V_{f,s}(K_1, K_2)$, for example by using a construc-
tion as [18, p. 114].

4. Input/output description of internal stability

One of the requirements that should be met by the to-be-designed compensator $\Sigma_c$ is that the closed loop system is internally stable, i.e. that $\sigma(\Delta_c) \subseteq \mathcal{C}_s$. In this section we will see that this requirement has an equivalent formulation in terms of certain transfer matrices associated with the closed loop system. Using this fact we shall show that every pair of strictly proper $s$-stable real rational matrices $(X,U)$ such that $(\mathcal{I}s - A)X(s) - BU(s) = I$ gives rise to an $s$-stabilizing dynamic compensator $U(s)X(s)^{-1}$.

Consider the system $(A,B)$ and let its input to state transfer matrix be given by

$$P(s) := (\mathcal{I}s - A)^{-1}B \quad .$$

Let the transfer matrix of the compensator (2.2) be given by

$$F(s) := N + M(\mathcal{I}s - K)^{-1}L \quad .$$

The following result states that the internal stability of the closed system formed by interconnecting the system $\dot{x} = Ax + Bu$ with the compensator (2.2) can be characterized in terms of expressions involving their transfer matrices:

**LEMMA 4.1** Assume that $(A,B)$ is $s$-stabilizable, $(K,L)$ is $s$-stabilizable and $(M,K)$ is $s$-detectable. Then we have

$$\sigma \left[ \begin{array}{cc} A + BN & BM \\ L & K \end{array} \right] \subseteq \mathcal{C}_s$$

if and only if $(\mathcal{I} - FP)^{-1}P$, $(\mathcal{I} - FP)^{-1}PF$, $(\mathcal{I} - FP)^{-1}F$ and $(\mathcal{I} - FP)^{-1}FP$ are $s$-stable.

**PROOF** A proof of this can be found in [14, p. 103].

The latter result will be very important to us. It means that once we have a $s$-stabilizable and $s$-detectable candidate compensator (2.2) for one of the (almost) non interacting control problems we want to solve, we can check whether is makes the closed loop system internally $s$-stable simply by checking whether the four transfer matrices appearing in LEMMA 4.1 are $s$-stable. The typical compensator construction that will be used in this paper is the following. Suppose that $(A,B)$ is $s$-stabilizable and let $X$ and $U$ be $s$-stable strictly proper real rational matrices satisfying

$$(\mathcal{I}s - A)X(s) - BU(s) = I \quad .$$

Since $sX(s) = I + BU(s) + AX(s)$, which is bicausal, the rational matrix $sX(s)$ has a proper inverse, say $L(s)$. Consequently also $X(s)$ is invertible with $X(s)^{-1} = sL(s)$ (not necessarily proper). Now define

$$F(s) := U(s)X(s)^{-1} \quad .$$

We claim that $F$ is proper. Indeed, this is easy since $F(s) = sU(s)L(s)$ with $sU(s)$ proper and $L(s)$ proper. We contend that $F(s)$ is the transfer matrix of a stabilizing compensator:

**LEMMA 4.2** Assume that $(A,B)$ is $s$-stabilizable. Let $X$ and $U$ be strictly proper $s$-stable real rational matrices such that $(\mathcal{I}s - A)X(s) - BU(s) = I$. Let $F(s) := U(s)X(s)^{-1}$. Then for every realization $N + M(\mathcal{I}s - K)^{-1}L$ of $F(s)$ such that $(K,L)$ $s$-stabilizable and $(M,K)$ $s$-detectable we have
By straightforward calculation it can be seen that

\[(I-P(s)F(s))^{-1}P(s) = X(s)B\]
\[(I-P(s)F(s))^{-1}P(s)F(s) = X(s)(Is-A) - I\]
\[(I-F(s)P(s))^{-1}F(s) = U(s)(Is-A)\]

and finally that
\[(I-F(s)P(s))^{-1}F(s)P(s) = U(s)B\]

Since all these are \(s\)-stable the conclusion follows from LEMMA 4.1.

5. Non-interacting control: main results

In this section we shall formulate and prove necessary and sufficient conditions for solvability of problem IV in terms of the subspaces that we considered in section 3. Subsequently, as corollaries we shall state conditions for solvability of problems I, II and III.

Before starting off, we shall introduce one more important concept, the concept of radial, [18]. Given a finite collection \(\{L_i \mid i \in \mathbb{R}\}\) of subspaces of a linear space \(X\), its radical is defined as the subspace

\[L_0 := \bigcap_{i=1}^{r} L_i \cap \bigcup_{j \neq i}^{r} L_j\]

The collection \(\{L_i \mid i \in \mathbb{R}\}\) is said to be independent if the radical \(L_0\) is equal to \(\{0\}\). For an extensive discussion on the various properties of the radical and its application to the "extended decoupling problem" we refer to [18]. In the sequel we shall make use of the following lemma:

**LEMMA 5.1** Let \(\{L_i \mid i \in \mathbb{R}\}\) be a collection of subspaces of \(X\). Let \(\tilde{L}_i \subset X\) be subspaces such that

\[L_0 \oplus \tilde{L}_i = L_0 + L_i, \quad i \in \mathbb{R}\]

Then the collection \(\{L_0, \tilde{L}_i \mid i \in \mathbb{R}\}\) is independent.

**PROOF** For a proof of this lemma we refer to App. A

Now consider the to-be-controlled system (2.1). Denote im \(G_i\) by \(F_i\). The radical of the collection \(\{G_i \mid i \in \mathbb{k}\}\) will be denoted by \(G_0\). Furthermore, define

\[K := \bigcap_{j=1}^{k} \ker D_j, \quad K_i := \bigcap_{j \neq i} \ker D_j, \quad i \in \mathbb{k}\]

The following theorem is the main result of this section:
THEOREM 5.2 Problem IV is solvable if and only if \((A, B)\) is \(s\)-stabilizable,

\[
G_i \subseteq V_{f,s}(K_i, K) \text{ for all } i \in k
\]  

(5.4)

and

\[
G_0 \subseteq V_{s}(K).
\]  

(5.5)

In order to establish a proof of this result we shall proceed as follows. Define

\[
M_j(s) := D_j((sI - A)^{-1}B)
\]

\[
N_i(s) := (sI - A)^{-1}G_i
\]

and

\[
W_{ij}(s) := D_j((sI - A)^{-1}G_i).
\]

the open loop transfer matrices form \(u\) to \(z_j\), \(v_i\) to \(x\) and \(v_i\) to \(z_j\) respectively. Given a dynamic compensator (2.2) denote its transfer matrix by \(F\). Let \(P\) be defined by (4.1). A straightforward calculation shows that the closed loop transfer matrix from \(v_i\) to \(z_j\) (as defined by (2.4)) is equal to

\[
T_{ij} = W_{ij} + M_j((I - FP)^{-1}FN_i)
\]

and thus the solvability of problem IV is equivalent to the existence of a proper real rational \(F\) such that \(W_{ij} + M_j((I - FP)^{-1}FN_i) = 0\) for all \(i, j \in k\) with \(i \neq j\), such that \(W_{ii} + M_i((I - FP)^{-1}FN_i)\) is \(f\)-stable for all \(i \in k\) and such that the closed loop system is internally \(s\)-stable. This brings us in a position to establish a proof of TH.5.2.

PROOF OF THEOREM 5.2 \((\Leftarrow)\) Assume that \((A, B)\) is \(s\)-stabilizable and that (5.4) and (5.5) hold. By LEMMA 3.3 for all \(i\) there exist \(s\)-stable strictly proper real rational matrices \(X_i, U_i\) such that

\[
(sI - A)X_i(s) - BU_i(s) = I,
\]

(5.7)

and

\[
D_jX_iG_i = 0 \text{ for all } j \neq i
\]

(5.8)

Moreover, there exist \(s\)-stable strictly proper real rational matrices \(X_0\) and \(U_0\) such that

\[
(sI - A)X_0(s) - BU_0(s) = I
\]

and

\[
D_jX_0G_0 = 0 \text{ for all } j.
\]

(5.9)

Here, \(G_0\) is any linear map such that \(G_0 = \text{im } G_0\). Now, we shall construct \(s\)-stable strictly proper real rational matrices \(X\) and \(U\) such that

\[
(sI - A)X(s) - BU(s) = I,
\]

(5.10)

and

\[
D_jXG_i = 0 \text{ for all } i, j \text{ with } i \neq j
\]

(5.11)

After doing this we shall argue that any \(s\)-stabilizable and \(s\)-detectable realization of the proper rational matrix \(U(s)X(s)\) gives us a compensator which establishes the requirements of problem IV.

