$L^p$-inversion of the diffusion equation

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$L^p$-INVERSION OF THE
DIFFUSION EQUATION
by
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Abstract

The ranges of the propagation operator for the diffusion equation are characterized. Thus, the formal inversion formula for the diffusion equation is made precise in $L^p$ space setting.

Consider the diffusion equation in the space of tempered distributions $S'$:

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.
$$

(1)

Here the differentiation with respect to $x$ is in the sense of that of tempered distributions, while the differentiation with respect to $t$ is in the topology of $S'$. Since the Fourier transform $F$ and its inverse $F^{-1}$ are continuous on $S'$, equation (1) is equivalently transformed into

$$
\frac{\partial (Fu)}{\partial t} = -x^2 Fu.
$$

(2)

From this we get immediately the propagator of the initial value problem of (1):

$$
T(t) u = e^{i \frac{x^2}{4t}} F^{-1} [e^{-i\alpha^2} F u] = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} *_{(x)} u
$$

(3)

where $*_{(x)}$ denotes convolution with respect to $x$. One may directly check that the family

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\{T(t) \mid t \geq 0\} of operators defined in (3) is indeed a \(C_0\) semigroup on each of the Banach spaces \(L^p(\mathbb{R})\) (\(1 \leq p < \infty\)). Since \(S\), the Schwartz test space of rapidly decreasing functions, is dense in \(L^p(\mathbb{R})\) and \(T(t)S \subset S\) for all \(t \geq 0\), it serves as a core for the generator \(A_p\) of \(\{T(t) \mid t \geq 0\}\) in \(L^p(\mathbb{R})\). Of course, one can also show straightforwardly that the operator \(\frac{\partial^2}{\partial x^2}\) as defined on \(S\) is an essentially dissipative operator in \(L^p(\mathbb{R})\) and so generates a \(C_0\) semigroup on \(L^p(\mathbb{R})\).

The purpose of this note is to answer the twin questions: Which are the possible states of the system at time \(t\) if the system starts evolution according to equation (1) from initial states in \(L^p(\mathbb{R})\)? What is the initial state in \(L^p(\mathbb{R})\) if the state at time \(t\) is known?

**Proposition 1.** If \(u \in L^p(\mathbb{R})\), then \(v = T(t)u\) extends to an entire function \(v(z) (z = x + iy)\) such that

\[
\|v(x+iy)\|_{p,s} \leq e^{\frac{\gamma^2}{4t}} \|u\|_p.
\]  

**(Proof.** Define

\[
v(z) = v(x + iy) = \frac{1}{2^{\sqrt{\pi t}}} \int e^{-\frac{(x+iy)^2}{4t}} u(x) \ast (x) \, dx
= \frac{1}{2^{\sqrt{\pi t}}} \int e^{-\frac{(x-z)^2 + 2i(x-z)y}{4t}} u(z) \, dz.
\]

Since the integral in (5) converges uniformly on each compact set of \(z\), \(v(z)\) is a well defined entire function. In view of (3) it is an extension of \(v(x) = T(t)u\). From equality (5) by Young's inequality we have immediately (4). Indeed

\[
\frac{1}{2^{\sqrt{\pi t}}} \int e^{-\frac{(x-z)^2 + 2i(x-z)y}{4t}} \, dz = \frac{1}{2^{\sqrt{\pi t}}} \int e^{-\frac{x^2}{4t}} \, dx = 1.
\]

\[\]

**Definition 2.** Given \(1 \leq p \leq \infty\) and \(s > 0\). Let \(A_{p,s}\) denote the normed space of entire functions \(v(\zeta)\) such that

\[
\|v\|_{p,s} = \sup_{x \in \mathbb{R}} e^{-\gamma^2} \left( \int e^{-\frac{(x+iy)^2}{4t}} dx \right)^{1/p} < \infty.
\]

