

Solution to problem 94-12 : A nonharmonic trigonometric series

Citation for published version (APA):

Boersma, J., & Chapman, R. J. (1995). Solution to problem 94-12 : A nonharmonic trigonometric series. *SIAM Review*, 37(3), 443-448.

Document status and date:

Published: 01/01/1995

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Furthermore, by using [2, p. 230, (1)] a computable representation for the G-function in (1) (and hence also for $F_{\nu-1/2}(s)$) may be given in terms of essentially a linear combination of four hypergeometric functions ${}_0F_3[b^2s^2/16]$ provided that no two of the parameters $\{1, \frac{1}{2}, \frac{1+\nu-\rho}{2}, \frac{1-\nu-\rho}{2}\}$ differ by an integer. This result for $F_{\nu-1/2}(s)$ is given (correctly) in [3, p. 353, (9)] so that *the proposed integral is in fact already tabulated in terms of known computable functions.*

REFERENCES

[1] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, Academic Press, San Diego, 1994.
 [2] Y. L. LUKE, *The Special Functions and Their Approximations*, Vol. I, Academic Press, San Diego, 1969.
 [3] A. P. PRUDNIKOV, YU. A. BRYCHKOV, AND O. I. MARICHEV, *Integrals and Series*, Vol. 2, Gordon and Breach, New York, 1986.

Also solved by R. G. BUSCHMAN (Langlois, Oregon), M. L. GLASSER (Clarkson University), CARL C. GROSJEAN (University of Ghent, Ghent, Belgium), NORBERT ORTNER AND PETER WAGNER (University of Innsbruck, Innsbruck, Austria), and JET WIMP (Drexel University), and the preparers.

A Nonharmonic Trigonometric Series

Problem 94-12, by M. L. GLASSER (Clarkson University).

The following function has been encountered in studying the dielectric function for a “quantum dot”:

$$f(a) = \sum_{n=1}^{\infty} \frac{\sin(a\sqrt{n})}{n}.$$

Find $\lim_{a \downarrow 0} f(a)$ and $\lim_{a \downarrow 0} f'(a)$.

Solution by J. BOERSMA (Eindhoven University of Technology, Eindhoven, The Netherlands).

The two limits are determined by a Mellin transform method with the results

$$\lim_{a \downarrow 0} f(a) = \pi, \quad \lim_{a \downarrow 0} f'(a) = \zeta\left(\frac{1}{2}\right),$$

where ζ stands for the Riemann zeta function.

We start with the Mellin transform

$$\mathcal{M}[\sin x; s] = \int_0^{\infty} \sin x x^{s-1} dx = \sin(\pi s/2)\Gamma(s),$$

valid for $-1 < \text{Re } s < 1$, and absolutely convergent for $-1 < \text{Re } s < 0$. By Mellin’s inversion formula [1, p. 46, Thm. 28] we have

$$\sin x = \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \sin(\pi s/2)\Gamma(s) x^{-s} ds, \quad (x > 0),$$

in which $-1 < c_1 < 0$, so that the integration path is in the strip of absolute convergence. In fact, the path may be shifted to $\text{Re } s = c_2$ with $-1 < c_2 \leq 1/2$. To show this, we use the asymptotic expansion of the Γ -function to establish the estimate

$$\sin(\pi s/2)\Gamma(s) = O(|t|^{\sigma-1/2}), \quad (t \rightarrow \pm\infty),$$

where $s = \sigma + it$. Then it follows that the contributions of the closing line segments $[c_1 \pm iT, c_2 \pm iT]$ vanish as $T \rightarrow \infty$, if $c_1 < c_2 \leq 1/2$.

