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ON THE ASYMPTOTIC BEHAVIOUR OF SOME DIRICHLET SERIES WITH A COMPLICATED SINGULARITY

BY

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1. *Introduction.* Let $\alpha(n)$ denote the *kernel* of n $(n = 1, 2, 3, \ldots)$, i.e. the product of all different primes dividing n. The sum $S(x) = \sum_{n\leqslant x} (\alpha(n))^{-1}$ was studied in [1] and [2]. In [1] it was shown that the corresponding Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} (\alpha(n))^{-1} n^{-s} \tag{1}$$

satisfies

$$\log f(s) \sim s^{-1}(\log s^{-1})^{-1} \qquad (s > 0,\ s \to 0), \tag{2}$$

and using this and a tauberian theorem of HARDY and RAMANUJAN [3] it was shown in [1] that

$$\log S(x) \sim \left(\frac{8 \log x}{\log\log x}\right)^{\frac{1}{2}} \qquad (x \to \infty). \tag{3}$$

In [2] we proved that $S(x)$ is slowly oscillating (in the sense of Karamata), i.e. that for each positive c we have

$$\lim_{x\to\infty} S(cx)/S(x) = 1.$$

This was derived from

$$\sum_{n\leqslant x} \frac{n}{\alpha(n)} = o(xS(x)) \qquad (x \to \infty).$$

In the proof of this, the following idea played a rôle. If a sequence of positive numbers $a_1, a_2, a_3, \ldots$ is such that for every prime number p we have

$$\sum_{n\leqslant x,\, p\nmid n} a_n = o(\sum_{n\leqslant x} a_n),$$

then also

$$\sum_{n\leqslant x} n^{-1}\varphi(n) a_n = o(\sum_{n\leqslant x} a_n).$$

This means that the effect of the factor $n^{-1}\varphi(n)$ is not given merely by multiplication by its average (which is $6\pi^{-2}$). In order to get a more precise idea of the effect of this extra factor, we study, in sec. 2, its effect on the series (1). That is, we study the series

$$g(s) = \sum_{n=1}^{\infty} n^{-1}\varphi(n)(\alpha(n))^{-1}n^{-s}. \tag{4}$$

Whereas we do not know the asymptotic behaviour of $f(s)$, but only of $\log f(s)$, we are more succesful with the quotient $g(s)/f(s)$, of which the asymptotic behaviour can be determined (Theorem 1).

A similar question, to be studied in secs. 3 and 4, considers the effect of omitting certain terms of the series for f. If λ is a positive integer, the positive integer m is called *λ-full* if m is divisible by at least the λ-th power of each prime which it contains. We shall study the effect of restricting the summations to λ-full integers, e.g.

$$S_\lambda(x) = \sum_{n\leqslant x}^{(\lambda)} (\alpha(n))^{-1}, \tag{5}$$

where the upper suffix (λ) indicates that only λ-full values of n are taken. If $\lambda = 1$, there is no restriction, so $S_1(x) = S(x)$.

We are interested in the quotient $S_\lambda(x)/S_{\lambda+1}(x)$ as $x \to \infty$. In [2] it was shown that this quotient lies between 1 and $\log (x + 1)$. We conjecture that

$$S_\lambda(x) \sim \lambda^{-1} e^{-\gamma} (2 \log x/\log \log x)^{\frac{1}{2}} S_{\lambda+1}(x) \tag{6}$$

(throughout the paper, γ denotes Euler's constant).

We were able to establish the behaviour of the quotient of the corresponding Dirichlet series $f_\lambda(s)/f_{\lambda+1}(s)$ (theorem 3), but if we want to apply this to a proof of (6), we need an unproved assumption on the smoothness of $S_\lambda(x)/S_{\lambda+1}(x)$ (see sec. 4).

In a final section (sec. 5) we shall devote some attention to the series $\sum_{n=1}^{\infty} (\alpha(n))^{-\beta}n^{-s}$ for $s \to 0$, if β is fixed and > 1. It shows a different asymptotic behaviour.