Our construction will be based on a suitable direct sum decomposition of the state space \(X\). For \(i \in k\) let \(G_i \subseteq G_i\) be subspaces such that \(G_i \oplus G_0 = G_i + G_0\). According to
LEMMA 5.1 the collection of subspaces \( \{ G_0, G_i \mid i \in k \} \) is independent. Let \( G_{k+1} \) be any complement of \( G_0 + \sum_{i=1}^{k} G_i \) in \( X \). This yields a direct sum decomposition

\[
X = G_0 \oplus G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus G_{k+1} .
\]  

(5.12)

Let \( P_0: X \rightarrow X \) be the projector onto \( G_0 \) along \( \sum_{i=1}^{k} G_i + G_{k+1} \), let \( P_i \) be the projector onto \( G_i \) along the other members of the decomposition (5.12) and let \( P_{k+1} \) be the projector onto \( G_{k+1} \) along \( G_0 + \sum_{i=1}^{k} G_i \). Note that \( \sum_{i=1}^{k} P_i = I \) and that \( G_i \subset \ker P_j \cap \ker P_{k+1} \) for all \( i, j \in k \) with \( i \neq j \).

Now let \( X_{k+1}, U_{k+1} \) be any pair of \( s \)-stable strictly proper real rational matrices satisfying \((ls-A)r_{k+1}(s) - BU_{k+1}(s) = I\) and define

\[
X := \sum_{i=0}^{k+1} X_i P_i, \quad U := \sum_{i=0}^{k+1} U_i P_i .
\]

Then we have

\[
(ls - A)X(s) - BU(s) = \sum_{i=0}^{k+1} ((ls - A)X_i(s) - BU_i(s)) P_i = I ,
\]

which shows that (4.3) holds for the \( X, U \) defined by (5.13). Moreover, for all \( i, j \in k \) we have

\[
D_j X G_i = D_j \left( \sum_{i=0}^{k+1} X_i P_i \right) G_i = D_j X_0 P_0 G_i + D_j X_i P_i G_i .
\]

(5.14)

Now note that, since \( \im P_0 G_i \subset G_0 \), the first term on the right of (5.14) vanishes. Thus, since \( \im P_i G_i \subset G_i \), we conclude from (5.7) and (5.8) that \( D_j X G_i = 0 \) for \( i \neq j \) and \( D_j X G_i \) is \( f \)-stable. Define \( F := UX^{-1} \).

As already noted in the proof of LEMMA 4.2 we then have

\[
U(s) = (I-F(s)P(s))^{-1}F(s)(ls-A)^{-1} .
\]

Also, from \((ls-A)X(s) - BU(s) = I\), we have

\[
X(s) = (ls-A)^{-1} + (ls-A)^{-1}BU(s) .
\]

Combining these two expressions we immediately obtain

\[
D_j X G_i = W_{ij} + M_j (l-FP)^{-1}FN_i = T_{ij} .
\]

We conclude that any realization \( N + M (ls-K)^{-1} L \) of \( X(s)U(s)^{-1} \) defines a compensator that achieves non interaction with \( i/o \) \( f \)-stability. Finally, since \( X \) and \( U \) are \( s \)-stable, it follows from LEMMA 4.2 that in addition any such realization with \((K, L)\) \( s \)-stabilizable and \((M, K)\) \( s \)-detectable yields an internally \( s \)-stable closed loop system.

\( \Rightarrow \) Assume problem IV is solvable, i.e. there exists a compensator (2.2) such that \( D_{j,\alpha}(ls-A_{\alpha})^{-1}G_{i,\alpha} = 0 \) for all \( i, j \) with \( i \neq j \). \( D_{i,\alpha}(ls-A_{\alpha})^{-1}G_{i,\alpha} \) is \( f \)-stable for all \( i \) and \( \sigma(A_{\alpha}) \subset \mathcal{C}_s \). We will first show that \( G_i \subset V_{f,\alpha}(K_i, K) \). For every \( v \in V_i \) define

\[
\begin{bmatrix}
\xi_v(s) \\
\nu_v(s)
\end{bmatrix} := (ls-A_{\alpha})^{-1} \begin{bmatrix}
G_i \\
0
\end{bmatrix} v .
\]

(5.15)

Then it is immediate that \( G_i v = (ls-A)\xi_v(s) - B(Mv(s) + N\xi(s)) \), which is a regular \( \alpha \)-stable\((\xi_v, w)\)-representation of \( G_i v \). Since also \( D_j \xi = 0 \) for all \( j \neq i \) and \( D_j \xi \) is \( f \)-stable (so consequently \( D_j \xi \) is \( f \)-stable for all \( j \)), we conclude that \( G_i v \in V_{f,\alpha}(K_i, K) \).
Next, we shall prove that the radical $G_0 \subseteq V^*_s(K)$. By (5.1), it suffices to show that $G_i \cap \bigoplus_{j \neq i} G_j \subseteq V^*_s(K)$ for all $i$. Let $G_i = \sum G_j w_j$. We contend that $D_{j,e} (Is-A_e)^{-1} G_{i,e} w_j = 0$ for all $j$. Now, for $j \neq i$ this is immediate. On the other hand,

$$D_{i,e} (Is-A_e)^{-1} G_{i,e} w_j = \sum_{j \neq i} D_{i,e} (Is-A_e)^{-1} G_{j,e} w_j = 0,$$

which proves our claim. Now define $\xi$ and $\nu$ by (5.15). Then $G_i \nu = (Is-A) \xi(s) - B (M \nu(s) + N \xi(s))$, which is a regular $s$-stable $(\xi, \nu)$-representation of $G_i \nu$ such that $D_{j,e} \xi = 0$ for all $j$. It follows that $G_i \nu \in V^*_s(K)$.

Finally, we will show that $(A, B)$ is $s$-stabilizable. Define

$$\begin{bmatrix} X(s) \\ L(s) \end{bmatrix} := (Is-A)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$ 

Then $I = (Is-A)X(s) - B (ML(s) + NX(s))$. Since $X$ and $ML + NX$ are $s$-stable and strictly proper, the conclusion follows from [4, cor. 2.20]. This completes the proof of TH.5.2

We stress that, since $V_{f,s}(K_i, K) = V^*_s(K_i) + V^*_s(K)$ (see TH.3.2), the conditions established above can indeed be checked constructively. An actual check would involve the calculation of $f$ $f$-stabilizability subspaces $V^*_s(K_i)$ and of the $s$-stabilizability subspace $V^*_s(K)$. A conceptual algorithm for this is described in [18, p. 114].

**REMARK 5.3** A few words on the dynamic order of the compensator as constructed in the proof of TH.5.2 are at order here. Using the fact that $X$ and $U$ are related by $(Is-A)X(s) - B(U(s))$ it may be shown that the McMillan degrees of the compensator $UX^{-1}$ and the rational matrix $X$ respectively are related by $\deg(UX^{-1}) \leq \deg(X)$ (see APP.B, LEMMA B.1). Thus, in order to obtain an upper bound to the McMillan degree of the compensator, it is interesting to obtain such bound for $X$. Now, it can be shown that the $X_i$ in terms of which $X$ is defined (see (5.13)) can in fact be constructed in such a way that we have $\deg(X_i P_i) \leq \dim V_{f,s}(K_i, K)$ ($i \in k$) and such that $\deg(X_0 P_0 + X_{k+1} P_{k+1}) \leq n$ ($= \dim X$). Consequently, if problem IV is solvable then a required compensator (2.2) exists with dynamic order satisfying

$$\dim W \leq n + \sum_{i=1}^k \dim V_{f,s}(K_i, K). \quad (5.16)$$

Note that this upper bound increases as the number of to-be-decoupled input/output channels increases.

As already noted in section 2, the main theorem TH.5.2 immediately provides necessary and sufficient conditions for solvability of the simpler problems I, II and III:

**COROLLARY 5.4**

(i) Problem III is solvable if and only if $(A, B)$ is $g$-stabilizable, $G_i \subseteq V^*_g(K_i)$ for all $i \in k$ and $G_0 \subseteq V^*_g(K)$.

