

# Global and local asymptotics for the busy period of the M/G/1 queue

**Citation for published version (APA):**

Denisov, D. E., & Shneer, V. (2007). *Global and local asymptotics for the busy period of the M/G/1 queue*. (Report Eurandom; Vol. 2007001). Eurandom.

**Document status and date:**

Published: 01/01/2007

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

# Global and local asymptotics for the busy period of the M/G/1 queue

Denis Denisov and Vsevolod Shneer

Eurandom  
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

January 8, 2007

## Abstract

We consider an M/G/1 queue with subexponential service times. We give a simple derivation of the global and local asymptotics for the busy period. This analysis relies on the explicit formula for the joint distribution for the number of customers and the length of the busy period of the M/G/1 queue.

*2000 Mathematics Subject Classification:* 60K25, 60G50, 60F10.

*Keywords & Phrases:* Busy period, busy cycle, heavy-tailed distributions.

## 1 Introduction

Let  $A_1, A_2, \dots$  and  $B_1, B_2, \dots$  be independent sequences of independent and identically distributed random variables. We call  $\{A_i\}$  inter-arrival times and  $\{B_i\}$  service times. We assume throughout that  $\rho := \mathbf{E}\{B_1\}/\mathbf{E}\{A_1\} < 1$ , so that the system is stable. Denote also  $\xi_i = B_i - A_i$  and  $S_n = \sum_{i=1}^n \xi_i$ . We shall denote by  $A$ ,  $B$  and  $\xi$  random variables with the same distributions as  $A_1$ ,  $B_1$  and  $\xi_1$ , respectively. We use the usual notation: we denote by M/G/1 the system with exponential service times  $A_i$ . We are interested in the asymptotics for the busy period

$$\tau = B_1 + \dots + B_\nu$$

where  $\nu = \inf\{n : S_n < 0\}$ . The random variable  $\nu$  is the number of customers which arrive during the busy cycle.

In [11], it was shown that if  $B$  has a regularly varying distribution then

$$\mathbf{P}\{\tau > t\} \sim \mathbf{E}\{\nu\}\mathbf{P}\{B > (1 - \rho)t\}, \quad t \rightarrow \infty. \quad (1)$$

This result has been generalized in [12] to the case of the GI/G/1 queue and under the assumption that the tail  $\mathbf{P}\{B > t\}$  is of intermediate regular variation. Later on, it has been shown in [3] and [10], that asymptotics (1) hold for the GI/G/1 model for another subclass of heavy-tailed distributions which includes the Weibull distributions with parameter  $\alpha < 1/2$ . The tails of the distributions considered in [3] and [10] are heavier than  $e^{-\sqrt{t}}$ . As is shown in [2], the latter condition is crucial for asymptotics (1) to hold.

It was shown in [7] that in the  $M/G/1$  queue

$$\mathbf{P}(\tau > t) \sim C \frac{\mathbf{P}(X_t > 0)}{t} \quad (2)$$

with a constant  $C$ , where  $X_t = \sum_{i=0}^{N(t)} B_i - t$ , and  $N$  is a Poisson process with intensity  $\lambda = 1/\mathbf{E}A$ . It was also shown in [7] that in the case when the tail of  $B$  is heavier than  $e^{-\sqrt{t}}$  (and under some further conditions) the asymptotic equivalence (2) reduces to (1). There are two aims of this note. The first one is to give a simple proof of (1) for the  $M/G/1$  queue for some classes of distributions with tails lighter than  $e^{-\sqrt{t}}$ . This proof is based on the explicit formula for the joint distribution of  $\tau$  and  $\nu$  (see e.g. [6, (4.63)]):

$$\mathbf{P}(\nu = n, \tau > t) = \int_t^\infty \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u} \mathbf{P}(B_1 + \dots + B_n \in du) \quad (3)$$

Conditions imposed on the right tail  $\mathbf{P}\{B > t\}$  are close to minimal. These conditions include as particular cases all known conditions on the right tail that lead to (1).

The second (and probably more important) aim of this note consists in providing new results on local asymptotics for  $\tau$  as well as for  $\nu$  (the number of customers served during the busy period). These results are similar to (2):

$$\mathbf{P}(\tau \in (t, t+T]) \sim C \frac{\mathbf{P}(X_t \in (0, T])}{t}$$

with a constant  $C$ , if  $0 < T < \infty$  is fixed.