The next step is to replace $\sin(a\sqrt{n})$ by the integral representation presented above, and to interchange the order of summation and integration in the series for $f(a)$. As a result it is found that

$$(1) \quad f(a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sin(\pi s/2) \Gamma(s) \zeta(1+s/2) a^{-s} ds,$$

where the factor $\zeta(1+s/2)$ stems from $\sum_{n=1}^{\infty} n^{-1-s/2} = \zeta(1+s/2)$, valid for $\text{Re } s > 0$; accordingly, one should take $0 < c \leq 1/2$. The integrand (1) is an analytic function of s , except for simple poles at $s = 0$ with residue π , and at $s = -2k - 1$ with residues $(-1)^k \zeta(1/2 - k) a^{2k+1} / (2k+1)!$, where $k = 0, 1, 2, \dots$. By shifting the path of integration to $\text{Re } s = -2N$ (with $N = 1, 2, 3, \dots$), we are led to the representation

$$f(a) = \pi + \sum_{k=0}^{N-1} \frac{(-1)^k \zeta(\frac{1}{2} - k)}{(2k+1)!} a^{2k+1} + R_N(a),$$

where the remainder term $R_N(a)$ is given by

$$(2) \quad R_N(a) = \frac{1}{2\pi i} \int_{-2N-i\infty}^{-2N+i\infty} \sin(\pi s/2) \Gamma(s) \zeta(1+s/2) a^{-s} ds.$$

To justify the shift of the integration path, we need the estimate [2, pp. 81–82]

$$\zeta(1+s/2) = \zeta(1 + \frac{1}{2}(\sigma + it)) = O(|t|^{\mu(\sigma)+\varepsilon}), \quad (t \rightarrow \pm\infty)$$

valid for every $\varepsilon > 0$, in which $\mu(\sigma)$ is determined by

$$\mu(\sigma) = 0 \text{ if } \sigma \geq 0; \quad \mu(\sigma) \leq -\sigma/4 \text{ if } -2 < \sigma < 0; \quad \mu(\sigma) = -(1+\sigma)/2 \text{ if } \sigma \leq -2.$$

It is now clear that $\sigma - \frac{1}{2} + \mu(\sigma) < 0$ if $\sigma < \frac{1}{2}$, and the integrand in (1) tends to zero as $\text{Im } s \rightarrow \pm\infty$, for fixed $\text{Re } s < -\frac{1}{2}$. Hence, the contributions of the closing line segments $[-2N \pm iT, c \pm iT]$ vanish as $T \rightarrow \infty$.

Changing the integration variable s into $-s$ in (2), we use the functional equation of the zeta function and some standard properties of the Γ -function to obtain the representation

$$R_N(a) = \frac{2\pi^{3/2}}{2\pi i} \int_{2N-i\infty}^{2N+i\infty} \frac{\cos(\pi s/4)}{\cos(\pi s/2)} \frac{\zeta(s/2)}{s\Gamma((1+s)/2)} \left(\frac{a}{2\sqrt{2\pi}}\right)^s ds.$$

In the latter integral we set $s = 2N(1 + it)$. Next we establish the inequalities

$$\left| \frac{\cos(N\pi(1+it)/2)}{\cos(N\pi(1+it))} \right| \leq 2 \exp[-N\pi|t|/2], \quad |\zeta(N(1+it))| \leq \zeta(N),$$

and, by use of the asymptotic expansion of the Γ -function,

$$\left| \frac{1}{\Gamma(\frac{1}{2} + N(1+it))} \right| \leq (2\pi)^{-1/2} K_N N^{-N} e^N |1+it|^{-N} \exp[N\pi|t|/2],$$

where $K_N = 1 + O(N^{-1})$. The remainder term $R_N(a)$ can now be estimated by

$$\begin{aligned} |R_N(a)| &\leq 2^{1/2} K_N \zeta(N) N^{-N} \left(\frac{a}{2\sqrt{2\pi e}}\right)^{2N} \int_{-\infty}^{\infty} \frac{dt}{|1+it|^{N+1}} \\ &= 2^{1/2} K_N \zeta(N) N^{-N} \left(\frac{a}{2\sqrt{2\pi e}}\right)^{2N} \frac{\Gamma(1/2)\Gamma(N/2)}{\Gamma((N+1)/2)}. \end{aligned}$$