(ii) Problem II is solvable if and only if $G_i \subseteq V^*_g(K_i) + V^*(K)$ for all $i \in k$ and $G_0 \subseteq V^*(K)$.

(iii) Problem I is solvable if and only if $G_i \subseteq V^*(K)$ for all $i \in k$ and $G_0 \subseteq V^*(K)$.  

We conclude this section by noting that in certain situations it is desirable instead of a proper compensator to design a strictly proper compensator $\Sigma$ that achieves non-interaction. Indeed, using the methods developed here it can for example be shown that there exists a
compensator (2.2) with $N=0$ such that $T_{ij} = 0$ for $i \neq j$ if and only if $G_i + AG_i \subset V^*(K_i)$ for $i \in k$ and $G_0 + AG_0 \subset V^*(K)$.

6. Almost non interacting control: problem formulation

If instead of requiring the off-diagonal blocks in the closed loop transfer matrix to be exactly equal to zero we only require these blocks to be arbitrarily small in some appropriate norm, we arrive at problems in the context of approximate or 'almost' non interacting control. In the present section we shall formulate the 'almost' analogues of the synthesis problems we studied in the foregoing. In the following the magnitudes of the closed loop transfer matrices involved will always be measured in $H^\infty$-norm. Let $C^-$ denote $\{s \in C \mid \text{Re} s < 0\}$. Given a $C^-$-stable proper real rational $p \times m$ matrix $W$ its $H^\infty$-norm is defined as

$$\| W \|_\infty := \sup_{\text{Re} s \geq 0} \| W(s) \|.$$

Here, for $s \in C$, $\| W(s) \|$ denotes the operator norm of the complex matrix $W(s)$ considered as a linear map from $C^n$ to $C^n$ or, equivalently, the largest singular value of the complex matrix $W(s)$. For more details we refer to [14] or [3].

Consider the system (2.1). If we require the off-diagonal blocks in the closed loop transfer matrix to have arbitrarily small $H^\infty$-norm we arrive at: for all $\varepsilon > 0$ determine a compensator (2.2) such that $\| T_{ij} \|_\infty \leq \varepsilon$ for all $i \neq j$. In this form it is required for all $\varepsilon > 0$ to find a suitable compensator state space $W_e$ together with suitable linear maps $K_e, L_e, M_e$ and $N_e$. Now, in practice one would like to exclude the possibility that as $\varepsilon$ becomes smaller and smaller (i.e. as the decoupling accuracy increases), the dynamic order $\text{dim} W_e$ of the compensator increases unboundedly. Therefore, we shall require not only that the off-diagonal blocks in the closed loop transfer matrix can be made arbitrarily small, but in addition that this can be done without having to increase the dynamic order of the required compensators unboundedly. In this way, denoting the dynamic order $\text{dim} W_e$ of the compensator (2.2) by $\text{dim} \Sigma_c$, we arrive at the following formulation:

**PROBLEM V** (Almost non interacting control). Problem V is said to be solvable if there exists an integer $N$ and if for all $\varepsilon > 0$ there exists a compensator $\Sigma_c$ with $\text{dim} \Sigma_c \leq N$ such that $\| T_{ij} \|_\infty \leq \varepsilon$ for all $i, j \in k$ with $i \neq j$.

If apart from approximate non interaction up to any desired degree of accuracy we require input/output stability of the closed loop system with respect to a given stability set $C_8$, we arrive at:

**PROBLEM VI** (Almost non interacting control with i/o-stability). Given a symmetric subset $C_8$ of $C$, problem VI is said to be solvable if there exists an integer $N$ and if for all $\varepsilon > 0$ there exists a compensator $\Sigma_c$ with $\text{dim} \Sigma_c \leq N$ such that $\| T_{ij} \|_\infty \leq \varepsilon$ for all $i, j \in k$ with $i \neq j$ and such that $T_{ij}$ is $g$-stable for all $i, j \in k$.

Note that by requiring $\| T_{ij} \|_\infty \leq \varepsilon$ for $i \neq j$ it is of course already implicitly assumed that $T_{ij}$ is $C^-$-stable for $i \neq j$. Thus, in the particular case that in the above we take $C_8$ equal to $C^-$ the requirement "$T_{ij}$ is $g$-stable for all $i, j \in k$" can be replaced by "$T_{ii}$ is $g$-stable for all $i \in k$" without changing the problem. Also note that a necessary condition for problem VI to be solvable is that $C_8 \cap C^- = \emptyset$, a condition that will of course be satisfied for any reasonable choice of $C_8$.

If instead of input/output stability we require internal stability of the closed loop system
we obtain the following:

**PROBLEM VII** (Almost non interacting control with internal stability). Given a symmetric subset \( C_g \) of \( C \), problem VII is said to be solvable if there exists an integer \( N \) and if for all \( \varepsilon > 0 \) there exists a compensator \( \Sigma \) with \( \text{dim} \Sigma \leq N \) such that \( \| T_{ij} \|_\infty \leq \varepsilon \) for all \( i, j \in k \) with \( i \neq j \) and such that \( \sigma(A_{\varepsilon}) \subseteq C_g \).

Finally, by combining the requirements of internal stability and input/output stability into one synthesis problem we can formulate:

**PROBLEM VIII** (Almost non interacting control with i/o and internal stability). Given two symmetric subsets \( C_f \subset C_g \) of \( C \), problem VIII will be said to be solvable if there exists an integer \( N \) and if for all \( \varepsilon > 0 \) there exists a compensator \( \Sigma \) with \( \text{dim} \Sigma \leq N \) such that \( \| T_{ij} \|_\infty \leq \varepsilon \) for all \( i, j \in k \) with \( i \neq j \). \( T_{ij} \) is \( f \)-stable for all \( i, j \in k \) and \( \sigma(A_{\varepsilon}) \subseteq C_g \).

In the sequel again we shall concentrate on the last of these four problems, problem VIII, as the first three can be obtained from this one as special cases.

### 7. Almost invariant subspaces

Given a system \((A, B)\) and a subspace \( K = \ker H \) of the state space \( X \) we shall denote by \( V_b^*(K) \) the supremal \( L_1 \)-almost controlled invariant subspace of \( K \) and by \( R_b^*(K) \) the supremal \( L_1 \)-almost controllability subspace of \( K \). For the exact definitions and extensive treatments of these subspaces, see [15], [16] and [12].

In the following, a subset \( C_g \) of \( C \) will be said to contain minus infinity if it has the property that there exists \( c \in \mathbb{R} \) such that \( (-\infty, c] \subseteq C_g \). In the context of almost invariant subspaces the latter is a natural assumption to be made on stability sets (see also [9] and [12]). The family of all symmetric subsets of \( C \) that contain minus infinity will be denoted by \( S_{-\infty} \). We recall that for a given proper rational matrix or vector \( X \) its McMillan degree is denoted by \( \text{deg}(X) \). In [12] the following characterizations in terms of regular \((c, \omega)\)-representations were established:

**PROPOSITION 7.1**

\[
V_b^*(K) = \{ x_0 \in X \mid \text{for all } \varepsilon > 0 \text{ there is a regular } (c, \omega)\text{-representation of } x_0 \\
\quad \text{with } \| H_{\xi} \|_\infty \leq \varepsilon \}.
\]

\[
R_b^*(K) = \{ x_0 \in X \mid \text{there is } r \in N \text{ and for all } \varepsilon > 0 \text{ and for every } C_g \in S_{-\infty} \text{ there is a regular } g\text{-stable } (c, \omega)\text{-representation of } x_0 \text{ with } \text{deg}(c) \leq r \\
\quad \text{such that } \| H_{\xi} \|_\infty \leq \varepsilon \}.
\]