The paper is organised as follows. In Section 2 we present a simple proof of (1) for the  $M/G/1$ . In Section 3 we give results on local asymptotics for  $\tau$  and  $\nu$ .

## 2 Global asymptotics for the busy period

Put  $\bar{F}(x) = \mathbf{P}\{B > x\}$ . We consider 2 subclasses of subexponential distribution.

**Definition 2.1.** *We say that distribution function  $F$  is intermediate regularly varying ( $F \in \mathcal{IRV}$ ) if*

$$\limsup_{\kappa \downarrow 1} \limsup_{x \rightarrow \infty} \frac{\bar{F}(x)}{\bar{F}(\kappa x)} = 1.$$

If particular a regularly varying function  $F$  belongs to this class. Let  $h_n = \sqrt{n}$ . Let

$$\varepsilon(n) \equiv \sup_{x \geq 2h_n} \frac{\mathbf{P}\{\xi_1 > h_n, \xi_2 > h_n, S_2 > x\}}{\bar{F}(x)}.$$

We will use following conditions

(A)  $\mathbf{E}B < \infty$  and  $F \in \mathcal{IRV}$

(B)  $\mathbf{E}B^2 < \infty$ ,  $\bar{F}(x - \sqrt{x}) \sim \bar{F}(x)$  and

$$\varepsilon(n) = o(1/n).$$

The following proposition follows from the results of [8].

**Proposition 2.1.** *Let either condition (A) or (B) hold. Then*

$$\mathbf{P}\{S_n > 0\} \sim n\bar{F}(n|\mathbf{E}\xi|).$$

Note that square-root insensitivity condition  $\bar{F}(x) \sim \bar{F}(x - \sqrt{x})$  implies that

$$\frac{-\ln \bar{F}(x)}{\sqrt{x}} \rightarrow 0.$$

A simple sufficient condition for (B) to hold is the following proposition (see [8] for the proof).

**Proposition 2.2.** *Let  $\bar{F}(x) = x^{-2}e^{-g(x)}$ . If  $g(x)/x^\delta$  is eventually non-increasing for some  $\delta < 1$ , then  $\varepsilon(n) = o(1/n)$ .*

In particular, this Proposition implies that lognormal distributions and Weibull distributions with parameter  $\alpha < 1/2$  satisfy (B).

We now state the main result of this Section.

**Theorem 2.1.** *Let either condition (A) or (B) hold. Then,*

$$\mathbf{P}\{\tau > t\} \sim \mathbf{E}\{\nu\}\mathbf{P}\{B > (1 - \rho)t\}.$$

We also need the following result.

**Proposition 2.3.** *Let either condition (A) or (B) hold. Then,*

$$\mathbf{P}\{\nu > t\} \sim \mathbf{E}\{\nu\}\mathbf{P}\{B > |\mathbf{E}\{\xi\}|t\}, \quad t \rightarrow \infty. \quad (4)$$

PROOF OF PROPOSITION 2.3. It follows from the result of [5] (proof can be found, for example, in [9]) that

$$\mathbf{P}\{\nu > n\} \sim \mathbf{E}\{\nu\} \frac{\mathbf{P}\{S_n > 0\}}{n} \quad (5)$$

if the sequence  $a_n = \frac{\mathbf{P}\{S_n > 0\}}{n}$  is subexponential, i.e., if  $\sum_{k=0}^n a_k a_{n-k} \sim 2a_n \sum_{k=0}^{\infty} a_k$  as  $n \rightarrow \infty$ . It is proved in [8] that under the assumptions of the Proposition,

$$\frac{\mathbf{P}\{S_n > 0\}}{n} \sim \bar{F}(|\mathbf{E}\xi|n), \quad n \rightarrow \infty.$$

It is also proved there that the sequence  $\bar{F}(|\mathbf{E}\xi|n)$  is subexponential. □

PROOF OF THEOREM 2.1. For condition (B), the estimate from below is a corollary of the CLT and condition  $\bar{F}(x - \sqrt{x}) \sim \bar{F}(x)$ . The proof may be proved in [12]. For condition (A), the estimate follows similarly from the Law of Large Numbers.