From the latter estimate it is clear that $R_N(a) \rightarrow 0$ as $N \rightarrow \infty$. Consequently, we have for $f(a)$ the representation by a convergent power series

$$f(a) = \pi + \sum_{k=0}^{\infty} \frac{(-1)^k \zeta(\frac{1}{2} - k)}{(2k + 1)!} a^{2k+1},$$

valid for all $a > 0$. The limit results stated at the outset are now obvious. Alternatively, $\zeta(\frac{1}{2} - k)$ may be expressed in terms of $\zeta(\frac{1}{2} + k)$ by means of the functional equation of the zeta function, leading to

$$f(a) = \pi + 2\pi^{1/2} \sum_{k=0}^{\infty} \frac{\sin((k + \frac{1}{2})\frac{\pi}{2})\zeta(k + \frac{1}{2})}{(k + \frac{1}{2})k!} \left(\frac{a}{2\sqrt{2\pi}}\right)^{2k+1}.$$

The latter series is convergent for all complex a .

Remark 1. For a quick derivation of $f(0+) = \pi$, write $f(a)$ as a Riemann sum: $f(a) = \sum_{n=1}^{\infty} g(a^2n)a^2$, where $g(x) = \sin(\sqrt{x})/x$. It can be shown that $\lim_{a \downarrow 0} f(a)$ is equal to the Riemann integral

$$\int_0^{\infty} g(x) dx = \int_0^{\infty} \frac{\sin(\sqrt{x})}{x} dx = 2 \int_0^{\infty} \frac{\sin t}{t} dt = \pi.$$

Remark 2. Similar results can be derived for the cosine-series, namely,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\cos(a\sqrt{n})}{n} &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \cos(\pi s/2)\Gamma(s)\zeta(1+s/2)a^{-s} ds \\ &= -2 \log a - \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(1-k)}{(2k)!} a^{2k} \\ &= -2 \log a - \gamma + \frac{a^2}{4} + \pi^{1/2} \sum_{m=1}^{\infty} \frac{(-1)^m \zeta(2m)}{m\Gamma(2m + \frac{1}{2})} \left(\frac{a}{2\sqrt{2\pi}}\right)^{4m}, \quad (a > 0), \end{aligned}$$

where γ is Euler's constant.

REFERENCES

- [1] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, 3rd ed., Chelsea, New York, 1986.
- [2] ———, *The Theory of the Riemann Zeta Function*, Clarendon Press, Oxford, 1951.

Solution by ROBIN JOHN CHAPMAN (University of Exeter, UK).

We start by proving a more general result. Let

$$(1) \quad f_{\beta}(a) = \sum_{n=1}^{\infty} \frac{\sin(an^{\beta})}{n}, \quad (0 < \beta < 1).$$

I claim that $\lim_{a \downarrow 0} f_{\beta}(a) = \pi/(2\beta)$. I remark that this is also true when $\beta = 1$, for by the theory of Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin(na)}{n} = \frac{\pi - a}{2}$$

when $0 < a < 2\pi$.

Note that if $\phi(x)$ is a continuously differentiable function then integration by parts gives

$$\int_{n-1}^n \phi(x) dx = \phi(n) - \int_{n-1}^n (x - n + 1)\phi'(x) dx.$$

Applying this to $\phi(x) = \sin(ax^\beta)/x$ and summing gives

$$(2) \quad \sum_{n=1}^N \frac{\sin(an^\beta)}{n} = \int_0^N \frac{\sin(ax^\beta)}{x} dx + \int_0^N \{x\} \left(a\beta \frac{\cos(ax^\beta)}{x^{2-\beta}} - \frac{\sin(ax^\beta)}{x^2} \right) dx,$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x . Putting $y = ax^\beta$ gives

$$\int_0^N \frac{\sin(ax^\beta)}{x} dx = \frac{1}{\beta} \int_0^{aN^\beta} \frac{\sin y}{y} dy,$$

