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Citation for published version (APA):

Slikker, M. (2003). *An m-sequencing game with an empty core*. (BETA publicatie : working papers; Vol. 94). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/2003

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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- The final published version features the final layout of the paper including the volume, issue and page numbers.

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An m -sequencing game with an empty core

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April 25, 2003

Abstract

We study m -sequencing games, which were introduced by Hamers et al. (1999). We answer the open question whether all these games are balanced in the negative. We do so, by an example of a 3-sequencing situation with 5 jobs, whose associated 3-sequencing game has an empty core. The counterexample finds its basis in an inconsistency in Hamers et al. (1999), which was probably overlooked by the authors. This observation demands for a detailed reconsideration of their proofs.¹

KEYWORDS: Game theory, cooperative games, sequencing situations

1 Introduction

Hamers et al. (1999) consider the example of a multidivisional firm in which the divisions share a repair and maintenance facility. Taking into consideration that the (financial) impact of a non-repaired item need not be the same for the different divisions, it might be beneficial for the firm to rearrange repair requests at the repair and maintenance facility rather than just using a first-come-first-serve principle.

The focus of Hamers et al. (1999) is on the allocation of the costs that the different divisions incur. Specifically, they analyze cost allocation in sequencing situations with possible cost-savings by rearrangements in a setting with several parallel (and identical) machines. Their main results deal with balancedness of cooperative games associated with sequencing situations with m parallel machines. To be precise, they prove balancedness in case the number of machines is at most two and in case all processing times are equal or all (financial) impact factors are equal. They end their cooperative analysis with the statement that it is an open problem whether cooperative games associated with sequencing situations with m parallel machines are balanced.

In this note we first show that the open problem can be answered in the negative. We do so, by an example of a 3-sequencing situation with 5 jobs, whose associated 3-sequencing game has an empty core. The basis for this counterexample can be interpreted as an inconsistency in Hamers et al. (1999). This inconsistency requires a detailed reexamination of their proofs.

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¹We will not deal with their noncooperative approach.

The set-up of the remainder of this paper is as follows. Section 2 deals with preliminaries. In section 3 we present our counterexample and prove that the example is minimal with respect to the number of jobs and the number of machines. Finally, in section 4 we focus on the inconsistency in Hamers et al. (1999) and carefully reexamine their proofs.²

2 Preliminaries

For a description of the model and the introduction of the notation applied in this paper we refer the reader to Hamers et al. (1999).

3 Balancedness

This section contains the main result of this paper, i.e., an example that illustrates that m -sequencing situations need not be balanced. Subsequently, we will show that this example is minimal in the sense that each m -sequencing situation with less jobs or less machines results in a balanced m -sequencing game.

We start with the example that shows that not all m -sequencing games are balanced.

Example 3.1 Let $M = \{1, 2, 3\}$, $N = \{1, 2, 3, 4, 5\}$, $p = (1, 3, 2, 1, 3)$, and $\alpha = (1, 3, 20, 1, 3)$. The initial order is given by

$$b^0(i) = \begin{cases} (1, 1) & \text{if } i = 1; \\ (1, 2) & \text{if } i = 2; \\ (2, 1) & \text{if } i = 3; \\ (3, 1) & \text{if } i = 4; \\ (3, 2) & \text{if } i = 5. \end{cases} \quad (1)$$

Let (N, v) be the associated 3-sequencing game. Obviously, one-player coalitions have value 0. Then, consider coalition $\{1, 2\}$. The optimal order for this coalition is obtained by a switch of its two jobs, followed by a replacement of the job that is last after the switch, to machine 2. Associated cost savings equal

$$v(\{1, 2\}) = 1 \times 1 + 3 \times 4 - 1 \times 3 - 3 \times 3 = 1.$$

Similarly, we find that $v(\{4, 5\}) = 1$. Finally, it is easily checked that $v(N) = 1$ as well. Suppose x is a core-allocation. Then $1 = v(N) = x(N) = x(\{1, 2\}) + x(\{3\}) + x(\{4, 5\}) \geq v(\{1, 2\}) + v(\{3\}) + v(\{4, 5\}) = 1 + 0 + 1$. This contradiction shows that the core of game (N, v) is empty. Hence, the game (N, v) is not balanced.³ \diamond

²We deal with their cooperative approach only. It was shown by Mitra (2000) that the noncooperative results of Hamers et al. (1999) were not correct.