Therefore, we will concentrate on the estimate from above. We will consider condition (B). The proof for condition (A) is similar. Note, that it follows from Proposition 2.3 and condition (B) that

$$\mathbf{P}(\nu > n) \sim \mathbf{P}(\nu > n - \sqrt{n}).$$

Therefore, there exists a function  $l(n) \uparrow \infty$  such that

$$\mathbf{P}(\nu > n) \sim \mathbf{P}(\nu > n - \sqrt{nl(n)}).$$

By the total probability formula,

$$\begin{aligned} \mathbf{P}(\tau > x) &= P_1 + P_2 + P_3 \\ &= \mathbf{P}(\tau > x, \nu \leq (1 - \varepsilon)\frac{x}{\mathbf{EA}}) \\ &+ \mathbf{P}(\tau > x, (1 - \varepsilon)\frac{x}{\mathbf{EA}} < \nu \leq \frac{x}{\mathbf{EA}} - \sqrt{x}l(x)) \\ &+ \mathbf{P}(\tau > x, \nu > \frac{x}{\mathbf{EA}} - \sqrt{x}l(x)). \end{aligned}$$

First,

$$\begin{aligned}
P_1 &= \mathbf{P}(B_1 + \cdots + B_\nu > x, \nu \leq (1 - \varepsilon) \frac{x}{\mathbf{E}A}) \\
&\leq \mathbf{P}(B_1 + \cdots + B_{(1-\varepsilon)\frac{x}{\mathbf{E}A}} > x, A_1 + \cdots + A_{(1-\varepsilon)\frac{x}{\mathbf{E}A}} > x) \\
&\leq \mathbf{P}(B_1 + \cdots + B_{\frac{x}{\mathbf{E}A}} > x) \mathbf{P}(A_1 + \cdots + A_{(1-\varepsilon)\frac{x}{\mathbf{E}A}} > x) \\
&= o(1/x) \mathbf{P}(B_1 + \cdots + B_{\frac{x}{\mathbf{E}A}} > x),
\end{aligned}$$

where that latter equivalence follows from the fact that, when  $\mathbf{E}A^2 < \infty$ ,

$$\mathbf{P}\left(\sum_{i=1}^n (A_i - \mathbf{E}A_i) > \varepsilon n\right) = o(1/n).$$

Then,

$$P_2 = \sum_{n=(1-\varepsilon)\frac{x}{\mathbf{E}A}}^{\frac{x}{\mathbf{E}A} - \sqrt{x}l(x)} \int_x^\infty \frac{\mathbf{P}(N(u) = n - 1)}{n} \mathbf{P}(B_1 + \cdots + B_n \in du).$$

For  $u \geq n - 1$ , probability  $\mathbf{P}(N(u) = n - 1)$  is non-increasing in  $u$ . Therefore,  $\mathbf{P}(N(u) = n - 1) \leq \mathbf{P}(N(x) = n - 1)$ , and this fact implies that

$$\begin{aligned}
P_2 &\leq \sum_{n=(1-\varepsilon)\frac{x}{\mathbf{E}A}}^{\frac{x}{\mathbf{E}A} - \sqrt{x}l(x)} \frac{\mathbf{P}(N(x) = n - 1)}{n} \mathbf{P}(B_1 + \cdots + B_n > x) \\
&\leq \frac{\mathbf{P}(B_1 + \cdots + B_{\frac{x}{\mathbf{E}A}} > x)}{(1 - \varepsilon) \frac{x}{\mathbf{E}A}} \mathbf{P}\left((1 - \varepsilon) \frac{x}{\mathbf{E}A} \leq N(x) \leq \frac{x}{\mathbf{E}A} - x^{1/\kappa}l(x)\right) \\
&\leq \frac{\mathbf{P}(B_1 + \cdots + B_{\frac{x}{\mathbf{E}A}} > x)}{(1 - \varepsilon) \frac{x}{\mathbf{E}A}} \mathbf{P}\left((1 - \varepsilon) \frac{x}{\mathbf{E}A} \leq N(x) \leq \frac{x}{\mathbf{E}A} - x^{1/2}l(x)\right)
\end{aligned}$$

