

Some remarks on the gap metric

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SOME REMARKS ON THE GAP METRIC

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Abstract

This paper presents the following results on the gap metric defined on the space of closed linear operators: 1) the topology introduced by the gap metric is a diagonal product topology; 2) if the gap of two closed operators is smaller than one, then the two directed gaps are the same; 3) a normalized right coprime factorization is constructed for given densely defined closed linear operators, and using this result an equivalent form of the gap metric is presented.

KEY WORDS : Closed linear operator, the gap metric, normalized
right coprime factorization.

0 INTRODUCTION The gap metric was introduced by [Kre.1947] in order to measure perturbations of closed linear operators. It was shown [Cor.1963, Kat.1966, Kra.1972, Zam.1980, Zhu.1987*] that the gap metric is a valuable tool for the perturbation analysis of closed linear operators and related problems. Many properties and some of their applications were collected in [Kat.1966]. In 1980, [Zam.1980] used the gap metric in system and control theory for the analysis of robust stabilization. [Zhu.1987] proved that the topology introduced by the gap metric, restricted to the set of linear time invariant (LTI) systems, which is a subspace of the space of the closed linear operators, has the diagonal product property. It was shown in [Zhu1988] that, if the gap of two LTI systems is smaller than one, then the two directed gaps are the same (for

simplicity, we refer to this property as coincidence property). The diagonal product property of the gap metric on the set of systems can be extended easily to the space of closed linear operators. However, we have to give a new procedure to prove that the gap metric defined on the space of the closed linear operators has the coincidence property. A normalized right coprime factorization is constructed for a given densely defined closed linear operator. Using this result, we present an equivalent form of the gap metric. In section 1, the definition of the gap metric is introduced. In section 2, we present the diagonal product property of the gap metric on the space of closed linear operators. The coincidence property is proved in section 3. Finally, in section 4, we construct a normalized right coprime factorization for a given densely defined closed linear operator and we give an equivalent form of the gap metric.

1 PRELIMINARY We first define the gap metric on the space consisting of all closed subspaces in a Hilbert space H . Suppose that φ and ψ are closed subspaces in H and let $\Pi(\varphi)$ denotes the orthogonal projection onto φ . Then the gap between φ and ψ is defined as

$$\delta(\varphi, \psi) = \max \{ \overrightarrow{\delta}(\varphi, \psi) , \overrightarrow{\delta}(\psi, \varphi) \}$$

where $\overrightarrow{\delta}(\varphi, \psi)$ is called the directed gap from φ to ψ and defined as

$$\overrightarrow{\delta}(\varphi, \psi) = \sup_{\substack{x \in \varphi \\ \|x\|=1}} \inf_{y \in \psi} \|x - y\|$$

It is easy to see that the directed gap has the following equivalent form

$$\overrightarrow{\delta}(\varphi, \psi) = \| (I - \Pi(\psi)) \Pi(\varphi) \|$$

One can easily prove that $\delta(\cdot, \cdot)$ is a metric on the space consisting of all closed subspaces in H .

Let X and Y be two Hilbert spaces and denote by $C(X, Y)$ the space of all closed linear operators mapping X to Y . We define the gap metric on the space $C(X, Y)$. For each element $P \in C(X, Y)$, the graph of P denoted by $G(P)$ is a closed subspace of $X \times Y$. For any two elements P_1 and P_2 in $C(X, Y)$, the gap $\delta(P_1, P_2)$ between P_1 and P_2 is defined as the gap $\delta(G(P_1), G(P_2))$ between their graphs i.e.

$$\delta(P_1, P_2) := \delta(G(P_1), G(P_2))$$

and the directed gap $\overrightarrow{\delta}(P_1, P_2)$ from P_1 to P_2 defined as the directed gap $\overrightarrow{\delta}(G(P_1), G(P_2))$ from the graph $G(P_1)$ of P_1 to the graph $G(P_2)$ of P_2 . In the sequel, we denote the orthogonal projection mapping $X \times Y$ onto $G(P)$ by $\Pi(P)$.

2 THE DIAGONAL PRODUCT PROPERTY In this section, we first introduce two inequalities related to the diagonal form of the closed linear operators. Their proofs can be found in [Zhu.1987]. The diagonal product property, which is our main result in this section, follows directly from the two inequalities.