In the case \(p = \infty\) the above equality should be understood as follows:
Lemma 3. \( A^{p,s} \hookrightarrow A^{\infty,s'} \) for any \( 1 \leq p < \infty \) and \( 0 < s < s' < \infty \). Namely \( A^{p,s} \subset A^{\infty,s'} \) and there exists a constant \( \alpha \) depending only on \( p,s \) and \( s' \) such that

\[
\sup_{x+iy \in \mathcal{C}} |v(x+iy)| \leq \alpha |u|_{p,s} e^{s'y^2}.
\]

(8)

**Proof.** Let \( R > 0 \) be fixed. By the mean value theorem we have

\[
v(x+iy) = \frac{1}{\pi R^2} \int_{|\zeta+i\eta| < R} v[(x+\zeta)+i(y+\eta)] d\zeta d\eta.
\]

So

\[
|v(x+iy)| \leq \frac{1}{\pi R^2} (\pi R^2)^{1-p} \left( \sup_{|\zeta+i\eta| < R} |v[(x+\zeta)+i(y+\eta)]|^p d\zeta d\eta \right)^{1/p}
\]

\[
\leq (trR)^{-1/p} (2R) \sup_{|\eta-y| < R} |v(\zeta+i\eta)|^p d\zeta
\]

\[
\leq (2/\pi R)^{1/p} |u|_{p,s} e^{s'(y+R)^2}.
\]

(9)

This implies that \( A^{p,s} \subset A^{\infty,s'} \) and inequality (8) holds, for \( 2|y| R \leq e y^2 + e^{-1} R^2 \) (\( e > 0 \) arbitrary). That is, \( A^{p,s} \hookrightarrow A^{\infty,s'} \).

Corollary 4. For \( 1 \leq p \leq \infty \) and \( s > 0 \) the normed space \( A^{p,s} \) is complete so it is a Banach space.

**Proof.** Let us first show that \( A^{\infty,s} \) is complete. Let \( \{v_n\} \) be a Cauchy sequence in \( A^{\infty,s} \). Then for any \( \varepsilon > 0 \) there exists \( N \) such that

\[
e^{-sy^2} |v_n(x+iy)-v_m(x+iy)| \leq \varepsilon \text{ for all } n,m \geq N.
\]

(10)

Hence the sequence \( \{v_n(x+iy)\} \) of functions converges uniformly on each compact subset of \( \mathcal{C} \), so to an entire function \( v(x+iy) \). Letting \( m \to \infty \) in the last inequality we have

\[
e^{sy^2} |v_n(x+iy)-v(x+iy)| \leq \varepsilon \text{ for } n \geq N.
\]

Therefore \( v_n - v \) belongs to \( A^{\infty,s} \), so does \( v \). Moreover \( \{v_n\} \to v \) in \( A^{\infty,s} \); \( A^{\infty,s} \) is complete.
Next let $1 \leq p < \infty$. Let $\{v_n\}$ be a Cauchy sequence in $A^{p,s}$. So, for any $\varepsilon > 0$ there exists $N$ such that

$$e^{-\varepsilon y^2} \left( \int_{\mathbb{R}} |v_n(x+iy) - v_m(x+iy)|^p \, dx \right)^{1/p} \leq \varepsilon \text{ for all } n, m > N \text{ and } y \in \mathbb{R}.$$  

(11)

Lemma 3 shows that $\{v_n\}$ is a Cauchy sequence in $A^{\infty,s'}$ for $s' > s$. By the completeness of $A^{\infty,s'}$ proved above $\{v_n\}$ converges to an entire function $v$. Letting $m \to \infty$ in (11) by Lebesgue's dominance convergence theorem we then obtain

$$e^{-\varepsilon y^2} \left( \int_{\mathbb{R}} |v(x+iy) - v(x+iy)|^p \, dx \right)^{1/p} \leq \varepsilon \text{ for } n > N.$$  

Therefore $v_n - v$ belongs to $A^{p,s}$, so does $v$ and $\{v_n\} \to v$ in $A^{p,s}$. We have thus proved that $A^{p,s}$ is complete.

