

# Non-embeddable quasi-residual designs with large K

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## Note

# Nonembeddable Quasi-residual Designs with Large $K$

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### 1. INTRODUCTION

We shall use the notation and definitions of [3] in which the first author studied nonembeddable quasi-residual designs with small  $k$ . In that paper a table of possible values of the parameters with  $v \leq 16$  is given, with the restriction  $k \leq \frac{1}{2}v + 1$  (this restriction is not clearly indicated in the paper). In [3] an example of a nonembeddable design with  $v = 12$ ,  $k = 6$ ,  $\lambda = 5$  was given, which at that time was the smallest example. The case  $v = 11$  was left open and subsequently settled by the second author (cf. [4]). If we allow large values of  $k$  we can easily find trivial nonembeddable quasiresidual designs. We mention two infinite sequences.

First, consider  $BD(k + 1, k; k(k + 1), k^2, k(k - 1))$ . This design is the unique design with  $k$  copies of  $J - I$  as incidence matrix. This design is therefore a residual design if and only if a  $2 - (k^2 + k + 1, k^2, k(k - 1))$  exists. Since this design is the complementary design of a projective plane of order  $k$  the Bruck-Ryser theorem provides us with infinitely many nonembeddable examples, e.g.,  $BD(7, 6; 42, 36, 30)$  is a nonembeddable design with the smallest possible value of  $v$ .

In [3] the sequence  $BD(k + 2, k; \frac{1}{2}(k + 2)(k + 1), \frac{1}{2}k(k + 1), \frac{1}{2}k(k - 1))$  which contains infinitely many nonembeddable designs was mentioned. To complement the analysis of [3] we shall study quasi-residual designs with values of  $k$  satisfying  $(v/2) + 1 < k < v - 2$ .

Table I lists the parameter values of these designs with  $v \leq 16$ . We also list the complementary designs by giving the number in the table given in [2]. The table shows that at least one construction (and, in fact, usually very

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TABLE I

| No. | $v$ | $k$ | $b$ | $r$ | $\lambda$ | Complementary Design |
|-----|-----|-----|-----|-----|-----------|----------------------|
| 1   | 9   | 6   | 24  | 16  | 10        | 2 copies of #2       |
| 2   | 10  | 7   | 30  | 21  | 14        | #26                  |
| 3   | 12  | 8   | 33  | 22  | 14        | #48                  |
| 4   | 12  | 9   | 44  | 33  | 24        | #47                  |
| 5   | 13  | 9   | 39  | 27  | 18        | 3 copies of #3       |
| 6   | 13  | 10  | 52  | 40  | 30        | 2 copies of #9       |
| 7   | 15  | 9   | 35  | 21  | 12        | #83                  |
| 8   | 15  | 10  | 42  | 28  | 18        | #82                  |
| 9   | 15  | 12  | 70  | 56  | 44        | 2 copies of #14      |
| 10  | 16  | 10  | 40  | 25  | 15        | #96                  |
| 11  | 16  | 11  | 48  | 33  | 22        | #95                  |
| 12  | 16  | 12  | 60  | 45  | 33        | 3 copies of #6       |
| 13  | 16  | 13  | 80  | 65  | 52        | #94                  |

many) for each of these designs is known. It is not unlikely that for each of the parameter sets a design exists which is nonembeddable. There are 13 nonisomorphic designs without repeated blocks with the parameter set No. 1 of the table (cf. [1]). We shall show below that the complementary design of  $I$  in [1] is nonembeddable. In fact, this is the first case of an infinite sequence which we treat in Section 2.

2. A CLASS OF NONEMBEDDABLE QUASI-RESIDUAL DESIGNS

Let  $l \equiv 0$  or  $1 \pmod{3}$  and let  $A$  be the  $l$  by  $l(l-1)/3$  incidence matrix of a  $2 - (l, 3, 2)$  design. As usual we denote by  $E_i$  the matrix with 1's in row  $i$  and 0's elsewhere. The permutation matrix corresponding to the permutation  $(1\ 2\ 3 \dots l+1)$  is denoted by  $C$ . Define  $B$  by

$$B := \begin{pmatrix} A & E_1 & E_2 & \dots & E_l \\ 0 & I + C & I + C^2 & \dots & I + C^l \end{pmatrix}.$$