**PROOF.** See [12, cor. 3.33] and [12, cor. 3.37] \( \Box \)

We stress that in the above the upper bound \( r \) to the McMillan degrees of the \( \xi \)'s is allowed to depend on \( x_0 \) but is independent of \( \varepsilon \). In the time domain, loosely speaking the above states that \( R_b^*(K) \) is equal to the subspace of \( X \) with the property that starting in it one may travel along trajectories such that their distance to \( K \) is arbitrarily small and their characteristic values are located arbitrarily in the complex plane.
As in our previous considerations on the exact non interacting control problem, in the sequel an important role will be played by almost controlled invariant subspaces that are defined in terms of pairs of stability sets and pairs of subspaces of the state space. Let \( \mathcal{C}_f \subset \mathcal{C}_s \) be symmetric subsets of \( \mathcal{C} \) and let \( \ker H_2 = K_2 \subset K_1 = \ker H_1 \) be subspaces of \( X \). The following definition is the 'almost' analogue of the subspace \( V_{f,s}(K_1,K_2) \) as defined in section 3:

**DEFINITION 7.2**

\[
W_{f,s}(K_1,K_2) := \{ x_0 \in X \mid \text{there is } r \in \mathbb{N} \text{ and for all } \varepsilon > 0 \text{ there is a regular } s\text{-stable } (\xi,0)\text{-representation of } x_0 \text{ with } \deg(\xi) \leq r \\
\quad \text{such that } \|H_1\xi\|_\infty \leq \varepsilon \text{ and } H_2^2\xi \text{ is } f\text{-stable} \}.
\]

It is clear that the above defines a linear subspace of \( X \), which by PROP. 7.1 is contained in \( V_{f,s}(K_1) \). It also follows immediately from the definitions that \( V_{f,s}(K_1,K_2) \subset W_{f,s}(K_1,K_2) \). In addition, if \( \mathcal{C}_f \subset S_\infty \) then we have \( R_{f,s}^t(K_1) \subset W_{f,s}(K_1,K_2) \). This follows by taking \( \mathcal{C}_f = \mathcal{C}_f \) in the characterization of \( R_{f,s}^t(K_1) \) given in PROP. 7.1 and by noting that \( \mathcal{C}_f \subset \mathcal{C}_s \). Thus we find that if \( \mathcal{C}_f \subset S_\infty \) then

\[
V_{s}^t(K_2) + V_{f}^t(K_1) + R_{f,s}^t(K_1) \subset W_{f,s}(K_1,K_2) \quad (7.1)
\]

It will be proven in section 8 that under some fairly mild additional assumptions on the set \( \mathcal{C}_f \) the inclusion (7.1) is in fact an equality. Since the three subspaces on the left in (7.1) can be calculated using simple algorithms (see [18, p. 114] and [16]) this means that for those \( \mathcal{C}_f \)'s we will actually be able to calculate explicitly the subspace \( W_{f,s}(K_1,K_2) \). Consequently, we will also be able to check every subspace inclusion involving \( W_{f,s}(K_1,K_2) \) constructively. Keeping the latter fact in mind as a motivation, we now state the following analogue of LEMMA 3.3:

**LEMMA 7.3** Let \( E \) be a linear subspace of \( X \) and let \( E \) be a linear map such that \( \text{im } E = E \). Then \( (A,B) \) is \( s\)-stabilizable and \( E \subset W_{f,s}(K_1,K_2) \) if and only if there exists \( r \in \mathbb{N} \) and if for all \( \varepsilon > 0 \) there exist strictly proper \( s\)-stable real rational matrices \( X \) and \( U \) with \( \deg(X) \leq r \) such that \( (Is-A)X(s)-BU(s) = I, \|H_1XE\|_\infty \leq \varepsilon \) and \( H_2XE \) is \( f\)-stable.

**PROOF** This follows immediately from DEF. 7.2 and can be proven completely analogously as LEMMA 3.3.

In addition to DEF. 7.2, for given symmetric subsets \( \mathcal{C}_f \subset \mathcal{C}_s \) of \( \mathcal{C} \) and a given single subspace \( K = \ker H \) of \( X \) we will define

\[
W_{f,s}(K) := W_{f,s}(K,K).
\]

It follows from LEMMA 7.3 that \( (A,B) \) is \( s\)-stabilizable and \( \text{im } E \subset W_{f,s}(K) \) if and only if there exists \( r \in \mathbb{N} \) and if for all \( \varepsilon > 0 \) there exist strictly proper \( s\)-stable real rational matrices \( X \) and \( U \) with \( \deg(X) \leq r \) such that \( (Is-A)X(s)-BU(s) = I, \|HXE\|_\infty \leq \varepsilon \) and \( HXE \) is \( f\)-stable.

We shall now return to the almost non interacting control problem. Consider the system (2.1) and let \( K \) and \( K_i \) be defined by (5.3). Again, let \( G_0 \) denote the radical of the family of subspaces \( \{G_i \mid i \in k \} \). The following result provides necessary and sufficient conditions for solvability of problem VIII in terms of the subspaces introduced in this section:
THEOREM 7.4 Problem VIII is solvable if and only if \((A \cdot B)\) is \(s\)-stabilizable,
\[
G_i \subseteq \mathcal{W}_{f,s}(K_i, K) \quad \text{for all } i \in k
\]
and
\[
G_0 \subseteq \mathcal{W}_{f,s}(K).
\]

PROOF. This can be proven in an entirely analogous way as its 'exact' version, TH. 5.2. Similar as in the proof of TH. 5.2, the idea is to apply LEMMA 7.3 to each of the \(k+1\) subspace inclusions (7.2) and (7.3) and to 'glue together' (this time for each \(\epsilon\)) the \(X_i's\) and \(U_i's\) into one pair of rational matrices \(X\) and \(U\) in order to obtain a compensator \(UX^{-1}\) (depending of course on \(\epsilon\)). A detailed proof is given in Appendix B. 

Before we continue, we want to stress that the above theorem is of course of little use unless we find a way to express the subspaces \(\mathcal{W}_{f,s}(K_1, K)\) and \(\mathcal{W}_{f,s}(K)\) in terms of subspaces that can in principle be calculated. As already announced previously in this section, under some mild assumptions on the stability set \(\mathcal{C}_f\) it turns out that these subspaces can indeed be characterized as the sums of stabilizability subspaces and \(L_1\)-almost controllability subspaces. Since such characterization is independent from the non interacting control context the main importance of TH. 7.4 is that it reduces our main problem to a problem of obtaining a satisfactory characterization of the single subspace \(\mathcal{W}_{f,s}(K_1, K_2)\) defined in DEF. 7.2. The latter will be the subject of section 8.

8. Almost non interacting control: main results

In the present section we shall establish conditions under which the subspace inclusion (7.1) (that was shown to hold under the assumption that \(\mathcal{C}_f \in S_{\omega}\)) can be replaced by equality. Again consider a system \((A, B)\) and let \(\ker H_2 = K_2 \subseteq K_1 = \ker H_1\) be subspaces of the state space \(X\). Let \(\mathcal{C}_f \subseteq \mathcal{C}_f\) be symmetric subsets of \(\mathcal{C}\). In the following, let \(\overline{\mathcal{C}}_f\) denote the topological closure of \(\mathcal{C}_f\) in \(\mathcal{C}\). Furthermore, let \(\sigma^*(A, B, H_1)\) denote the set of invariant zeros associated with the system \((A, B, H_1)\). (I.e. the fixed spectrum \(\sigma(A + BF)\)\(V^*(K_1)/R^*(K_1)\) (see [18, p. 112] or [1]). We will show that the inclusion (7.1) is in fact an equality if either one of the following two assumptions hold: (i) \(\mathcal{C}_f\) is symmetric, contains minus infinity and \(\overline{\mathcal{C}}_f\setminus\mathcal{C}_f\) contains no invariant zeros of \((A, B, H_1)\), or (ii) \(\mathcal{C}_f\) is equal to the open left half plane \(\mathcal{C}^-\). Note in particular that if \(\mathcal{C}_f \in S_{\omega}\) is closed then (i) above will of course trivially hold. Also note that the fact that (ii) is a sufficient condition in order to have equality in (7.1) is particularly pleasing since in many applications \(\mathcal{C}_f\) will be taken to be equal to \(\mathcal{C}^-\). Now, at first glance the fact that both "\(\mathcal{C}_f\) is closed" as well as "\(\mathcal{C}_f\) is equal to the open left half plane" are sufficient conditions seems somewhat strange, because they are so different in nature. However, the fact that \(\mathcal{C}_f = \mathcal{C}^-\) plays a special role here is connected with the fact that the subspace \(\mathcal{W}_{f,s}(K_1, K_2)\) is defined in terms of \(H^\infty\)-functions that, by definition, are already \(\mathcal{C}^-\)-stable (independent of any other stability considerations).