Proposition 5. Assume that $v \in A^{p,s}$ ($1 \leq p < \infty$, $s > 0$) and $t < 1/4s$. Then

(i) The function

$$u(x) = \frac{1}{2^{\sqrt{-1}t}} \int_{\mathbb{R}} e^{-\frac{1}{4t}(x-i\eta)^2} \, v(i\eta) \, d\eta, \quad x \in \mathbb{R}$$  

(12)

is well defined and is equivalently given by

$$u(x) = \frac{1}{2^{\sqrt{-1}t}} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{4t}(x-\zeta)^2} \, v(\zeta) \, d\zeta, \quad x \in \mathbb{R}$$  

(13)

($c \in \mathbb{R}$ arbitrary), in particular

$$u(x) = \frac{1}{2^{\sqrt{-1}t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t} \eta^2} \, v(x+i\eta) \, d\eta.$$  

(14)

(ii) $u(x) \in L^p(\mathbb{R})$, and for any $a \in (s, 1/4t)$ there holds the estimate ($q = p^* = \frac{p}{p-1}$):

$$\|u\|_p \leq \frac{\|v\|_{p,s}}{2^{\sqrt{-1}t}} \|e^{-\frac{1}{4t}(a-x)^2}\|_q \|e^{-\frac{1}{4t}a^2}\|_p.$$  

(15)

(iii) $T(t) u = e^{t \frac{\partial}{\partial x}} u = v$.

Proof:

(i) For any $s' \in (s, 1/4t)$ Lemma 3 ensures the existence of some constant $\alpha$ such that inequality
(8) holds. Therefore a function \( u \) is well defined by equality (12). Furthermore, by Cauchy’s contour integral theorem we have

\[
u(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{4t}(x-\zeta)^2} v(\zeta) \, d\zeta
\]

\[
u(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{4t}(x-\zeta)^2} v(\zeta) \, d\zeta.
\]

In particular \( u \) is given by (14) if \( c \) assumes \( x \).

(ii) By Hölder’s inequality and Fubini’s theorem we have

\[
(2\sqrt{\pi t})^p \int_{-\infty}^{+\infty} |u(x)|^p \, dx
\]

\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\pi|\eta|^2} \eta^2} \eta \, d\eta \, dx \quad \text{by (14)}
\]

\[
\leq \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} e^{-\frac{1}{2\pi|\eta|^2} \eta^2} \eta \, d\eta \right)^{p/q} \left( \int_{-\infty}^{+\infty} |v(x+i\eta)|^p \, d\eta \right)^{q/p}
\]

\[
= \|e^{-(\frac{1}{2\pi} |\eta|^2)}\|_q^p \int_{-\infty}^{+\infty} e^{\frac{1}{p} |\eta|^2} \left( \int_{-\infty}^{+\infty} |v(x+i\eta)|^p \, d\eta \right) \, d\eta
\]

\[
\leq \|e^{-(\frac{1}{2\pi} |\eta|^2)}\|_q^p \|v\|_{L^p}^p \int_{-\infty}^{+\infty} e^{-p(|\eta|^2)} \, d\eta
\]

\[
= \|e^{-(\frac{1}{2\pi} |\eta|^2)}\|_q^p \|v\|_{L^p}^p.
\]

Thus \( u \in L^p(\mathbb{R}) \) and there holds the estimate (15).

(iii) Put \( w(x) = e^{-\frac{1}{4t^2} x^2} u(x) \). Then, by (5) and (14) we have

\[
w(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{1}{4t^2} x^2} u(x)
\]

\[
= \frac{1}{4\pi t} \int e^{-\frac{1}{4t^2} (x-\zeta)^2} \frac{1}{4t^2} |\eta|^2 v(\zeta+i\eta) d\zeta d\eta.
\]
\[ \begin{align*}
\frac{1}{4\pi t} & \int_0^\infty e^{-\frac{r^2}{4t}} \int_0^{2\pi} v(x+re^{i\theta}) d\theta \\
& = \frac{1}{4\pi t} \int_0^\infty 2\pi v(x) re^{-\frac{r^2}{4t}} dr \\
& = v(x) \quad (x \in \mathbb{R}).
\end{align*} \tag{17} \]

We remark that the form of the above inversion formula (12) was suggested by Widder in [2]. See also [1], Section 5.4.