³We remark that the game (N, v) is not even superadditive, since $v(\{1, 2\}) + v(\{3\}) + v(\{4, 5\}) > v(N)$.

The main idea behind the example is the following. The initial schedule satisfies condition (2) in Hamers et al. (1999), i.e., each job that is in the last position of a machine cannot make any profit by joining the end of a queue of any other machine. However, a switch between jobs 1 and 2, which does not change the total costs for these two players puts job 1 in a position that was excluded for *the initial schedule* by condition (2) in Hamers et al. (1999). Consequently, with the aid of job 2, job 1 can move to machine 2 to obtain cost-savings. Obviously, with the help of job 5, job 4 can obtain similar cost-savings. However, it is impossible for the two pairs of jobs to obtain the profit *simultaneously*. Hence, the grand coalition can only obtain the cost-savings that could be obtained by the pair job 1 and job 2 or by the pair job 4 and job 5, but not both.

The m -sequencing situation in example 3.1 is minimal in the sense that one cannot come up with an m -sequencing situation with less machines and/or less jobs that results in an m -sequencing game with an empty core. The fact that one cannot come up with an example with less machines follows from theorem 3.2 in Hamers et al. (1999), which states that 2-sequencing games are balanced, and the balancedness of 1-sequencing games, which was established in Curiel et al. (1989).⁴ Obviously, any m -sequencing games resulting from a situation with 3 or more machines and at most 3 jobs is balanced, since condition (2) of Hamers et al. (1999) implies that this associated m -sequencing game is a zero-game, i.e., a game that assigns zero to every coalition. Similarly, m -sequencing situations with 4 or more machines and 4 jobs result in balanced m -sequencing games. It remains to prove that any 3-sequencing situation with 4 jobs is balanced. This is covered in the following lemma.

Lemma 3.1 Let (M, N, b^0, p, α) be an m -sequencing situation such that $|M| = 3$ and $|N| = 4$. Then the corresponding 3-sequencing game (N, v) is balanced.

Proof: The only nontrivial case deals with situations in which all jobs have a (strictly) positive processing time, since otherwise, using condition (2) in Hamers et al. (1999), the zero-game will result.

For this case, it follows by the same condition that 2 machines have 1 job in their queue and 1 machine has two jobs. Without loss of generality assume that

$$b^0(i) = \begin{cases} (1, 1) & \text{if } i = 1; \\ (1, 2) & \text{if } i = 2; \\ (2, 1) & \text{if } i = 3; \\ (3, 1) & \text{if } i = 4. \end{cases} \quad (2)$$

It is a straightforward exercise to check that $v(S) = 0$ for all S with $2 \notin S$. By monotonicity of (N, v) we conclude subsequently that $(0, v(N), 0, 0)$ is a core-element of (N, v) . This implies that (N, v) is balanced. \square

⁴Curiel et al. (1989) proved that 1-sequencing games are convex, which implies that they are balanced. See section 4 in this paper for some remarks on theorem 3.2 in Hamers et al. (1999).

Combining this lemma with the arguments before the lemma provides the proof of the following theorem.

Theorem 3.1 Any m -sequencing situation with $|M| \leq 2$ or $|N| \leq 4$ results in a balanced m -sequencing game.

Proof: Follows directly from lemma 3.1 and the arguments before this lemma. □

4 Some remarks on Hamers et al. (1999)

In this section we will focus on an inconsistency in Hamers et al. (1999). First we will describe this inconsistency and its influence on implicit assumptions in Hamers et al. (1999). The direct influence on their results is then captured in three remarks. Subsequently, we show that the eventual influence is negligible in the sense that their proofs can be repaired.

On page 681 in Hamers et al. (1999) admissible schedules for a coalition are described. These schedules are determined by specifying conditions on switches between pairs of jobs. Consequently, this puts no restrictions on a replacement of a job that is at the end of the queue behind one machine to the end of the queue of another machine. Stated differently, cooperation of the last job on a machine is NOT needed to put another job behind him. This interpretation is confirmed by example 3.2 in Hamers et al. (1999) where these replacements are considered explicitly.