Let X^i and Y^i ($i=1,2$) be Hilbert spaces and define the Hilbert spaces X and Y as

$$X := X^1 \times X^2 \qquad Y := Y^1 \times Y^2$$

Now suppose that $P_k \in C(X, Y)$ ($k=1,2$) have the following diagonal form

$$P_k = \begin{bmatrix} P_k^1 & 0 \\ 0 & P_k^2 \end{bmatrix} \qquad (k=1,2)$$

where $P_k^i \in C(X^i, Y^i)$ ($i=1,2$). Then we have

LEMMA 2.1

$$(2.1) \quad \delta(P_1^1, P_1^1) + \delta(P_1^2, P_1^2) \geq \delta(P_1, P_2) \geq \max \{ \delta(P_1^1, P_2^1), \delta(P_1^2, P_2^2) \}$$

Lemma 2.1 can be proved completely in the same way as the methods presented in [Zhu.1987]

for proving the corresponding results for LTI systems.

We introduce a parameter λ which is in a Hausdorff-topological space Γ . Assume that, for each $\lambda \in \Gamma$, $P_\lambda \in C(X, Y)$ has the following diagonal form

$$P_\lambda = \begin{bmatrix} P_\lambda^1 & 0 \\ 0 & P_\lambda^2 \end{bmatrix}$$

where $P_\lambda^i \in C(X^i, Y^i)$ ($i=1,2$). The following result follows from (2.1).

THEOREM 2.2

$$\delta(P_\lambda, P_{\lambda_0}) \longrightarrow 0 \quad (\lambda \longrightarrow \lambda_0)$$

if and only if

$$\delta(P_\lambda^i, P_{\lambda_0}^i) \longrightarrow 0 \quad (\lambda \longrightarrow \lambda_0)$$

holds for both $i=1$ and $i=2$.

It is easy to show that lemma 2.1, and theorem 2.2 are still valid for the case when $P = \text{diag} (P^1, P^2 \dots P^1)$. This is the diagonal product property of the gap topology on the space of the linear closed operators. An application of this property can be seen in [Zhu.1987]. Note that the gap topology is not a product topology in the sense that if P_λ has the following form

$$P_\lambda = \begin{bmatrix} P_\lambda^1 & P_\lambda^3 \\ P_\lambda^4 & P_\lambda^2 \end{bmatrix},$$

then $\delta(P_\lambda, P_{\lambda_0}) \longrightarrow 0$ neither implies nor is implied by $\delta(P_\lambda^i, P_{\lambda_0}^i) \longrightarrow 0$ simultaneously for $i = 1, 2, 3$, and 4 . This can be seen as follows: In [Zhu. 1987] it was proved that for finite dimensional LTI systems the gap topology is equivalent to the graph topology, and in [Vid. 1985, pp 247] it was shown that the graph topology defined for finite dimensional LTI systems is not a product topology.

3 THE COINCIDENCE PROPERTY In this section, we directly prove that, if the gap between two closed subspaces is smaller than 1, then the two directed gaps are the same.

THEOREM 3.1 Suppose that φ and ψ are closed subspaces in a Hilbert space H . If $\delta(\varphi, \psi) < 1$, then $\overline{\delta}(\varphi, \psi) = \overline{\delta}(\psi, \varphi)$.

To prove this theorem, we need

LEMMA 3.2 [Kra.1972, pp206] $\delta(\varphi, \psi) < 1$ if and only if $\Pi(\varphi)$ maps ψ bijectively onto

φ .

PROOF OF THEOREM 3.1 Suppose $\delta(\varphi, \psi) < 1$. Define $[\Pi(\psi)]_{\Gamma}$ as the restriction of $\Pi(\psi)$ to φ . By lemma 3.2, $[\Pi(\psi)]_{\Gamma}$ has a bounded inverse. Accordingly, we have

$$\begin{aligned} \overrightarrow{\delta}(\varphi, \psi)^2 &= \| (I - \Pi(\psi)) \Pi(\varphi) \|^2 \\ &= \sup_{\substack{x \in H \\ \|x\|=1}} \| (I - \Pi(\psi)) \Pi(\varphi) x \|^2 \\ &= \sup_{\substack{x \in \varphi \\ \|x\|=1}} \| (I - \Pi(\psi)) x \|^2 \\ &= \sup_{\substack{x \in \varphi \\ \|x\|=1}} \| x - [\Pi(\psi)]_{\Gamma} x \|^2 \\ &= 1 - \inf_{\substack{x \in \varphi \\ \|x\|=1}} \| [\Pi(\psi)]_{\Gamma} x \|^2 \\ &= 1 - \| [\Pi(\psi)]_{\Gamma}^{-1} \|^2 \end{aligned}$$