By the Central Limit Theorem for renewal processes,

$$\mathbf{P}\left((1 - \varepsilon) \frac{x}{\mathbf{E}A} \leq N(x) \leq \frac{x}{\mathbf{E}A} - x^{1/2}l(x)\right) = o(1),$$

and, therefore,

$$P_2 = o(1/x) \mathbf{P}(B_1 + \cdots + B_{\frac{x}{\mathbf{E}A}} > x).$$

As a result, we have

$$P_1 + P_2 = o(1/x) \mathbf{P}(B_1 + \cdots + B_{\frac{x}{\mathbf{E}A}} > x) = o(\mathbf{P}(B > x - \rho x)).$$

Third term,

$$P_3 \leq (1 + o(1)) \mathbf{P}(\nu > \frac{x}{\mathbf{E}A}) = (1 + o(1)) \mathbf{E}\{\nu\} \mathbf{P}(B > x - \rho x),$$

where the latter equivalence follows from (4).

Finally, we have

$$\mathbf{P}(\tau > x) \leq (1 + o(1)) \mathbf{E}\{\nu\} \mathbf{P}(B > x - \rho x).$$

□

### 3 Local asymptotics for the busy period

It appears that explicit formula (3) is extremely convenient for the derivation of the local asymptotics. Recall that  $S_n = \sum_{i=1}^n (B_i - A_i)$  and  $X_t = \sum_{i=1}^{N(t)} B_i - t$ . We will start with the following simple result.

**Theorem 3.1.** *Assume that random variable  $B$  is absolutely continuous. Then,*

$$\mathbf{P}\{\nu = n\} = \frac{f_{S_n}(0)}{\lambda n}. \quad (6)$$

and

$$f_\tau(t) = \frac{f_{X_t}(0)}{\lambda t}. \quad (7)$$

Here  $f_\xi(x)$  denotes the density of random variable  $\xi$  at point  $x$ . Note that this Theorem holds not only for subexponential distributions, but for all distributions. It reduces the original problem to the simpler and well-studied problem of large deviations probabilities for random walks and Levy processes.

The next result holds for non-absolutely continuous random variables.

**Theorem 3.2.** *Let  $T > 0$  be a fixed number and  $\Delta = (0, T]$ . Assume that  $\mathbf{P}\{X_t \in \Delta\}e^{\delta t} \rightarrow \infty$  for any  $\delta > 0$ . Then,*

$$\mathbf{P}\{\tau \in t + \Delta\} \sim \frac{\mathbf{P}\{X_t \in \Delta\}}{\lambda t} \sim \frac{\mathbf{P}\{X_{[t]} \in \Delta\}}{\lambda [t]}, \quad \text{as } t \rightarrow \infty,$$

where  $[t]$  denotes the integer part of  $t$ .

We are going to give explicit asymptotics following from Theorem 3.1 and Theorem 3.2. First, we will give counterparts of condition (A) and (B). Fix  $T > 0$  and put  $\Delta = (0, T]$ .

**Definition 3.1.** *We say that function  $F(x + \Delta)$  is intermediate regularly varying if*

$$\limsup_{\kappa \downarrow 1} \limsup_{x \rightarrow \infty} \sup_{x \leq y \leq \kappa x} \left| \frac{F(y + \Delta)}{F(x + \Delta)} - 1 \right| = 0. \quad (8)$$

In particular, (8) holds when  $F(x + \Delta)$  is regularly varying at infinity. Let  $h_n = \sqrt{n}$ . Let

$$\varepsilon_\Delta(n) \equiv \sup_{x \geq 2h_n} \frac{\mathbf{P}\{\xi_1 > h_n, \xi_2 > h_n, S_2 \in x + \Delta\}}{F(x + \Delta)}.$$

We will use the following conditions.

**(A $_\Delta$ )**  $\mathbf{E}B < \infty$  and  $F(x + \Delta)$  is intermediate regularly varying

**(B $_\Delta$ )**  $\mathbf{E}B^2 < \infty$ ,

$$\sup_{y \leq \sqrt{x}} \left| \frac{F(x - y + \Delta)}{F(x + \Delta)} - 1 \right| \rightarrow 0$$

and

$$\varepsilon_\Delta(n) = o(1/n).$$

A simple sufficient condition for (B) to hold can be stated analogously to Proposition 2.2. Then, one can see that  $F(x + \Delta) \sim e^{-x^\beta}$ ,  $\beta < 1/2$  satisfy condition (B) as well as  $F(x + \Delta) \sim e^{-\ln^\beta x}$ ,  $\beta > 1$ . The following proposition follows from the results of [8].