Before going into the details, we shall first give an example of a situation in which the inclusion (7.1) is strict:

EXAMPLE 8.1 \(A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, K_1 = \ker H_1\) with \(H_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}\), \(K_2 = \ker I_{2 	imes 2}\). Let \(\mathcal{C}_f = \{s \in \mathcal{C} \mid \text{Re } s < -1\}\) and let \(\mathcal{C}_f\) be any symmetric subset of \(\mathcal{C}\) containing \(\mathcal{C}_f\). Note that \(\mathcal{C}_f \in S_{\omega}\). Using the algorithms in [18, p. 114] and [16] we calculate that
We claim however that $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ lies in $W_{f,s}(K_1, K_2)$. To see this, let $0 < \varepsilon < 1$ and define a state feedback by

$$F_\varepsilon := (\begin{matrix} -1/4 & \varepsilon^2 \\ -\varepsilon \end{matrix})$$

Define $\xi_\varepsilon(s) := (is - A - BF_\varepsilon)^{-1}x_0$ and $\omega_\varepsilon(s) := F_\varepsilon \xi_\varepsilon(s)$. This yields a regular $(\xi, \omega)$-representation of $x_0$ that obviously has the property that $\deg(\xi_\varepsilon) \leq 2$. Moreover, $\sigma(A + BF_\varepsilon) = \{ -1 - \varepsilon, -1 - \varepsilon \} \subset \mathcal{C}_f$ and therefore $\xi_\varepsilon$ is $s$-stable and $H_2 \xi_\varepsilon = \xi_\varepsilon$ is $f$-stable. Finally, we calculate that

$$\|H_1 \xi_\varepsilon\|_\infty = \frac{\varepsilon^2}{1 + \varepsilon + \frac{1}{4} \varepsilon^2} \leq \frac{1}{4} \varepsilon^2 < \varepsilon$$

and conclude that $x_0 \in W_{f,s}(K_1, K_2)$. In connection with condition (i) as stated in the introduction to this section: it can be shown that $\sigma = 1$ is an invariant zero of $(A, B, H_1)$. This invariant zero is contained in $\mathcal{C}_f \setminus \mathcal{C}_f = \{ s \in \mathcal{C} \mid \Re s = 1 \}$.

In order to proceed, let $W_f(K) := W_{f,f}(K, K)$ denote the subspace obtained by taking $\mathcal{C}_s = \mathcal{C}_f$ and $K_1 = K_2 = K$ in DEFINITION 7.2. Consequently, if $K = \ker H$ then we have:

$$W_f(K) = \{ x_0 \in X \mid \text{there is } r \in \mathbb{N} \text{ and for all } \varepsilon > 0 \text{ there exists a regular } f\text{-stable } (\xi, \omega)\text{-representation of } x_0 \text{ with } \deg(\xi) \leq r \text{ such that } \|H \xi\|_\infty \leq \varepsilon \} . \quad (8.1)$$

The following lemma reduces the problem of finding an in principle computable expression for $W_{f,s}(K_1, K_2)$ to the problem of finding such expression for $W_f(K_1)$. Since this subspace depends on only one stability set and one subspace the latter is expected to be easier:

**LEMMA 8.2** $W_{f,s}(K_1, K_2) = \mathcal{V}_s^*(K_2) + W_f(K_1)$.

**PROOF** $(\subset)$ Let $x_0 \in W_{f,s}(K_1, K_2)$ and let $r \in \mathbb{N}$ be the integer associated with $x_0$. Let $\varepsilon > 0$. There are strictly proper $s$-stable rational vectors $\xi$ and $\omega$ with $\deg(\xi) \leq r$ such that $x_0 = (is - A) \xi(s) - B \omega(s)$, $\|H_1 \xi\|_\infty \leq \varepsilon$ and $H_2 \xi$ is $f$-stable. Since $\xi$ and $\omega$ are strictly proper, they can be decomposed uniquely as $\xi = \xi_1 + \xi_2$ and $\omega = \omega_1 + \omega_2$, where $\xi_1$ and $\omega_1$ are strictly proper and $f$-stable and $\xi_2$, $\omega_2$ are strictly proper and $\mathcal{C}_s \setminus \mathcal{C}_f$-stable. Consequently we have

$$x_0 = (is - A) \xi_1(s) - B \omega_1(s) + (is - A) \xi_2(s) - B \omega_2(s) ,$$

where the left hand side is proper and $\mathcal{C}_f$-stable and the right hand side is proper and $\mathcal{C}_s \setminus \mathcal{C}_f$-stable. This however implies that both sides must in fact be equal to the same constant vector, say $x_{02}$. This yields

$$x_{02} = (is - A) \xi_2(s) - B \omega_2(s) ,$$

$$x_0 - x_{02} = (is - A) \xi_1(s) - B \omega_1(s) .$$

We also have $H_2 \xi_2 = H_1 \xi_2 - H_2 \xi_1$. The right hand side of this equation is $\mathcal{C}_f$-stable, the left hand side is $\mathcal{C}_s \setminus \mathcal{C}_f$-stable. Consequently, $H_2 \xi_2 = 0$. Since also $\xi_2$ is $s$-stable it follows from (3.2) that $x_{02} \in V_s^*(K_2)$. As for $x_0 - x_{02}$, note that since $K_2 \subset K_1$ we have $H_1 \xi_2 = 0$. Hence we have

$$\|H_1 \xi_1\|_\infty = \|H_1 \xi - H_1 \xi_2\|_\infty = \|H_1 \xi\|_\infty \leq \varepsilon .$$
Since also $\xi_1$ is $f$-stable and since $\deg(\xi_1) \leq \deg(\xi) \leq r$ it follows that $x_0 - x_{02} \in W_f(K_1)$. (\Rightarrow) The converse inclusion follows immediately from (3.2) and the definition of $W_f(K_1)$. 

Motivated by the previous lemma we shall concentrate on studying the subspace given by (8.1). Let $K = \ker H$ be a subspace of $X$ and let $\mathcal{C}_f$ be a symmetric subset of $\mathcal{C}$. By (3.2) it is immediate that $V^*_{f}(K) \subset W_f(K)$. In addition, from PROP. 7.1 it follows that if $\mathcal{C}_f$ is symmetric and contains minus infinity then $R^*_s(K) \subset W_f(K)$. Thus, if $\mathcal{C}_f \in S_\infty$ then we have

$$V^*_{f}(K) + R^*_s(K) \subset W_f(K) .$$

(8.2)

Note that by combining LEMMA 8.2 and (8.2) we reobtain (7.1). Also note that EX. 8.1 provides an example of a situation in which the inclusion in (8.2) is strict (take $K = K_1$).

In the sequel, if $\mathcal{C}_f$ is a given stability set then $V^*_{f}(K)$ will denote the supremal stabilizability subspace in $K$ with respect to $\mathcal{C}_f$ (the closure of $\mathcal{C}_f$). We have the following:

**LEMMA 8.3** Assume that $\mathcal{C}_f$ is a symmetric subset of $\mathcal{C}$. Then

$$W_f(K) \subset V^*_{f}(K) + R^*_s(K) .$$

(8.3)

**PROOF** For a proof of this lemma we refer to APP. C

By combining the two inclusions obtained above we immediately see that if $V^*_{f}(K) = V^*_{f}(K)$ then the inclusion (8.2) is in fact an equality. It is well known that the structure of stabilizability subspaces is closely connected to the notion of invariant zero. In fact, we obtain:

**COROLLARY 8.4** Let $\mathcal{C}_f \in S_\infty$. Assume that $\sigma^*(A,B,H) \cap (\overline{\mathcal{C}_f} \setminus \mathcal{C}_f) = \emptyset$. Then we have $W_f(K) = V^*_{f}(K) + R^*_s(K)$.

**PROOF** Let $R^*(K)$ be the supremal controllability subspace in $K$. Let $F$ be such that $(A+BF)V^*(K) \subset V^*(K)$. Denote $\sigma^*(A,B,H)$ by $\sigma^*$. Denote by $E_s$ the generalized eigenspace of the mapping $(A+BF) \mid V^*(K)$ associated with the eigenvalue $s$. By [18, p. 114] for a given symmetric subset $\mathcal{C}_g$ of $\mathcal{C}$ we have

$$V^*_g(K) = R^*(K) \oplus \bigoplus_{s \in \sigma^* \cap \mathcal{C}_f} E_s$$

Since $\sigma^* \cap \mathcal{C}_f = \sigma^* \cap \overline{\mathcal{C}_f}$ this implies that $V^*_{f}(K) = V^*_{f}(K)$. 