Similarly

$$\overrightarrow{\delta}(\psi, \varphi)^2 = 1 - \| [\Pi(\varphi)]_{\Gamma}^{-1} \|^2$$

Since

$$\langle [\Pi(\varphi)]_{\Gamma} x - x, y \rangle = \langle x, [\Pi(\psi)]_{\Gamma} y - y \rangle = 0 \quad \forall x \in \psi \quad \forall y \in \varphi$$

we have

$$\langle [\Pi(\varphi)]_{\Gamma} x, y \rangle = \langle x, y \rangle = \langle x, [\Pi(\psi)]_{\Gamma} y \rangle \quad \forall x \in \psi \quad \forall y \in \varphi$$

Hence $[\Pi(\varphi)]_{\Gamma}^* = [\Pi(\psi)]_{\Gamma}$. Consequently, $([\Pi(\varphi)]_{\Gamma}^{-1})^* = ([\Pi(\psi)]_{\Gamma}^{-1})$ and $\|([\Pi(\varphi)]_{\Gamma}^{-1})\| = \|([\Pi(\psi)]_{\Gamma}^{-1})\|$.

This implies that $\overrightarrow{\delta}(\varphi, \psi) = \overrightarrow{\delta}(\psi, \varphi)$.

Q.E.D..

Since the gap of two closed operators is defined as the gap of their graphs, it is a trivial consequence that, if the gap of two closed operators is smaller than 1, then the two directed gaps are the same.

4 NORMALIZED RIGHT COPRIME FACTORIZATION OF THE CLOSED LINEAR OPERATOR

For a linear operator P mapping Hilbert space X to Hilbert space Y , we say P has a right coprime factorization over the bounded linear operators if there exist bounded linear operators $D \in \hat{B}(X)$ and $N \in B(X, Y)$ such that

- 1) D is invertible;
- 2) there exist bounded linear operators $F \in B(X)$ and $H \in B(Y, X)$ such that
 $FD + HN = I$
- 3) $P = N D^{-1}$.

A right coprime factorization (D, N) of P is said to be normalized if

$$D^* D + N^* N = I.$$

In this section we construct a normalized right coprime factorization for an operator $P \in C(X, Y)$ which has a dense domain. For convenience, let $DC(X, Y)$ denote the subset consisting of all densely defined elements in $C(X, Y)$.

THEOREM 4.1 Suppose $P \in DC(X, Y)$, then there exists a bounded linear operator A mapping X onto $G(P)$ which satisfies

$$(4.2) \quad A^* A = I$$

$$(4.3) \quad \Pi(P) = A A^*.$$

If we partition A as

$$(4.4) \quad A = \begin{bmatrix} D \\ N \end{bmatrix}$$

such that $D \in B(X)$ and $N \in B(X, Y)$, then (D, N) is a normalized right coprime factorization of P .

The proof consists of the following lemmas.

LEMMA 4.2 [Rie.1953, pp. 307–] If $P \in DC(X, Y)$, then $(I + P^* P)^{-1}$ (which will be denoted by R_P) exists as a bounded self-adjoint positive operator mapping X to $\text{Dom}(P)$ bijectively. Moreover, $P R_P$ is also bounded and

$$\|R_P\| \leq 1 \qquad \|PR_P\| \leq 1.$$

LEMMA 4.3 [Cor.1963] If $P \in DC(X, Y)$, then the bounded self-adjoint positive operator $R_P := (I + P^*P)^{-1}$ has a unique bounded self-adjoint positive square root, which we denote by S_P , i.e.

$$R_P = S_P \cdot S_P$$

Moreover, S_P maps X to $\text{Dom}(P)$ bijectively and PS_P is bounded and

$$\|S_P\| \leq 1 \qquad \|PS_P\| \leq 1.$$

PROOF OF THEOREM 4.1 Suppose $P \in C(X, Y)$ and define

$$(4.5) \quad A := \begin{bmatrix} I \\ P \end{bmatrix} S_P$$

One can easily check that A satisfies the required conditions: (4.2) follows directly from (4.5) and (4.3) follows from the facts that $(AA^*)^2 = AA^*$, AA^* is self-adjoint, and $\text{Range } AA^* = G(P)$. Finally, according to (4.5) the (D, N) in partition (4.4) is (S_P, PS_P) , and one can easily show that (S_P, PS_P) is a normalized right coprime factorization of P . QED

In the rest of this section, we present an equivalent form of the gap metric using the operator A .