Then  $B$  is the  $2l+1$  by  $\frac{2}{3}l(2l+1)$  incidence matrix of a  $2 - (2l+1, 3, 2)$  design. Replacing all 0's by 1's and vice versa yields the incidence matrix  $\bar{B}$  of a  $2 - (2l+1, 2l-2, \frac{2}{3}(l-1)(2l-3))$  design. The parameters of the design  $\bar{B}$  are those of the residual of a symmetric  $2 - (\frac{2}{3}l(2l+1) + 1, \frac{4}{3}l(l-1), \frac{2}{3}(l-1)(2l-3))$  design. However, we claim that  $\bar{B}$  is not a residual design if  $l$  is even. Suppose that  $\bar{B}$  is embeddable in a design

$$\begin{pmatrix} 0 & \bar{B} \\ 1 & D \end{pmatrix}.$$

Consider the binary code  $\mathcal{C}$  generated by the rows of  $\bar{B}$ . Since  $\lambda = \frac{2}{3}(l-1)(2l-3)$  is even, every row of  $D$  is a codeword in  $\mathcal{C}^\perp$ . By our construction the sum of the first  $l$  rows of  $\bar{B}$  is the all one vector, i.e., every word in  $\mathcal{C}^\perp$  has even weight. Since every vector in  $D$  has weight  $\frac{4}{3}l(l-1) - 1$ , which is odd, we have a contradiction. Therefore, we have constructed a sequence of nonembeddable quasi-residual designs with parameters  $v = 2l + 1$ ,  $k = 2l - 2$ ,  $\lambda = \frac{2}{3}(l-1)(2l-3)$ , where  $l \equiv 0$  or  $4 \pmod{6}$ . The first case is  $l = 4$ , where we take  $A := (J - I)_4$ . This yields a nonembeddable design with the parameter set No. 1 of Table I. If we take  $l = 6$  we find No. 6 of the table. From our construction of the  $2 - (9, 3, 2)$  design in the case  $l = 4$  we see that it has an automorphism of order 5. Therefore it must be number I in the list [1]. We checked by computer that in fact only number I and II in the list [1] have nonembeddable complements.

There are several ways of generalizing the idea used above. We demonstrate this by an example. Let  $H$  be the 19 by 57 incidence matrix of design number 57 in Hall's list. Since this is a cyclic  $2 - (19, 4, 2)$  the matrix consisting of two copies of  $H$  can be regarded as  $(M_1 M_2 M_3 M_4 M_5 M_6)$ , where each  $M_i$  has row sum 4. Then

$$B := \begin{pmatrix} (J - I)_6 & E_1 E_2 \cdots E_6 \\ 0 & M_1 M_2 \cdots M_6 \end{pmatrix}$$

is the incidence matrix of a  $2 - (25, 5, 4)$ . Hence  $\bar{B}$  is the incidence matrix of a quasi-residual design corresponding to a  $2 - (121, 96, 76)$ . Since the first six rows of  $\bar{B}$  add up to  $\underline{1}$  we see that the quasi-residual design is not embeddable by the same argument we used. In fact we can generalize the whole idea as follows:

Let  $A_1$  be the  $n$  by  $n(n-1)/(2m+1)$  incidence matrix of a  $2 - (n, 2m+1, 2m)$  design  $D_1$ . Let  $D_2$  be a  $2 - ((2m-1)n+1, 2m, 2m)$  design whose blocks can be divided into  $n$  groups each with  $(2m-1)n+1$  blocks so that each point occurs exactly  $2m$  times in each group. In other words  $D_2$  has an incidence matrix of the form

$$A_2 = (C_1, \dots, C_n),$$

where  $C_i$  ( $1 \leq i \leq n$ ) is a square matrix of order  $(2m-1)n+1$  with row sum  $2m$ . Then

$$B = \begin{pmatrix} A_1 & E_1 \cdots E_n \\ 0 & C_1 \cdots C_n \end{pmatrix}$$

is the incidence matrix of a  $2 - (2mn+1, 2m+1, 2m)$  design  $D$ . The complement of  $D$  is a quasi-residual  $2 - (2mn+1, 2m(n-1), 2m(n-1)(2m(n-1)-1)/(2m+1))$  design  $\bar{D}$ , which is nonembeddable if  $n$  is even.

A  $2 - ((2m - 1)n + 1, 2m, 2m)$  design  $D_2$  with  $n \equiv 0 \pmod{2}$  has the desired property if for instance  $((2m - 1)n + 1, 2m) = (n - 1, m) = 1$  and  $D_2$  admits a regular automorphism group (e.g., if  $D_2$  is cyclic).

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