**Definition 6.** For \( s \in (0,\infty] \) let \( A_*^{p,s} = \bigcup_{\sigma < s} A_*^{p,\sigma} \) be the inductive limit of the family of Banach spaces \( \{ A_*^{p,\sigma} \mid \sigma < s \} \). For \( s \in [0,\infty) \) let \( A_*^{p,s} = \bigcap_{\sigma > s} A_*^{p,\sigma} \) be the projective limit of the family of Banach spaces \( \{ A_*^{p,\sigma} \mid \sigma > s \} \).

Let \( (L^p)_t^{(\frac{\partial^2}{\partial x^2})} \) be the range at time \( t \) of the propagator of the diffusion equation (1) in \( L^p(\mathbb{R}) \), i.e., \( R(T(t)) = R(e^{t \frac{\partial^2}{\partial x^2}}) \). With the graph norm it is a Banach space. Moreover, \( (L^p)_t^{(\frac{\partial^2}{\partial x^2})} \rightarrow (L^p)^{\sigma}(\frac{\partial^2}{\partial x^2}) \) if \( t > s \). Let \( (L^p)^{\sigma}(\frac{\partial^2}{\partial x^2}) = \bigcup_{\sigma > t} (L^p)^{\sigma}(\frac{\partial^2}{\partial x^2}) \) \( (0 \leq t < \infty) \) be the inductive limit, and \( (L^p)_t^{(\frac{\partial^2}{\partial x^2})} = \bigcap_{\sigma < t} (L^p)^{\sigma}(\frac{\partial^2}{\partial x^2}) \) \( (0 < t \leq \infty) \) be the projective limit.

Summarizing Propositions 1 and 5 we obtain the characterization:

**Theorem 7.** For \( t \in [0,\infty) \), \( (L^p)^{1+}(\frac{\partial^2}{\partial x^2}) = A_*^{p,1/4t} \) topologically. For \( t \in (0,\infty) \), \( (L^p)^{1-}(\frac{\partial^2}{\partial x^2}) = A_*^{p,1/4t} \) topologically.

Therefore, in an obvious sense we have \( (L^p)^{1}(\frac{\partial^2}{\partial x^2}) \approx A_*^{p,1/4t} \). In the special case \( p = 2 \), however, we can characterize each of the Hilbert spaces \( (L^2)^{1}(\frac{\partial^2}{\partial x^2}) \) exactly.

**Theorem 8.** \( (L^2)^{1}(\frac{\partial^2}{\partial x^2}) \) is isometrically equivalent to the Hilbert space of entire functions \( v \) such that
\[ |v|_2^2 = \frac{1}{\sqrt{2\pi t}} \int \int_{\mathbb{C}} |v(x+iy)|^2 e^{-\gamma^2 u} \, dx \, dy < \infty. \]

**Proof.** For \( u \in L^2 \) and \( v = e^{\frac{\partial^2}{\partial x^2}} u \), using Plancherel's theorem we have

\[
\frac{1}{\sqrt{2\pi t}} \int \int_{\mathbb{C}} |v(x+iy)|^2 e^{-\gamma^2 u} \, dx \, dy
\]

\[
= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\gamma^2 u} \, dy \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\alpha k} e^{-ky-k\gamma^2 u} \, (F u) \, dk \, |F u|^2
\]

\[
= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-2k^2} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ky-k\gamma^2 u} \, (F u) \, dk \, |F u|^2
\]

\[
= \int_{\mathbb{R}} |F u|^2 \, dk
\]

\[
= \int_{\mathbb{R}} |u|^2 \, dx. \tag{18}
\]

This together with Theorem 3 in [Z-S] completes the proof. \( \Box \)
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References


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