THEOREM 4.4 Suppose $P_i \in DC(X, Y)$ and A_i is defined by (4.5) with respect to P_i for $i = 1, 2$. Then $\delta(P_1, P_2) < 1$ if and only if $M := A_1^* A_2$ maps X to X bijectively. Further, if $\delta(P_1, P_2) < 1$, then

$$\delta(P_1, P_2)^2 = 1 - \|M^{-1}\|^{-2}$$

To prove this theorem we need

LEMMA 4.5 Let $P \in DC(X, Y)$ and B be a linear bounded operator mapping X to $X \times Y$. Then B maps X bijectively to $G(P)$ if and only if then there exists a unique linear bounded operator U mapping X to X bijectively such that $B = A U$, where A is defined by (4.5)

PROOF \Leftarrow This is obvious, because A maps X bijectively to $G(P)$.

\implies Since both B and A map X to $G(P)$ bijectively, for each $x \in X$

there is a unique $y \in X$ such that

$$(*) \quad Bx = Ay$$

and vice versa.

By (*) we can define a linear mapping U

$$Ux = y$$

It is obvious that U maps X to X bijectively (hence bounded) and $B = AU$. The uniqueness comes from the bijection of B and A . This completes the proof.

LEMMA 4.6 Assume P_i ($i=1,2$) $\in DC(X, Y)$ and define the operator B_1 as

$$(4.6) \quad B_1 := \Pi(P_1) A_2$$

where A_2 is defined by (4.5) corresponding to P_2 . Then $\delta(P_1, P_2) < 1$ if and only if B_1 maps X bijectively to $G(P_1)$.

PROOF This result follows from lemma 3.2 and the fact, that $\Pi(P_1)$ maps $G(P_2)$ bijectively to $G(P_1)$ if and only if $\Pi(P_1) A_2$ maps X bijectively to $G(P_2)$. Q.E.D..

PROOF OF THEOREM 4.4 According to (4.3) and (4.6), $B_1 = A_1 A_1^* A_2$. By lemma 4.6, $\delta(P_1, P_2) < 1$ if and only if B_1 maps X bijectively to $G(P_1)$. By lemma 4.5, B_1 maps X to $G(P_1)$ bijectively if and only if $A_1^* A_2$ maps X to X bijectively. This proves the first part. Suppose that $\delta(P_1, P_2) < 1$, then M is bijective, hence M^{-1} exists as a bounded operator. By definition

$$\begin{aligned} \overline{\delta}^2(P_2, P_1) &= \| (I - \Pi(P_1)) \Pi(P_2) \|^2 \\ &= \sup_{\substack{x \in X \times Y \\ \|x\|=1}} \| (I - \Pi(P_1)) \Pi(P_2) x \|^2 \\ &= \sup_{\substack{x \in G(P_2) \\ \|x\|=1}} \| (I - \Pi(P_1)) x \|^2 \\ &= \sup_{\substack{x \in A_2 X \\ \|x\|=1}} \| (I - \Pi(P_1)) x \|^2 \quad (G(P_2) = A_2 X) \end{aligned}$$

$$\begin{aligned}
 &= \sup_{\substack{x \in X \\ \|x\|=1}} \| (I - \Pi(P_1)) A_2 x \|^2 \quad (\|A_2 x\| \equiv \|x\|) \\
 &= \sup_{\substack{x \in X \\ \|x\|=1}} (\| A_2 x \|^2 - \| \Pi(P_1) A_2 x \|^2) \quad (\Pi(P_1) \text{ is orthogonal}) \\
 &= 1 - \inf_{\substack{x \in X \\ \|x\|=1}} \| \Pi(P_1) A_2 x \|^2 \\
 &= 1 - \inf_{\substack{x \in X \\ \|x\|=1}} \| A_1 A_1^* A_2 x \|^2 \\
 &= 1 - \inf_{\substack{x \in X \\ \|x\|=1}} \| A_1^* A_2 x \|^2 \quad (\|A_1 x\| \equiv \|x\|) \\
 &= 1 - \| (A_1^* A_2)^{-1} \|^2 \\
 &= 1 - \| M^{-1} \|^2
 \end{aligned}$$

According to section 3, the two directed gaps are the same. Consequently, our claim is true.
 QED.

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