2. *The quotient* $g(s)/f(s)$

Theorem 1. $\dfrac{g(s)}{f(s)} \sim \dfrac{e^{-\gamma}}{\log (1/s)} \qquad (s > 0, s \to 0).$

Proof. The product expansions of f and g are

$$f(s) = \prod_p \left(1 + \frac{1}{p(p^s - 1)}\right), \quad g(s) = \prod_p \left(1 + \frac{1 - p^{-1}}{p(p^s - 1)}\right), \tag{7}$$

whence

$$\frac{g(s)}{f(s)} = \prod_p \left(1 - \frac{1}{p^2(p^s - 1) + p}\right). \tag{8}$$

Assuming $0 < s < 1$, we put $A = (s \log(1/s))^{-1}$, so $A \to \infty$ if $s \to 0$. Using $|\log(1-x)| \leqslant C|x|$ for $|x| \leqslant \frac{1}{2}$, we have

$$\left| \Sigma_{p>A} \log\left(1 - \frac{1}{p^2(p^s-1)+p}\right)\right| \leqslant Cs^{-1} \Sigma_{p>A}\, p^{-2}\log p.$$

A simple application of the prime number theorem shows that this is $O(s^{-1}A^{-1}(\log A)^{-2})$, and this tends to zero if $s \to 0$. On the other hand

$$\left| \Sigma_{p \leqslant A} \left\{ \log\left(1 - \frac{1}{p^2(p^s-1)+p}\right) - \log\left(1-\frac{1}{p}\right)\right\}\right| =$$
$$= \left| \Sigma_{p\leqslant A} \log\left(1 - \frac{p(p^s-1)}{p^2(p^s-1)+p-1}\right)\right| =$$
$$= O\left(\Sigma_{p\leqslant A} \frac{p(p^s-1)}{p-1}\right) = O\left(\frac{A(A^s-1)}{\log A}\right) = O(sA) = o(1)$$

if $s \to 0$. Now using the following formula, well known in prime number theory:

$$\Sigma_{p\leqslant A} \log(1-p^{-1}) = -\log\log A - \gamma + o(1),$$

we finally obtain

$$\log\frac{g(s)}{f(s)} = \Sigma_{p\leqslant A}\log(1-p^{-1}) + o(1) =$$
$$= -\log\log(s^{-1}) - \gamma + o(1) \qquad (s>0,\ s\to 0), \tag{9}$$

and the theorem follows.

If we insert the extra factor $n^{-1}\varphi(n)$ in a series like $\zeta(s+1) = \Sigma_1^\infty n^{-1-s}$, the effect is asymptotically (as $s \to 0$) multiplication by $6\pi^{-2}$. The fact that the effect of the extra factor is so much more severe in the case of $f(s)$, is connected with the fact that the separate factors of its product expansion do not tend to unity as $s \to 0$. We express this in the following

Theorem 2. Let $a(n)$ be a positive multiplicative function, such that $\Sigma_1^\infty a(n)n^{-s}$ converges for $s > 0$. For each prime p we put

$$M_p = \lim_{s\to 0, s>0} (a(p)p^{-s} + a(p^2)p^{-2s} + \ldots),$$

and we assume that there is a constant $c > 0$ such that $M_p > c$ for all p. Then we have, as $s \to 0$, $s > 0$,

$$\Sigma_{n=1}^\infty n^{-1}\varphi(n)a(n)n^{-s} = o(\Sigma_{n=1}^\infty a(n)n^{-s}). \tag{10}$$

Proof. The series involved have the product expansions

$$\Pi_p\left(1+\frac{p-1}{p}R(p,s)\right) \text{ and } \Pi_p\,(1+R(p,s)),$$

where $R(p, s) = a(p)p^{-s} + a(p^2)p^{-2s} + \ldots$. So for the quotient we obtain

$$\Pi_p\left(1-\frac{p^{-1}R(p,s)}{1+R(p,s)}\right).$$

Denoting the factors by $\theta(p, s)$, we have $0 < \theta(p, s) \leqslant 1$ for all p, and

$$\lim_{s>0,s\to 0}\theta(p,s) < 1-\frac{c}{c+1}p^{-1}$$

for each seperate p. As $\Pi_p\,(1 - c(c+1)^{-1}p^{-1})$ diverges, we infer that $\Pi_p\,\theta(p, s)$ tends to zero as $s \to 0$. This proves the theorem.

3. *The behaviour of* $f_\lambda(s)/f_{\lambda+1}(s)$.