However, reading the proofs of Hamers et al. (1999) it looks as if these replacements are not allowed. Another and maybe more plausible explanation might be that Hamers et al. (1999) assumed that these replacements would never occur once attention is restricted to m -sequencing situations that satisfy the following condition: *'the starting time of a job that is in the last position on a machine with respect to b^0 is smaller than or equal to the completion time of each job that is in the last position with respect to b^0 on the other machines.'* (see Hamers et al. (1999), page 680). As can be concluded from example 3.1 this is not true.

Strengthening the condition to make sure that the replacements do not occur for any schedule, rather than for the initial schedule only, might solve the problem, but seems unnatural. A first-come-first-serve process naturally results in the condition on b^0 only.

Next we turn to the consequences of the inconsistency in Hamers et al. (1999). This will result in three remarks. In the previous section we showed that m -sequencing need not be balanced. Example 3.1 that was used to show this, resulted in an m -sequencing game that was not even superadditive. Hamers et al. (1999) use explicitly superadditivity of m -sequencing games (see Hamers et al. (1999), page 687, line 4).

Remark 4.1 M -sequencing games need not be superadditive.

Example 3.1 illustrates that m -machine games need not be superadditive either since the value of machine 1 equals the value of jobs of 1 and 2, i.e., 1. Similarly, we have that the value

of machine 3 equals 1, and the value of machines 1 and 3 together equals 1. This contradicts the statement that m -machine games are superadditive *'by definition'* (see Hamers et al. (1999), page 682, line 10).

Remark 4.2 M -machine games need not be superadditive.

To illustrate our third remark, we consider the following example.

Example 4.1 Let $M = \{1, 2\}$, $N = \{1, 2, 3, 4\}$, $p = (2, 1, 2, 3)$, and $\alpha = (4, 3, 4, 9)$. The initial order is given by

$$b^0(i) = \begin{cases} (1, 1) & \text{if } i = 1; \\ (1, 2) & \text{if } i = 2; \\ (1, 3) & \text{if } i = 3; \\ (2, 1) & \text{if } i = 4. \end{cases} \quad (3)$$

Let (N, v) be the associated 2-sequencing game. Obviously, one-player coalitions have value 0. Furthermore, it is easily checked that

$$\begin{aligned} v(\{1, 2\}) &= 2 \times 4 + 3 \times 3 - 3 \times 4 - 1 \times 3 = 8 + 9 - 12 - 3 = 2; \\ v(\{1, 3\}) &= v(\{1\}) + v(\{3\}) = 0; \\ v(\{2, 3\}) &= 3 \times 3 + 5 \times 4 - 4 \times 3 - 4 \times 4 = 9 + 20 - 12 - 16 = 1; \\ v(\{1, 2, 3\}) &= v(\{1, 2\}) = 2. \end{aligned}$$

Note that coalition $\{2, 3\}$ optimizes its optimal cost reduction by switching places, followed by a replacement of player 2 from the end of the queue behind machine 1 to the end of the queue behind machine 2. Using these values, we have

$$v(\{1, 2, 3\}) - v(\{2, 3\}) = 1 < 2 = v(\{1, 2\}) - v(\{2\}).$$

So, $(\{1, 2, 3\}, v_{|\{1,2,3\}})$ is not convex, i.e., $(N_1(b^0), v_{|N_1(b^0)})$ is not convex. \diamond

Example 4.1 shows that for an m -sequencing game (N, v) it need not be the case that $(N_k(b^0), v_{|N_k(b^0)})$ for a specific machine k is convex. Hamers et al. (1999) argue that such a subgame is always convex *'since the subgame $(N_k(b^0), v_{|N_k(b^0)})$ is a 1-machine sequencing game'* (see Hamers et al. (1999), page 686, line 15/16). Example 4.1 shows that the subgame $(N_k(b^0), v_{|N_k(b^0)})$ need not be a 1-machine sequencing game. The reason for this stems from the fact that after a rearrangement on a specific machine, the (new) last player(s) may prefer to switch to the end of the queue in front of another machine. This option is obviously not available in a 1-machine sequencing game.