**Proposition 3.1.** *Let either condition  $(A_\Delta)$  or  $(B_\Delta)$  hold. Then*

$$\mathbf{P}\{S_n \in \Delta\} \sim n\mathbf{P}\{\xi \in n|\mathbf{E}\xi| + \Delta\}$$

and hence (since  $N(t)$  is a Poisson process),

$$\mathbf{P}\{X_{[t]} \in \Delta\} \sim [t]\mathbf{P}\{X_1 \in [t]|\mathbf{E}X_1| + \Delta\}.$$

A similar result holds for densities. Using Proposition 3.1, Theorem 3.1 and Theorem 3.2 immediately allows us to obtain local asymptotics:

**Corollary 3.1.** *Assume that  $B$  is absolutely continuous. Let one of conditions  $(A_\Delta)$  or  $(B_\Delta)$  hold for all  $T > 0$ . Then,*

$$\mathbf{P}\{\nu = n\} \sim (\lambda)^{-1}f(n\mathbf{E}(B - A))$$

and

$$f_\tau(t) \sim f(t - \rho t).$$

**Corollary 3.2.** *Let either condition  $(A_\Delta)$  or  $(B_\Delta)$  hold for some  $T > 0$ . Then,*

$$\mathbf{P}\{\tau \in t + \Delta\} \sim F(t - \rho t + \Delta).$$

PROOF OF THEOREM 3.1. Put  $a = \lambda^{-1} = \mathbf{E}A, b = \mathbf{E}B$ . Then,

$$\mathbf{P}\{\nu = n\} = \int_0^\infty \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u} f_B^{*n}(u) du = \lambda^{-1} \int_0^\infty \frac{f_A^{*n}(u)}{n} f_B^{*n}(u) du = \lambda^{-1} \frac{f_{B-A}^{*n}(0)}{n}.$$

This implies the first statement of the Theorem.

Now we proceed to the local asymptotics for  $\tau$ ,

$$\begin{aligned} f_\tau(t) &= \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{n!} e^{-\lambda t} f_B^{*n}(t) = \frac{1}{\lambda t} \sum_{n=1}^{\infty} \mathbf{P}\{N(t) = n\} f_B^{*n}(t) \\ &= \frac{1}{\lambda t} \sum_{n=1}^{\infty} \mathbf{P}\{N(t) = n\} \frac{\mathbf{P}\{B_1 + \dots + B_n \in t + du\}}{du} \\ &= \frac{1}{\lambda t} \frac{\mathbf{P}\{\sum_{i=1}^{N(t)} B_i \in t + du\}}{du} = \frac{f_{X_t}(0)}{\lambda t}. \end{aligned}$$

□

*Proof of Theorem 3.2.* We will prove the asymptotic equivalence  $\mathbf{P}\{\tau \in t + \Delta\} \sim \mathbf{P}\{X_t \in \Delta\}/t$ . The proof of the equivalence  $\mathbf{P}\{\tau \in t + \Delta\} \sim \mathbf{P}\{X_{[t]} \in \Delta\}/[t]$  may be given following the same lines. Fix any  $\varepsilon > 0$ . According to formula (3),

$$\begin{aligned} \mathbf{P}(\tau \in t + \Delta) &= \sum_{n=1}^{\infty} \int_t^{t+\Delta} \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u} \mathbf{P}(B_1 + \dots + B_n \in du) \\ &= \sum_{n=1}^{\infty} \int_t^{t+\Delta} \frac{\mathbf{P}(N(u) = n)}{\lambda u} \mathbf{P}(B_1 + \dots + B_n \in du) \\ &= \sum_{n=1}^{\lambda t - \varepsilon t} + \sum_{\lambda t - \varepsilon t}^{\lambda t + \varepsilon t} + \sum_{\lambda t + \varepsilon t}^{\infty} (\dots) \equiv S_1 + S_2 + S_3. \end{aligned}$$