Briefly summarizing the above we see that in general we do not have equality in (8.2). A counter example was provided by EX. 8.1. However, if we make an assumption on the position of the invariant zeros we do obtain equality in (8.2). In particular if we assume that $\mathcal{C}_f$ is closed then this assumption will always be satisfied. Another possibility to make sure that the assumption holds for a given system is to choose $\mathcal{C}_f$ 'sufficiently far too the left' in the open left half plane.

Now, by applying COR. 8.4 to the case that $\mathcal{C}_f = \mathcal{C}^-$ we obtain that equality holds in (8.2) if $(A,B,H)$ has no invariant zeros on the imaginary axis. Surprisingly however we do not need the latter assumption for this particular choice of stability set. This is due to the fact that in the definition of $W_f(K)$ (see (8.1)) there is a more or less 'hidden' assumption that $H_\xi$ is $\mathcal{C}^-$-stable. Thus it can be shown that:
PROOF A proof of this result is beyond the scope of this paper. For a detailed proof we refer to [13].

Collecting the above results the following is now immediate:

COROLLARY 8.6 Consider the system $(A, B)$ and let $\ker H_2 = K_2 \subset K_1 = \ker H_1$ be subspaces of the state space $X$. Let $\mathcal{C}_f \subset \mathcal{C}_s$ be symmetric subsets of $\mathcal{C}$. Assume that at least one of the following two conditions is satisfied:

(A) $\mathcal{C}_f \in S_{ss}$ and $\sigma^*(A, B, H_1) \cap (\overline{\mathcal{C}_f} \setminus \mathcal{C}_f) = \emptyset$.

(B) $\mathcal{C}_f = \mathcal{C}^-$.

Then we have $W_{f,s}(K_1, K_2) = V_f^*(K_2) + V_f^*(K_1) + R_b^*(K_2)$.

Let us now return to the almost non interacting control problem. Consider the system (2.1) and again let $K$ and $K_j$ be defined by (5.3). Let $G_0$ be the radical of the $G_j$'s. At this point we have all material needed to obtain the following results on the solvability of problems $V$ to $VIII$:

COROLLARY 8.7 Assume that one of the following two conditions holds: (A) $\mathcal{C}_f \in S_{ss}$ and is closed, (B) $\mathcal{C}_f = \mathcal{C}^-$. Then problem VIII is solvable if and only if $(A, B)$ is $s$-stabilizable,

$$G_i \subset V_f^*(K_i) + V_f^*(K) + R_b^*(K_i) \text{ for all } i \in \mathbb{k}$$

and

$$G_0 \subset V_f^*(K) + R_b^*(K)$$.

COROLLARY 8.8 Assume that one of the following two conditions holds: (A) $\mathcal{C}_g \in S_{ss}$ and is closed, (B) $\mathcal{C}_g = \mathcal{C}^-$. Then we have:

(i) Problem VII is solvable if and only if $(A, B)$ is $g$-stabilizable, $G_i \subset V_g^*(K_i) + R_b^*(K_i)$ for all $i \in \mathbb{k}$ and $G_0 \subset V_g^*(K) + R_b^*(K)$.

(ii) Problem VI is solvable if and only if $G_i \subset V_g^*(K_i) + V^*(K) + R_b^*(K_i)$ for all $i \in \mathbb{k}$ and $G_0 \subset V^*(K) + R_b^*(K)$.

(iii) Problem V is solvable if and only if $G_i \subset V^*(K_i) + R_b^*(K_i)$ for all $i \in \mathbb{k}$ and $G_0 \subset V^*(K) + R_b^*(K)$.

REMARK 8.9 For the sake of simplicity, in the statement of the above corollaries we have chosen for a closedness condition on the stability sets. Alternatively however it is possible to formulate a more general condition involving the invariant zeros. In fact if we define

$$\hat{D}_i := (D_1^T, D_2^T, \ldots, D_{i-1}^T, D_{i+1}^T, \ldots, D_{\mathbb{k}}^T)^T, i \in \mathbb{k},$$

and

$$\hat{D} := (D_1^T, D_2^T, \ldots, D_{\mathbb{k}}^T),$$

then it can be seen that for all $i$ we have $\sigma^*(A, B, \hat{D}_i) \subset \sigma^*(A, B, \hat{D})$. Using this fact it is easy to show that the statement of COR. 8.7 remains valid if we replace the condition (A) by
REMARK 8.10 Also in the 'almost' case it is possible to establish an upper bound to the required order of compensation. Under the assumptions of COR. 8.7 it is possible to show that for every decoupling accuracy $\varepsilon$ a compensator (2.2) can be found with dynamic order satisfying

$$\dim W \leq n + \sum_{i=1}^{k} \dim [V_{f,s}(K_i, K) + \langle A \mid \im B \rangle].$$

Here $\langle A \mid \im B \rangle$ denotes the reachable subspace of $(A, B)$. As was required in the definition of problem VIII this upper bound does not depend on the decoupling accuracy $\varepsilon$.

9. Concluding remarks

In this paper we have been able to find solvability conditions for two rather general problems in the context of non interacting control by dynamic state feedback. The first of these was a problem of exact block diagonal decoupling with internal stability and input/output stability, the second one its 'almost' analogue in which only approximate decoupling was required. As special cases we obtained conditions for solvability of the corresponding problems where only input/output stability, only internal stability and no stability was required.

There are several points that we did not consider in this paper. One interesting problem would be to find conditions for solvability of the problems treated here with an additional requirement of output controllability preservation (see also [5]). In our context, preservation of output trajectories would mean that we would restrict the class of admissible compensators to those compensators that have the property that the diagonal blocks do not lose rank. More concretely, for each diagonal block the normal rank after compensation should at least be equal to the rank of the corresponding block before compensation.

Another interesting problem would be to generalize the above theory to the case of dynamic measurement feedback. At this point however, even for the exact non interacting control problem without any stability requirements such extension seems to be a very hard problem.

APPENDIX A

PROOF OF LEMMA 5.1 We shall first show that the collection $\{L_i \mid i \in \mathcal{I}\}$ is independent. Assume the contrary. Then there is an index $i$ such that

$$L_i \cap \sum_{j \neq i} L_j \neq \{0\} \quad (A.1)$$

Now, on the one hand the subspace (A.1) is contained in $L_i$. On the other hand it is contained in the radical of the collection $\{L_0 + L_i \mid i \in \mathcal{I}\}$ which, by [18, lemma 10.1] is equal to $L_0$. Thus, (A.1) is contained in $L_i \cap L_0 = \{0\}$, which is a contradiction. To complete the proof it suffices to show that $L_0 \cap (L_i \oplus L_2 \oplus \cdots \oplus L_k) = \{0\}$. Assume that $x = \sum_{j \neq i} x_j \in L_0$ with $x_j \in L_j$. Then we have $\sum_{j \neq i} x_j \in L_0 \oplus L_i = L_0 + L_i$. Also, $\sum_{j \neq i} (L_0 + L_j)$ and consequently $\sum_{j \neq i} x_j$ is an element of the radical of $\{L_0 + L_j \mid j \in \mathcal{I}\}$. Since the latter equals $L_0$, we find that $\sum_{j \neq i} x_j \in L_0$. However, since $\sum_{j \neq i} x_j = x \in L_0$ it follows that $x_i \in L_0$. We conclude that $x_i \in L_i \cap L_0 = \{0\}$, whence $x = 0$. The latter argument holds for every $i \in \mathcal{I}$ and therefore $x = 0$. This completes the proof of the lemma.
In this appendix we will give a proof of TH.7.4. First we shall prove the following useful result (see also REMARK 5.3). Recall that for a given proper rational matrix X its McMillan degree is denoted by \( \text{deg}(X) \).

**LEMMA B.1** Consider the system \((A, B)\). Let \(X\) and \(U\) be strictly proper real rational matrices such that \((I - A)X(s) - BU(s) = I\). Then we have \(\text{deg}(UX^{-1}) \leq \text{deg}(X)\).