Remark 4.3 Let (N, v) be a sequencing game and let $N_k(b^0)$ be the players in front of machine k according to the initial ordering b^0 . Then $(N_k(b^0), v_{|N_k(b^0)})$ need not be convex.

Considering these remarks, one may wonder whether theorems 3.1 and 3.2 of Hamers et al. (1999) are still valid. In the following we will prove that these results still hold true.

Theorem 4.1 (Theorem 3.1 of Hamers et al. (1999)) Let (M, N, b^0, p, α) be an m -sequencing situation. Let (N, v) be the associated m -sequencing game and let (M, w) be the corresponding m -machine game. Then (N, v) is balanced if and only if (M, w) is balanced.

Proof: The proof goes along the lines of the proof of theorem 3.1 of Hamers et al. (1999). Two adaptations to their proof should be made.

1. At the start of the third step, use that $(N_k(b^0), v_{|N_k(b^0)})$ is σ_k -component additive, where σ_k is the original order on machine k according to b^0 .⁵ This implies that $(x_i)_{i \in N_k(b^0)} \in C(N_k(b^0), v_{|N_k(b^0)})$, which follows similarly to proposition 1 in Curiel et al. (1994) where it is shown that the average of two marginal vectors belongs to the core. In fact, Curiel et al. (1994) prove that both marginal vectors belong to the core.
2. The third inequality in the last display follows from

$$v\left(\bigcup_{k \in M: \tilde{T}_k \neq \emptyset} N_k(b^0)\right) \geq v\left(\bigcup_{k \in M: \tilde{T}_k \neq \emptyset} \tilde{T}_k\right) + \sum_{k \in M: \tilde{T}_k \neq \emptyset} v(N_k(b^0) \setminus \tilde{T}_k).$$

This equation holds since the schedules within $\bigcup_{k \in M: \tilde{T}_k \neq \emptyset} \tilde{T}_k$ and $N_k(b^0) \setminus \tilde{T}_k$ for all $k \in M$ such that $\tilde{T}_k \neq \emptyset$ can straightforwardly be combined into a schedule for $\bigcup_{k \in M: \tilde{T}_k \neq \emptyset} N_k(b^0)$ such that the requested cost savings to prove the inequality is achieved. Note that this is possible since all 'tails under consideration' originally already belong to the same coalition, i.e., $\bigcup_{k \in M: \tilde{T}_k \neq \emptyset} \tilde{T}_k$.

This completes the proof. □

Though the published proof of theorem 3.2 in Hamers et al. (1999) is incorrect, the same authors provided a correct proof in a preliminary version of the paper (Hamers et al. (1998)).

Theorem 4.2 (Theorem 3.2 of Hamers et al. (1999)) Let (M, N, b^0, p, α) be such that $|M| = 2$. Then the corresponding 2-sequencing game (N, v) is balanced.

Proof: (Taken from Hamers et al. (1998)) Let i_1, \dots, i_{m_1} be the jobs on machine 1 such that $b^0(i_x) \prec b^0(i_y)$ if $x < y$ and let i_n, \dots, i_{m_1+1} be the jobs on machine 2 such that $b^0(i_x) \prec b^0(i_y)$ if $x > y$. Take $\sigma \in \Pi(N)$ such that $\sigma(j) = i_j$ for all $j \in N$. From superadditivity of (N, v) together with the conditions of admissible schedules it follows that (N, v) is a σ -component additive game.⁶ Since σ -component additive games are balanced (cf. Curiel et al.

⁵We have previously argued that m -sequencing games need not be superadditive (see remark 4.1). However, it is a straightforward exercise to show that the restriction of such a game to the jobs on one machine is superadditive.

⁶Though we have previously argued that m -sequencing games need not be superadditive, this holds for $m \geq 3$ only. It is a straightforward exercise to show that 2-sequencing games are superadditive.

(1994)), we have that 2-sequencing games are balanced. \square

Remark 4.4 In fact, the proof of Hamers et al. (1998) needs some additional remarks on superadditivity. Equivalently, we could have noted that though m -machine games need not be superadditive, this holds for $|m| \geq 3$ only. Using superadditivity for $m = 2$ allows for using the arguments in Hamers et al. (1999) to prove theorem 3.2 of Hamers et al. (1999).

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