We define, for $\lambda = 1, 2, \ldots,$

$$f_\lambda(s) = \Sigma^{(\lambda)}\,(\alpha(n))^{-1}n^{-s},$$

where $\Sigma^{(\lambda)}$ denotes summation over all λ-full integers. In particular, $f_1(s) = f(s)$. For the quotient of consecutive f_λ's we get the product expansion

$$f_\lambda(s)/f_{\lambda+1}(s) = \Pi_p\left\{1+\left(p^{\lambda s+1}+\frac{1}{p^s-1}\right)^{-1}\right\}. \qquad (11)$$

Taking logarithms we obtain a sum over all primes, which we want to replace by an integral. This requires the following lemma:

Lemma 1. Let $g(x)$ be positive for $x \geqslant \frac{3}{2}$, increasing for $\frac{3}{2} \leqslant x < X$, decreasing for $x \geqslant X$, and assume

$$J = \textstyle\int_{\frac{3}{2}}^{\infty} g(x)(\log x)^{-1}\,dx < \infty.$$

Then we have

$$|\Sigma_p\, g(p) - J| < C\{2g(X)\textstyle\int_{\frac{3}{2}}^{X}(\log x)^{-2}\,dx + \int_X^{\infty} g(x)(\log x)^{-2}\,dx\}, \qquad (12)$$

where C is an absolute constant.

Proof. It is a consequence of well known facts from prime number theory that there exists a constant C such that, for all $x \geqslant \frac{3}{2}$,

$$|\pi(x) - \textstyle\int_{\frac{3}{2}}^{x}(\log t)^{-1}\,dt| < C\eta(x), \qquad (13)$$

where $\pi(x)$ stands for the number of primes $\leqslant x$, and $\eta(x) = \int_{\frac{3}{2}}^{x} (\log t)^{-2}\, dt$. Writing $\sum_p g(p) = \int_{\frac{3}{2}}^{\infty} g(x)\, d\pi(x)$, and applying (13), integrating by parts, we find that the left-hand side of (12) is at most

$$C \int_{\frac{3}{2}}^{X} \eta(x)\, dg(x) - C \int_{X}^{\infty} \eta(x)\, dg(x).$$

In the first integral we use $\eta(x) \leqslant \eta(X)$; in the second one we integrate by parts

$$\int_{X}^{\infty} \eta(x)\, dg(x) = -\eta(X) g(X) + \int_{X}^{\infty} \eta'(x) g(x)\, dx,$$

and the result follows. It should be remarked that justification of the partial integrations can be given either by proving that $g(x)\eta(x) \to 0$ $(x \to \infty)$ or by replacing the interval $(\frac{3}{2}, \infty)$ by $(\frac{3}{2}, Y)$ (with some $Y \to X$), and making $Y \to \infty$ only at the very end of the proof. In both cases we use the fact that $\int^{\infty} g(x)(\log x)^{-2}\, dx$ converges.

Theorem 3. If λ is a fixed positive integer, and if $s > 0$, $s \to 0$, we have

$$f_\lambda(s)/f_{\lambda+1}(s) \sim \lambda^{-1} e^{-\gamma} \left(s \log \frac{1}{s}\right)^{-1}. \tag{14}$$

Proof. According to (11), we shall apply (12) to the function

$$g(x) = \log \left\{ 1 + \left(x^{\lambda s+1} + \frac{1}{x^s - 1} \right)^{-1} \right\}.$$

This function is easily seen to have the monotonicity properties expressed in the lemma, and the number X is obtained from the equation $X^{(\lambda-1)s+1}(X^s - 1)^2 = s(\lambda s + 1)^{-1}$. Its asymptotic behaviour is $X \sim s^{-1}(\log s^{-1})^{-2}$ $(s > 0, s \to 0)$, and we easily obtain that $g(X) \sim X^s - 1 \sim s \log X$. It follows that

$$g(X) \int_{\frac{3}{2}}^{X} (\log x)^{-2}\, dx = O\{(s \log X) X (\log X)^{-2}\} = O((\log s^{-1})^{-3}).$$

Furthermore, as $g(x) < \log(1 + x^{-1}) < x^{-1}$ $(x > 1)$, we have

$$\int_{X}^{\infty} g(x)(\log x)^{-2}\, dx < \int_{X}^{\infty} x^{-1}(\log x)^{-2}\, dx = O((\log X)^{-1}).$$

It follows that the right-hand side of (12) tends to zero if $s \to 0$.