For the first sum we have:

$$\begin{aligned} S_1 &\leq \frac{1}{\lambda t} \sum_{n=1}^{\lambda t - \varepsilon t} \int_t^{t+T} \mathbf{P}(N(u) = n) \mathbf{P}(B_1 + \dots + B_n \in du) \\ &\leq \frac{1}{\lambda t} \mathbf{P}(N(t) \leq \lambda t - \varepsilon t) \sum_{n=1}^{\lambda t - \varepsilon t} \mathbf{P}\{B_1 + \dots + B_n \in t + \Delta\} = o(e^{-\delta t}) \end{aligned}$$

with some  $\delta > 0$ , since  $N(x)$  is Poisson distributed with parameter  $\lambda t$ . Indeed, it follows immediately from the exponential Chebyshev inequality that for some  $\delta > 0$ ,

$$\mathbf{P}\{|N(t) - \lambda t| \geq \varepsilon t\} = o(e^{-\delta t}).$$

It can be shown in a similar way that  $S_3 = o(e^{-\delta t})$ . Let us now investigate the asymptotic behaviour of the remaining sum:

$$S_2 = (1+o(1)) \frac{1}{\lambda t} \sum_{\lambda t - \varepsilon t}^{\lambda t + \varepsilon t} \frac{e^{-\lambda t} (\lambda t)^n}{n!} \int_t^{t+T} e^{-\lambda(u-t)} \left(1 + \frac{u-t}{t}\right)^n \mathbf{P}(B_1 + \dots + B_n \in du).$$

We now consider

$$\begin{aligned} e^{-\lambda(u-t)} \left(1 + \frac{u-t}{t}\right)^n &= e^{-\lambda(u-t) + n \log(1 + \frac{u-t}{t})} \leq \exp\left\{-\lambda(u-t) + n \frac{u-t}{t}\right\} \\ &\leq \exp\{-\lambda(u-t) + (\lambda + \varepsilon)(u-t)\} = \exp\{\varepsilon(u-t)\} \leq \exp\{\varepsilon T\}. \end{aligned}$$

We used here the facts that for the sum under consideration  $n \leq \lambda(1 + \varepsilon)t$ ,  $u \leq t + T$  and the inequality  $\log(1 + t) \leq t$ . In a similar way, with the help of the inequality  $\log(1 + t) \geq t/2$ ,  $t \leq 1$  one can prove that  $e^{-\lambda(u-t)} \left(1 + \frac{u-t}{t}\right)^n \geq \exp\{-\varepsilon T/2\}$ . Therefore,

$$\begin{aligned} S_2 &\leq (1 + o(1)) \frac{e^{\varepsilon T}}{\lambda t} \mathbf{P}(X_t \in \Delta, \lambda t - \varepsilon t \leq N(t) \leq \lambda t + \varepsilon t) \\ &\geq (1 + o(1)) \frac{e^{-\varepsilon T/2}}{\lambda t} \mathbf{P}(X_t \in \Delta, \lambda t - \varepsilon t \leq N(t) \leq \lambda t + \varepsilon t), \end{aligned}$$

where we denoted  $X_t = \sum_{i=0}^{N(t)} B_i - t$ . Note now that

$$\mathbf{P}(X_t \in \Delta, |N(t) - \lambda t| \geq \varepsilon t) \leq \mathbf{P}(|N(t) - \lambda t| \geq \varepsilon t) = o(e^{-\delta t}).$$

Hence, the sum under consideration is asymptotically equivalent to

$$\begin{aligned} \mathbf{P}\{\tau \in t + \Delta\} &\leq (1 + o(1)) \frac{e^{\varepsilon T}}{\lambda t} \mathbf{P}(X_t \in \Delta) + o(e^{-\delta t}) \\ &\geq (1 + o(1)) \frac{e^{-\varepsilon T/2}}{\lambda t} \mathbf{P}(X_t \in \Delta) + o(e^{-\delta t}). \end{aligned}$$

Using condition  $e^{\delta t} \mathbf{P}\{X_t \in \Delta\} \rightarrow \infty, \delta > 0$  and then letting  $\varepsilon \rightarrow 0$  we arrive at the conclusion of the Theorem. □

Proofs of Corollary 3.1 and Corollary 3.2 are absolutely analogous. Therefore, we only give the proof of Corollary 3.2.