**PROOF** Let \(F(s) := U(s)X(s)^{-1}\) and \(L(s) := sX(s)\). It was shown in section 4 that \(L\) is bicausal. We claim that \(\text{deg}(L) = \text{deg}(L^{-1})\). Indeed, if \((A_1, B_1, C_1, D_1)\) is a minimal realization of \(L^{-1}\), \((I - A - B - BF(s)) = X(s)^{-1} = sL_0 + sN(s)\) and consequently \((I - L_0)s = sN(s) + A + BF(s)\). In this equation the right hand side is proper so we must have \(L = L_0\) and \(sN(s) + A + BF(s) = 0\). Let \(B^*\) be a left inverse of \(B\). Then \(F(s) = -B^*A - sB^*N(s)\). If follows that \(\text{deg}(F) = \text{deg}(sB^*N) \leq \text{deg}(N) = \text{deg}(L^{-1})\).

**PROOF OF TH. 7.4\((\Leftarrow)\)** Let \(X\) be decomposed according to (5.12) and for \(i = 0, 1, \ldots, k + 1\) let \(P_i\) be the projectors associated with this direct sum decomposition. Let \(G_0\) be a linear map such that \(G_0 = \text{im} G_0\). Since \(\text{im} P_i \subseteq G_i\) \((i = 0, 1, \ldots, k + 1)\) there are linear maps \(T_i\) such that \(P_iG_i = G_iT_i\). Assume now that the hypotheses of TH. 7.4 hold. Then by LEMMA 7.3 for all \(i \in k\) there is \(r_i \in N\) such that for all \(\varepsilon > 0\) there are \(s\)-stable strictly proper real rational matrices \(X_i\) and \(U_i\) with \(\|D_jX_iG_i\|_{\infty} \leq \frac{\varepsilon}{2(\|T_i\| + 1)}\) for all \(j \in k, j \neq i\) (B.1) and \(D_jX_iG_i\) is \(s\)-stable for all \(j \in k\). (B.2)

Also, there is \(r_0 \in N\) such that for all \(\varepsilon > 0\) there are \(s\)-stable strictly proper real rational matrices \(X_0\) and \(U_0\) with \(\|D_jX_0G_0\|_{\infty} \leq \frac{\varepsilon}{2(\|T_0\| + 1)}\) for all \(j \in k\) (B.3) and \(D_jX_0G_0\) is \(s\)-stable for all \(j \in k\). (B.4)

Now, let \(\varepsilon > 0\) and let \(X_i, U_i\) be such that the above conditions are satisfied. Choose arbitrary \(s\)-stable strictly proper real rational matrices \(X_{k+1}\) and \(U_{k+1}\) such that \((I - A)X_{k+1}(s) + BU_{k+1}(s) = I\). Let \(r_{k+1} := \text{deg}(X_{k+1})\). As in the proof of TH. 5.2 define

\[
X := \sum_{i=0}^{k+1} X_iP_i, \quad U := \sum_{i=0}^{k+1} U_iP_i
\]

Then \(X\) and \(U\) are \(s\)-stable, \((I - A)X(s) - BU(s) = I\) and \(\text{deg}(X) \leq \sum_i r_i =: N\) (independent of \(\varepsilon\)). Moreover, for all \(i, j \in k\) with \(i \neq j\) it follows from (B.1) and (B.3) that
\[ \| D_jXG_i \|_\infty = \| D_jX_0P \delta G_i + D_jX_1P_i G_i \|_\infty \]
\[ \leq \| D_jX_0 \|_\infty \| T_0 \| + \| D_jX_1 G_i \|_\infty \| T_i \| < \varepsilon . \]

Also, for all \( i, j \in k \) we have
\[ D_jXG_i = D_jX_0G_0T_0 + D_jX_1G_1T_1 , \]
which by (B.2) and (B.4) is \( f \)-stable. Define now a compensator transfer matrix by
\[ F := UX^{-1}. \]
By the previous lemma we have \( \deg(F) \leq N \). Moreover, as in the proof of TH. 5.2, for all \( i, j \in k \), \( D_jXG_i \) is equal to \( T_{ij} \), the closed loop transfer matrix from \( v_i \) to \( z_j \).

Consequently, any minimal realization \( \Sigma_c \) of \( F(s) \) yields \( \| T_{ij} \|_\infty \leq \varepsilon \) (\( i \neq j \)) and makes \( T_{ij} \) \( f \)-stable for all \( i, j \). Since \( X \) and \( U \) are \( s \)-stable, by LEMMA 4.4 \( \Sigma_c \) also yields an internally \( s \)-stable closed loop system. Finally, \( \dim \Sigma_c = \deg(F) \leq N \) (independent of \( \varepsilon \)). \( \Rightarrow \) The converse implication of the theorem is also proven completely analogously to the corresponding proof of TH. 5.2. The proof is left to the reader.

### APPENDIX C

In this appendix a proof will be established of LEMMA 8.3. Until so far, all relevant subspaces have been characterized in terms of regular \((\xi, \omega)\)-representations. For most of the subspaces appearing in sections 7 and 8 however also characterizations in terms of not necessarily strictly proper \((\xi, \omega)\)-representations can be given. In the time domain such representations correspond to distributional state trajectories and controls (see also [16] and [12]). In the following, consider the system \((A, B)\) and let \( K \) be a subspace of \( X \). The following characterization will be useful:

**LEMMA C.1** Let \( \mathcal{C} \) be a symmetric subset of \( \mathcal{C} \). Then
\[ V_f^*(K) + R_b^*(K) = \{ x_0 \in X \mid x_0 \text{ has an } f \text{-stable } (\xi, \omega) \text{-representation with} \]
\[ H\xi = 0 \} . \]

For a proof of this we refer to [9, prop. 2.10] (see also [8]). Now, the idea of the proof of LEMMA 8.3 that we will give here is as follows. Given \( x_0 \in W_f(K) \), by definition there is an integer \( r \) and there are sequences \((\xi_n)\) and \((\omega_n)\) of \( f \)-stable strictly proper rational vectors with \( \deg(\xi_n) \leq r \) such that \( H\xi_n \|_\infty \to 0 \) (\( n \to \infty \)) and \( x_0 = (I - A)\xi_n(s) - B\omega_n(s) \). The idea is to analyse the limiting behaviour of the sequences \((\xi_n)\) and \((\omega_n)\) and to produce not necessarily strictly proper rational vectors \( \xi \) and \( \omega \) such that in a certain sense are limits of the \( \xi_n \)'s and \( \omega_n \)'s for \( n \to \infty \). For these vectors we will have \( x_0 = (I - A)\xi(s) - B\omega(s) \). Moreover we will have \( H\xi = 0 \) and, since the \( \xi_n \)'s are \( f \)-stable, it will turn out that \( \xi \) is \( \mathcal{C}_f \)-stable. It then follows from the previous lemma that \( x_0 \in V_f^*(K) + R_b^*(K) \). In the sequel we shall elaborate this idea.

Given a real rational function \( f = p/q \) with \( p \) and \( q \) coprime polynomials, define the degree of \( f \) as \( \deg(f) := \max(\deg(p), \deg(q)) \). Here \( \deg(p) \) and \( \deg(q) \) are the usual degrees of \( p \) and \( q \) as polynomials. If \( f \) is proper then \( \deg(f) \) coincides with the McMillan degree \( \deg(f) \). By \( \sigma(f) \) we shall denote the set of poles of \( f \) (i.e. the zeroes of the denominator \( q \)).