which tends to $\pi/(2\beta)$ as $N \rightarrow \infty$. If $\varepsilon > 0$ the convergence is uniform for $a \geq \varepsilon$. The integrand of the second integral is $O(x^{\beta-1})$ as $x \rightarrow 0$ and $O(x^{\beta-2})$ as $x \rightarrow \infty$. Hence

$$(3) \quad f_\beta(a) = \frac{\pi}{2\beta} + \int_0^\infty \{x\} \left(a\beta \frac{\cos(ax^\beta)}{x^{2-\beta}} - \frac{\sin(ax^\beta)}{x^2} \right) dx.$$

If $K > \varepsilon > 0$ the second integrand in (2) is bounded by $(K\beta + 1)x^{\beta-2}$ so the series (1) converges uniformly for $a \in [\varepsilon, K]$. The integrand in (3) tends pointwise to zero as $a \downarrow 0$. If $0 < a < 1$ it is bounded in absolute value by $2\psi(x)$, where

$$\psi(x) = \begin{cases} x^{\beta-1} & \text{if } x < 1, \\ x^{\beta-2} & \text{if } x \geq 1 \end{cases}$$

is a positive integrable function. Hence by Lebesgue's theorem of dominated convergence

$$\lim_{a \downarrow 0} f(a) = \frac{\pi}{2\beta}$$

as claimed.

We now return to the case $\beta = 1/2$. Using (3) we write

$$f(a) = f_{1/2}(a) = \pi + \frac{1}{2}g_1(a) - g_2(a),$$

where

$$g_1(a) = \int_0^\infty \{x\} \frac{\cos(a\sqrt{x})}{x^{3/2}} dx$$

and

$$g_2(a) = \int_0^\infty \{x\} \frac{\sin(a\sqrt{x})}{x^2} dx.$$

Differentiating the integral for g_2 formally with respect to a gives the integral for g_1 . As the integrand in the integral for g_1 is bounded above by the L^1 function $x \mapsto \{x\}/x^{3/2}$, then $g_2'(a) = g_1(a)$. Now consider g_1 . Integration by parts gives, for a suitable function ϕ ,

$$\int_{n-1}^n \{x\}\phi(x) dx = \frac{\phi(n)}{2} - \int_{n-1}^n \frac{\{x\}^2}{2} \phi'(x) dx.$$

Applying this to $\phi(x) = x^{-3/2} \cos(a\sqrt{x})$ and summing gives

$$g_1(a) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\cos(a\sqrt{n})}{n^{3/2}} + \int_0^{\infty} \frac{\{x\}^2}{2} \left(\frac{a \sin(a\sqrt{x})}{2x^2} + \frac{3 \cos(a\sqrt{x})}{2x^{5/2}} \right) dx.$$

The formal derivative of this with respect to a is

$$h(a) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(a\sqrt{n})}{n} + \int_0^{\infty} \frac{\{x\}^2}{4} \left(\frac{a \cos(a\sqrt{x})}{x^{3/2}} - \frac{2 \sin(a\sqrt{x})}{x^2} \right) dx.$$

We have already shown that the sum here is uniformly convergent when a lies in a set of the form $[\varepsilon, K]$; also the integrand is bounded by a multiple of the L^1 function $x \mapsto \{x\}^2(x^{-2} + x^{-3/2})$ when $a \leq K$. Hence $g'_1(a) = h(a)$, and so $f'(a) = -\frac{1}{2}g_1(a) + \frac{a}{2}h(a)$. By Lebesgue's dominated convergence theorem, and the fact that $\lim_{a \downarrow 0} f(a)$ exists, it follows that $h(a)$ tends to a finite limit as $a \downarrow 0$. Hence, again by Lebesgue's theorem,

$$\lim_{a \downarrow 0} f'(a) = -\frac{1}{2} \int_0^{\infty} \frac{\{x\}}{x^{2/3}} dx.$$