*Proof of Corollary 3.2.* It follows from Proposition 3.1 and Theorem 3.2 that

$$\mathbf{P}\{\tau \in t + \Delta\} \sim \lambda^{-1} \mathbf{P}\{X_1 \in [t] | \mathbf{E}X_1| + \Delta\} = \lambda^{-1} \mathbf{P}\{X_1 \in [t] - \rho[t] + \Delta\},$$



since  $\mathbf{E}X_1 = \mathbf{E}N(1)\mathbf{E}B - 1 = \rho - 1$ . Now we should note that

$$\begin{aligned} \mathbf{P}\{X_1 \in [t] - \rho[t] + \Delta\} &\sim \mathbf{P}\left\{\sum_{i=1}^{N(1)} B_i \in [t] - \rho[t] + \Delta\right\} \\ &\sim \mathbf{E}N(1)\mathbf{P}(B \in [t] - \rho[t] + \Delta) \sim \mathbf{E}N(1)\mathbf{P}(B \in t - \rho t + \Delta), \end{aligned}$$

by the theorem on local behaviour for randomly stopped sums from [1]. Indeed, either condition (A) or (B) implies that  $F \in \mathcal{S}_\Delta$ , see [1] for this Theorem and definitions.  $\square$

Note that the Corollaries hold only for some classes of distributions such that  $F(x + \Delta) \gg e^{-\sqrt{x}}$ . It is also possible to obtain asymptotics for the distributions lighter than Weibull(1/2), but we should use results for large deviation of random walks other than Proposition 3.1. There are a lot of known results in that direction, we refer to the recent paper of Borovkov and Mogulskii [4] for some new results and review. We would also like to mention that the paper [4] contains a proof of the asymptotic equivalence  $\mathbf{P}\{S_n \in \Delta\} \sim n\mathbf{P}\{\xi \in n|\mathbf{E}\xi| + \Delta\}$  for distributions such that  $\mathbf{P}\{\xi > x\} = e^{-x^\beta l(x)}$  with  $\beta < 1/2$ , under some conditions on the function  $l(x)$ .

**Acknowledgements.** The authors would like to thank Serguei Foss for drawing their attention to this problem and Onno Boxma for a number of useful discussions and for many suggestions that helped to improve this manuscript. The research of both authors is supported by the Dutch BSIK project (*BRICKS*) and the EURO-NGI project.

## References

- [1] ASMUSSEN, S., FOSS, S., KORSHUNOV, D. (2003). Asymptotics for sums of random variables with local subexponential behaviour. *J. Theoret. Probab.* **16**, 489–518.
- [2] ASMUSSEN, S., KLÜPPELBERG, C., SIGMAN, K. (1999). Sampling at sub-exponential times, with queueing applications. *Stochastic Process. Appl.* **79**, 265–286.
- [3] BALTRŪNAS, A., DALEY, D. J., KLÜPPELBERG, C. (2004). Tail behaviour of the busy period of a *GI/GI/1* queue with subexponential service times. *Stochastic Process. Appl.* **111**, 237–258.
- [4] BOROVKOV, A. A., MOGULSKII, A. A. (2006). Integro-local and integral theorems for sums of random variables with semiexponential distributions. *Submitted*.
- [5] CHOVER, J., NEY, P., WAINGER, S. (1973). Functions of probability measures. *J. Analyse Math.* **17**, 255–302.
- [6] COHEN, J. W. (1982). *The Single Server Queue*, 2nd edn. North-Holland Publishing Co., Amsterdam-New York.
- [7] DENISOV, D., SHNEER, V. (2006). Asymptotics for first passage times of Levy processes and random walks. *EURANDOM Report 2006-017*.
- [8] DENISOV, D., DIEKER, T., SHNEER, V. (2007). Large deviations for random walks under subexponentiality: the big-jump domain. *In preparation*.
- [9] DONEY, R. A. (1989). On the asymptotic behaviour of first passage times for transient random walks. *Probab. Theory Related Fields* **81**, 239–246.

- [10] JELENKOVIĆ, P. R., MOMČILOVIĆ, P. (2004). Large deviations of square root insensitive random sums. *Math. Oper. Res.* **29**, no. 2, 398–406.
- [11] DE MEYER, A., TEUGELS, J. L. (1980). On the asymptotic behaviour of the distributions of the busy period and service time in  $M/G/1$ . *J. Appl. Probab.* **17**, no. 3, 802–813.
- [12] ZWART, A. P. (2001). Tail asymptotics for the busy period in the  $GI/G/1$  queue. *Math. Oper. Res.* **26**, no. 3, 485–493.