The following result states that if \((f_n)\) is a sequence of rational functions of which the degrees are uniformly bounded from above and if \( f_n \) converges to a rational function \( f \) pointwise for infinitely many \( s \in \mathcal{C} \), then the poles of \( f \) lie in the closure of the union of the poles of all \( f_n \)'s:

**LEMMA C.2** Let \((f_n)_{n \in \mathbb{N}} \) be a sequence of real rational functions. Assume there exists \( r \in \mathbb{N} \) such that \( \deg(f_n) \leq r \) for all \( n \in \mathbb{N} \). Assume that \( f \) is a real rational function such that \( f_n(s) \to f(s) \) (\( n \to \infty \)) for infinitely many \( s \in \mathcal{C} \). Then
PROOF Using the existence of an upper bound $r$ to the degrees $\partial(f_n)$ and the pointwise convergence for infinitely many $s \in \mathcal{C}$, it was shown in [6, p. 169-171] that

(i) for every $n \in \mathbb{N}$ there exist coprime polynomials $p_n(s) = \sum_{i=0}^{r_n} a_i(n) s^i$ and $q_n(s) = \sum_{i=0}^{r_n} b_i(n) s^i$ such that $f_n(s) = p_n(s) q_n(s)^{-1}$,

(ii) there exist polynomials $p(s) = \sum_{i=0}^{r} a_i s^i$ and $q(s) = \sum_{i=0}^{r} b_i s^i$ such that $f(s) = p(s) q(s)^{-1}$,

(iii) the coefficients satisfy $a_j(n) \to a_j$ and $b_j(n) \to b_j \quad (n \to \infty)$ for all $j \in \mathbb{Z}$.

Since the coefficients of the polynomials $q_n$ converge to those of $q$, it may be shown that $q_n \to q$ uniformly on compact subsets of $\mathcal{C}$. Now, let $s_0 \in \sigma(f)$. Then $q(s_0) = 0$. Let $\mu > 0$ be such that $s_0$ is the only zero of $q$ in the disc $|s - s_0| < \mu$. Take an arbitrary $\varepsilon \in (0, \mu)$. We shall prove that the disc $|s - s_0| < \varepsilon$ contains an element of $\sigma(f_n)$ for some sufficiently large $N$.

Define $\alpha := \min \left\{ \|q(s)\| \mid |s - s_0| = \varepsilon \right\}$. Then $\alpha > 0$. Since $q_n \to q$ uniformly on the circle $\mathcal{C}_\varepsilon := \{ s \in \mathcal{C} \mid |s - s_0| = \varepsilon \}$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $\| q_n(s) - q(s) \| < \alpha$ for all $s$ on the circle $\mathcal{C}_\varepsilon$. Now define a polynomial $g := q_N - q$. Then $\|g(s)\| = \|q_N(s) - q(s)\| < \alpha \leq \|q(s)\|$ on $\mathcal{C}_\varepsilon$. Hence, by Rouché’s theorem (see [2, p. 116]), $g + q = q_N$ has a zero inside the disc $|s - s_0| < \varepsilon$. Since $q_N$ and $p_N$ are coprime this zero is a pole of $f_N$ and hence lies in $\sigma(f_N)$.

We are now in a position to give a proof of LEMMA 8.3. In the following, let $R^*(K)$ be the supremal controllability subspace contained in $K$ (see [18, p. 109]).

PROOF OF LEMMA 8.3 Let $x_0 \in W_f(K)$. There is $r \in \mathbb{N}$ and there are sequences $(\xi_n)$ and $(\omega_n)$ of strictly proper real rational vectors with $\xi_n \in f$-stable, deg($\xi_n) \leq r$, $\|H \xi_n\|_\infty \to 0 \quad (n \to \infty)$ and $x_0 = (Is - A) \xi_n(s) - B \omega_n(s)$. Let $F : X \to U$ be such that $(A + BF)R^*(K) \subset R^*(K)$ and $\sigma(A + BF) \cap R^*(K) \subset \mathbb{C}_F$. Define $\omega_n(s) := \omega_n(s) - F \xi_n(s)$. Denote $A + BF$ by $A_F$. Then we have

$$x_0 = (Is - A_F) \xi_n(s) - B \omega_n(s) \quad \forall n .$$

Make a direct sum decomposition $X = X_1 \oplus X_2$ with $X_1 := R^*(K)$ and $X_2$ any complement of $X_1$ in $X$. With respect to a basis compatible with this decomposition we have:

$$A_F = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad H = (0 \ H_2) .$$

Furthermore, $\sigma(A_{11}) \subset \mathbb{C}_F$. Next, make a direct sum decomposition $U = U_1 \oplus U_2$ with $U_1 := \ker B_2$ and $U_2$ any complement of $U_1$ in $U$. With respect to this decomposition, let $B_1 = (B_{11} \ B_{12})$ and $B_2 = (0 \ B_{22})$. Obviously $B_{22}$ is injective. By applying [18, ex. 4.4] and [18, ex. 5.8] it can be seen that system $(A_{22}, B_{22}, H_2)$ is left-invertible in the sense that the transfer matrix $T(s) = H_2(Is - A_{22})^{-1}B_{22}$ has a left inverse $T^*(s)$. Partition

$$x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, \quad \omega_n = \begin{bmatrix} \omega_{1n} \\ \omega_{2n} \end{bmatrix} \quad \text{and} \quad \xi_n = \begin{bmatrix} \xi_{1n} \\ \xi_{2n} \end{bmatrix} .$$

Then we have

$$\sigma(f) \subset \bigcup_{n=1}^{\infty} \sigma(f_n) .$$
which yields

$$\tilde{\omega}_{2a}(s) = T^+(s) (H_2 \xi_{2a}(s) + H_2 (I - A_{22})^{-1} x_{02}).$$

Now, obviously \( \|H_2 \xi_{2a}\|_\infty = \|H \xi_a\|_\infty \to 0 \) as \( n \to \infty \) and consequently for all \( s \in \mathcal{C}^+ = \{ s \in \mathcal{C} \mid \text{Re}\ s \geq 0 \} \) we have

$$\|H_2 \xi_{2a}(s)\| \leq \|H_2 \xi_{2a}\|_\infty \to 0 \quad (n \to \infty).$$

A fortiori this implies that \( \tilde{\omega}_{2a}(s) \to T^+(s)H_2(I - A_{22})^{-1} x_{02} =: \tilde{\omega}_2(s) \) for infinitely many \( s \in \mathcal{C} \). Since

$$\xi_{2a}(s) = (I - A_{22})^{-1}(B_{22} \tilde{\omega}_{2a}(s) + x_{02}), \quad \forall n,$$

this implies that \( \tilde{\omega}_{2a}(s) \to (I - A_{11})^{-1}(B_{12} \tilde{\omega}_{2}(s) + A_{12} \xi_{2}(s)) \) for infinitely many \( s \in \mathcal{C} \). Note that \( \tilde{\omega}_2 \) and \( \xi_2 \) are rational but not necessarily strictly proper. Also note that \( H_2 \xi_2 = 0 \) and that \( x_{02} = (I - A_{22})^{-1}(B_{12} \tilde{\omega}_{2}(s) + A_{12} \xi_{2}(s)) \). Since \( \xi_{2a} \) is \( f \)-stable for all \( n \) and since the degrees of all its components are bounded from above, it follows from LEMMA C.2 that \( \xi_2 \) has all its poles in the closure \( \mathcal{C}_f \). The same holds for \( \tilde{\omega}_2 \). Define now

$$\tilde{\omega} := \begin{bmatrix} 0 \\ \tilde{\omega}_2 \end{bmatrix} \quad \text{and} \quad \xi_1(s) := (I - A_{11})^{-1}(B_{12} \tilde{\omega}(s) + A_{12} \xi_2(s)).$$

Since \( \sigma(A_{11}) \subset \mathcal{C}_f \), \( \xi_1 \) is \( f \)-stable. Moreover,

$$\begin{bmatrix} 0 \\ x_{02} \end{bmatrix} = \begin{bmatrix} I - A_{11} & -A_{12} \\ I - A_{22} \end{bmatrix} \begin{bmatrix} \xi_1(s) \\ \xi_2(s) \end{bmatrix} - \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{\omega}_2(s) \end{bmatrix}.$$

The latter expression is a \((\xi, \omega)\)-representation of \( \begin{bmatrix} 0 \\ x_{02} \end{bmatrix} \) with \( \xi := (\xi_1^T, \xi_2^T)^T \) \( \mathcal{C}_f \)-stable and \( H \xi = 0 \). Consequently, it follows from LEMMA C.1 that the vector \( \begin{bmatrix} 0 \\ x_{02} \end{bmatrix} \) lies in \( V_f^*(K) + R_0^*(K) \). Since \( \begin{bmatrix} x_{01} \\ 0 \end{bmatrix} \in R^*(K) \subset R_0^*(K) \) (see e.g. [16]) this yields \( x_0 \in V_f^*(K) + R_0^*(K) \). This completes the proof of the lemma.