Our next step is replacing $g(x)$ in $\int_{\frac{3}{2}}^{\infty} g(x)(\log x)^{-1}\, dx$ by

$$(x^{\lambda s+1} + (x^s - 1)^{-1})^{-1}.$$

The error in g is less then $(x^s - 1)^2$ and also less than x^{-2}, whence the error in the integral is at most

$$\int_{\frac{3}{2}}^{X} (x^s - 1)^2 (\log x)^{-1}\, dx + \int_{X}^{\infty} x^{-2} (\log x)^{-1}\, dx = \\ = O(s^2 X \log X) + O(X^{-1}) = o(1) \qquad (s > 0, s \to 0).$$

Hence we have proved that

$$\log \frac{f_\lambda(s)}{f_{\lambda+1}(s)} = \int_{\frac{3}{2}}^{\infty} \left(x^{\lambda s+1} + \frac{1}{x^s - 1}\right)^{-1} \frac{dx}{\log x} + o(1). \qquad (15)$$

We split the integral into three parts, viz. from $\frac{3}{2}$ to X, from X to s^{-1}, and from s^{-1} to ∞. On these parts we use, respectively

$$\left(x^{\lambda s+1} + \frac{1}{x^s - 1}\right)^{-1} < X^s - 1,$$

$$\left(x^{\lambda s+1} + \frac{1}{x^s - 1}\right)^{-1} < x^{-1},$$

$$\left(x^{\lambda s+1} + \frac{1}{x^s - 1}\right)^{-1} = x^{-\lambda s-1} + O\left(\frac{x^{2(\lambda s+1)}}{x^s - 1}\right) =$$

$$= x^{-\lambda s-1} + O(x^{-2}(s^{-s} - 1)^{-1}).$$

The main contribution comes from the term $x^{-\lambda s-1}$, which produces $\int_{1/s}^{\infty} x^{-\lambda s-1}(\log x)^{-1}\, dx$. Putting $x = e^{-t/(\lambda s)}$ we obtain for this integral, as $s \to 0$,

$$\int_{\lambda s \log(1/s)}^{\infty} e^{-t}\, t^{-1}\, dt = -\gamma - \log(\lambda s \log(1/s)) + o(1). \qquad (16)$$

The other contributions to the integral in (15) are all $o(1)$:

$$\int_{\frac{3}{2}}^{X} (X^s - 1)\frac{dx}{\log x} = O\left((X^s - 1)\frac{X}{\log X}\right) = O(sX) = o(1);$$

$$\int_{X}^{1/s} x^{-1}\frac{dx}{\log x} = \log\left(\log\frac{1}{s}/\log X\right) = o(1);$$

$$(s^{-s} - 1)^{-1} \int_{1/s}^{\infty} x^{-2}(\log x)^{-1}\, dx = O((s^{-s} - 1)^{-1}s) = o(1).$$

This completes the proof of the theorem.

4. *Conclusions about* $S_\lambda(x)$. The Dirichlet series $f_\lambda(s)$ (sec. 2) is related to the coefficient sum $S_\lambda(x)$ (see (5)) by

$$f_\lambda(s) = \int_{\frac{1}{2}}^{\infty} x^{-s}\, dS_\lambda(x) = s\int_{\frac{1}{2}}^{\infty} S_\lambda(x)x^{-s-1}\, dx =$$
$$= s\int_{-\log 2}^{\infty} S_\lambda(e^t)\, e^{-st}\, dt = s\int_{2}^{\infty} S_\lambda(e^t)\, e^{-st}\, dt + O(s).$$

We know the asymptotic behaviour of $\log S_1(e^t)$ (see [1]):

$$\log S_1(e^t) \sim (8t/\log t)^{\frac{1}{2}} \qquad (t \to \infty),$$

and as $S_\lambda(e^t)$ and $S_{\lambda+1}(e^t)$ differ by a factor $\log(e^t + 1)$ at most (see [2], sec. 6), we have, for all λ ($\lambda = 1, 2, 3, \ldots$)

$$\log S_\lambda(e^t) \sim (8t/\log t)^{\frac{1}{2}} \qquad (t \to \infty). \qquad (17)$$

