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Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series A
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Vol. 37, No. 3, November 1984
Printed in Belgium

Note

Nonembeddable Quasi-residual Designs with Large K

J. H. VAN LINT* AND V. D. TONCHEV

Department of Mathematics, California Institute of Technology, Pasadena, California 91125, and
Institute of Mathematics, Bulgarian Academy of Sciences, Sofia 1090, P.O. Box 373, Bulgaria

Communicated by the Managing Editors

Received December 17, 1981

1. INTRODUCTION

We shall use the notation and definitions of [3] in which the first author studied nonembeddable quasi-residual designs with small k. In that paper a table of possible values of the parameters with $v \leqslant 16$ is given, with the restriction $k \leqslant \frac{1}{2}v + 1$ (this restriction is not clearly indicated in the paper). In [3] an example of a nonembeddable design with $v = 12$, $k = 6$, $\lambda = 5$ was given, which at that time was the smallest example. The case $v = 11$ was left open and subsequently settled by the second author (cf. [4]). If we allow large values of k we can easily find trivial nonembeddable quasiresidual designs. We mention two infinite sequences.

First, consider $BD(k+1, k; k(k+1), k^2, k(k-1))$. This design is the unique design with k copies of $J-I$ as incidence matrix. This design is therefore a residual design if and only if a $2-(k^2+k+1, k^2, k(k-1))$ exists. Since this design is the complementary design of a projective plane of order k the Bruck–Ryser theorem provides us with infinitely many nonembeddable examples, e.g., $BD(7, 6; 42, 36, 30)$ is a nonembeddable design with the smallest possible value of v.

In [3] the sequence $BD(k+2, k; \frac{1}{2}(k+2)(k+1), \frac{1}{2}k(k+1), \frac{1}{2}k(k-1))$ which contains infinitely many nonembeddable designs was mentioned. To complement the analysis of [3] we shall study quasi-residual designs with values of k satisfying $(v/2)+1 < k < v-2$.

Table I lists the parameter values of these designs with $v \leqslant 16$. We also list the complementary designs by giving the number in the table given in [2]. The table shows that at least one construction (and, in fact, usually very

*Present address: Eindhoven University of Technology, Eindhoven, The Netherlands.

0097-3165/84 $3.00

TABLE I

No.	v	k	b	r	λ	Complementary Design
1	9	6	24	16	10	2 copies of #2
2	10	7	30	21	14	#26
3	12	8	33	22	14	#48
4	12	9	44	33	24	#47
5	13	9	39	27	18	3 copies of #3
6	13	10	52	40	30	2 copies of #9
7	15	9	35	21	12	#83
8	15	10	42	28	18	#82
9	15	12	70	56	44	2 copies of #14
10	16	10	40	25	15	#96
11	16	11	48	33	22	#95
12	16	12	60	45	33	3 copies of #6
13	16	13	80	65	52	#94

many) for each of these designs is known. It is not unlikely that for each of the parameter sets a design exists which is nonembeddable. There are 13 nonisomorphic designs without repeated blocks with the parameter set No. 1 of the table (cf. [1]). We shall show below that the complementary design of I in [1] is nonembeddable. In fact, this is the first case of an infinite sequence which we treat in Section 2.

2. A Class of Nonembeddable Quasi-residual Designs

Let $l \equiv 0$ or $1 \pmod 3$ and let A be the l by $l(l-1)/3$ incidence matrix of a $2-(l, 3, 2)$ design. As usual we denote by E_i the matrix with 1's in row i and 0's elsewhere. The permutation matrix corresponding to the permutation $(1\ 2\ 3 \cdots l+1)$ is denoted by C. Define B by

$$B := \begin{pmatrix} A & E_1 & E_2 & \cdots & E_l \\ 0 & I+C & I+C^2 & \cdots & I+C^l \end{pmatrix}.$$