It only remains to identify this final integral. If $s = \sigma + it$ is a complex number with real part $\sigma > 1$, then

$$\sum_{n=1}^N \frac{1}{n^s} = 1 + \sum_{n=2}^N \frac{1}{n^s} = 1 + \int_1^N \frac{dx}{x^s} - s \int_1^N \frac{\{x\}}{x^{s+1}} dx$$

and so

$$(4) \quad \zeta(s) = 1 + \int_1^{\infty} \frac{dx}{x^s} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx.$$

But the integral on the right-hand side of this is an analytic function for $\sigma > 0$, and so by analytic continuation (4) is valid for all $s \neq 1$ with $\sigma > 0$. Suppose that s is real and $0 < s < 1$. Then

$$\int_0^1 \frac{\{x\}}{x^{s+1}} dx = \int_0^1 \frac{dx}{x^s} = \frac{1}{1-s}.$$

Hence for $0 < s < 1$ we have

$$\zeta(s) = -s \int_0^{\infty} \frac{\{x\}}{x^{s+1}} dx$$

and so

$$\zeta(1/2) = -\frac{1}{2} \int_0^{\infty} \frac{\{x\}}{x^{3/2}} dx$$

as required.

Also solved by CARL C. GROSJEAN (University of Ghent, Ghent, Belgium), W. B. JORDAN (Scotia, NY), KEE-WAI LAU (Hong Kong), G. LOHÖFER (German Aerospace Research Establishment, Köln, Germany), O. P. LOSSERS (Technical University, Eindhoven, The Netherlands), G. F. NEWELL (University of California, Berkeley), PETER WAGNER (University of Innsbruck, Innsbruck, Austria), JAMES A. WILSON (Iowa State University), and the proposer.

PETER WAGNER also found the generalization obtained by ROBIN CHAPMAN. The proposer incorporated both the results of Boersma and Chapman in the following generalization. Let

$$f_p(a) = \sum_{n=1}^{\infty} \frac{\sin(an^p)}{n}, \quad (a > 0).$$

Then the Mellin transform is

$$\begin{aligned} F_p(s) &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} x^{s-1} \sin(n^p x) dx \\ &= \Gamma(s) \sin(\pi s/2) \zeta(1 + ps), \quad (0 < \operatorname{Re} s < 1, \quad p > 0). \end{aligned}$$

For $p > 0$,

$$f_p(a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} a^{-s} \Gamma(s) \sin(\pi s/2) \zeta(1 + ps) ds, \quad (0 < c \leq \frac{1}{2}).$$

For $0 < p < 1$ we can close the contour to the left (justified as in Boersma's solution) and sum the residues of the simple poles to get

$$f_p(a) = \frac{\pi}{2p} + \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!} \zeta(1 - (2k+1)p).$$

For $p = 1$, this result reduces to $f_1(a) = (\pi - a)/2$, valid for $0 < a < 2\pi$. The generalization to the sum with n^γ in the denominator can be obtained in the same way, but there are several cases that must be looked at separately.

A Limit Problem from the Theory of Delay Equations

*Problem 94-13**, by YUNKANG LIU (Cambridge University, Cambridge, England).

Suppose that a and b are complex constants, $\operatorname{Re} b > 0$ and $\lambda \in (0, 1)$. Prove or disprove

$$(1) \quad \lim_{t \rightarrow \infty} y(t) e^{-bt} = \prod_{k=0}^{\infty} \left(1 + \frac{a}{b} \lambda^k \right),$$

where

$$y(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{k=0}^{n-1} (b + a\lambda^k)$$

is the unique solution of the delay differential equation

$$y'(t) = ay(\lambda t) + by(t), \quad t > 0, \quad y(0) = 1.$$

The existence of the limit was proved by Kato and McLeod [1] in the case of $b > 0$ and later extended by Kato [2] to the case of $\operatorname{Re} b > 0$. The proof that (1) holds in the case of $b > 0$ is trivial.