If we replace $S_\lambda(e^t)$ by its approximation in the integral for f_λ, the integrand becomes $\exp \omega(t)$, where

$$\omega(t) = (8t/\log t)^{\frac{1}{2}} - st.$$

The function $\omega(t)$ has a maximum at a point t_0, determined by

$$\tfrac{1}{2}(8t_0/\log t_0)^{\frac{1}{2}} t_0^{-1}(1 - (\log t_0)^{-1}) = s,$$

whence

$$t_0 \sim s^{-2}(\log s^{-1})^{-1} \qquad (s > 0,\, s \to 0). \tag{18}$$

Let c be a positive number $(0 < c < 1)$ which we are keeping constant for the time being. Considering large values of t_0, we notice that $\omega(t) - \omega(t_0)$ is quite large if $|t - t_0| > ct_0$, and accordingly it is not difficult to show that the contribution of these values to the integral are negligible. In fact we have

$$f_\lambda(s) \sim \int_{(1-c)t_0}^{(1+c)t_0} S_\lambda(e^t)\, e^{-st}\, dt \tag{19}$$

if $s > 0$, $s \to 0$, but we omit the details of the proof.

We shall now introduce a smoothness condition which we are unable to prove. We assume that the function $\psi(t) = \log S_\lambda(e^t) - \log S_{\lambda+1}(e^t)$ has the property that to each $\varepsilon > 0$ there exist numbers $\delta > 0$, $T > 0$ such that for all t, t' the inequalities

$$T \leqslant t \leqslant t' \leqslant (1 + \delta)t$$

imply

$$|\psi(t) - \psi(t')| \leqslant \varepsilon. \tag{20}$$

(This property has also been given the name "slowly oscillating", see [4], p. 124, but this notion is different from Karamata's as mentioned in sec. 1).

Under this assumption we can prove (6), i.e.

$$S_\lambda(e^t)/S_{\lambda+1}(e^t) \sim \lambda^{-1}\, e^{-\gamma}\, (2t/\log t)^{\frac{1}{2}}. \tag{21}$$

For we have, by (19),

$$f_\lambda(s) \sim \textstyle\int_{(1-c)t_0}^{(1+c)t_0} S_{\lambda+1}(e^t)\, e^{\psi(t)-st}\, dt;$$

taking $c < \frac{1}{2}\delta$ and making $s \to 0$, $t_0 \to \infty$ we infer that

$$f_\lambda(s)/f_{\lambda+1}(s) \sim \exp(\psi(t_0) + \theta(s)),$$

where $|\theta(s)| \leqslant \varepsilon$ for all s. As ε was arbitrary, theorem 3 gives

$$\exp(\psi(t_0)) \sim \lambda^{-1}\, e^{-\gamma}\, (s \log (1/s))^{-1} \qquad (s \to 0).$$

From (18) we derive $(s \log (1/s))^{-1} \sim (2t_0/\log t_0)^{\frac{1}{2}}$, and now (21) easily follows.

We can make a similar statement if we start from theorem 1. Define $T(x) = \sum_{n \leqslant x} \varphi(n) n^{-1} (\alpha(n))^{-1}$, that is, T is the partial sum of the Dirichlet series $g(s)$. And we take, in this case, $\psi(x) = \log T(x) - \log S(x)$. Now again assuming that this ψ is slowly oscillating in the sense expressed by (20), we can derive that

$$T(x)/S(x) \sim 2e^{-\gamma}/\log \log x,$$

but again we were unable to prove the assumption.

5. *The series* $\sum_{n=1}^{\infty} (\alpha(n))^{-\beta} n^{-s}$. The behaviour of these functions, for β fixed, $\beta > 1$, $s > 0$, $s \to 0$ can be determined by methods similar to those applied in sec. 3. We only mention the result:

$$\sum_{n=1}^{\infty} (\alpha(n))^{-\beta} n^{-s} = \frac{\pi}{\sin \pi/\beta} \beta^{2+1/\beta} \left\{ s^{1/\beta} \left(\log \frac{1}{s} \right)^{1+1/\beta} \right\}^{-1} + $$
$$+ O((\log \log s^{-1}) s^{-1/\beta} (\log s^{-1})^{-2}).$$

It is also possible to obtain results holding uniformly with respect to β, especially for $\beta \to 0$, but these are rather complicated and do not seem very illuminating.

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