Then B is the $2l+1$ by $\frac{2}{3}l(2l+1)$ incidence matrix of a $2-(2l+1, 3, 2)$ design. Replacing all 0's by 1's and vice versa yields the incidence matrix $\bar{B}$ of a $2-(2l+1,\ 2l-2,\ \frac{2}{3}(l-1)(2l-3))$ design. The parameters of the design $\bar{B}$ are those of the residual of a symmetric $2-(\frac{2}{3}l(2l+1)+1,$ $\frac{4}{3}l(l-1), \frac{2}{3}(l-1)(2l-3))$ design. However, we claim that $\bar{B}$ is not a residual design if l is even. Suppose that $\bar{B}$ is embeddable in a design

$$\begin{pmatrix} 0 & \bar{B} \\ \underline{1} & D \end{pmatrix}.$$

Consider the binary code $\mathscr{C}$ generated by the rows of $\bar{B}$. Since $\lambda = \frac{2}{3}(l-1)(2l-3)$ is even, every row of D is a codeword in $\mathscr{C}^{\perp}$. By our construction the sum of the first l rows of $\bar{B}$ is the all one vector, i.e., every word in $\mathscr{C}^{\perp}$ has even weight. Since every vector in D has weight $\frac{4}{3}l(l-1)-1$, which is odd, we have a contradiction. Therefore, we have constructed a sequence of nonembeddable quasi-residual designs with parameters $v = 2l+1$, $k = 2l-2$, $\lambda = \frac{2}{3}(l-1)(2l-3)$, where $l \equiv 0$ or $4 \pmod 6$. The first case is $l = 4$, where we take $A := (J-I)_4$. This yields a nonembeddable design with the parameter set No. 1 of Table I. If we take $l = 6$ we find No. 6 of the table. From our construction of the $2-(9,3,2)$ design in the case $l = 4$ we see that it has an automorphism of order 5. Therefore it must be number I in the list [1]. We checked by computer that in fact only number I and II in the list [1] have nonembeddable complements.

There are several ways of generalizing the idea used above. We demonstrate this by an example. Let H be the 19 by 57 incidence matrix of design number 57 in Hall's list. Since this is a cyclic 2-(19, 4, 2) the matrix consisting of two copies of H can be regarded as $(M_1 M_2 M_3 M_4 M_5 M_6)$, where each M_i has row sum 4. Then

$$B := \begin{pmatrix} (J-I)_6 & E_1\, E_2 \cdots E_6 \\ 0 & M_1 M_2 \cdots M_6 \end{pmatrix}$$

is the incidence matrix of a $2-(25,5,4)$. Hence $\bar{B}$ is the incidence matrix of a quasi-residual design corresponding to a $2-(121,96,76)$. Since the first six rows of $\bar{B}$ add up to $\underline{1}$ we see that the quasi-residual design is not embeddable by the same argument we used. In fact we can generalize the whole idea as follows:

Let A_1 be the n by $n(n-1)/(2m+1)$ incidence matrix of a $2-(n, 2m+1, 2m)$ design D_1. Let D_2 be a $2-((2m-1)n+1, 2m, 2m)$ design whose blocks can be divided into n groups each with $(2m-1)n+1$ blocks so that each point occurs exactly $2m$ times in each group. In other words D_2 has an incidence matrix of the form

$$A_2 = (C_1, \ldots, C_n),$$

where C_i $(1 \leqslant i \leqslant n)$ is a square matrix of order $(2m-1)n+1$ with row sum $2m$. Then

$$B = \begin{pmatrix} A_1 & E_1 \cdots E_n \\ 0 & C_1 \cdots C_n \end{pmatrix}$$

is the incidence matrix of a $2-(2mn+1, 2m+1, 2m)$ design D. The complement of D is a quasi-residual $2-(2mn+1, 2m(n-1), 2m(n-1)$ $(2m(n-1)-1)/(2m+1))$ design $\bar{D}$, which is nonembeddable if n is even.

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A $2-((2m-1)n+1, 2m, 2m)$ design D_2 with $n \equiv 0 \pmod 2$ has the desired property if for instance $((2m-1)n+1, 2m) = (n-1, m) = 1$ and D_2 admits a regular automorphism group (e.g., if D_2 is cyclic).

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium