Decision rules in Markovian decision processes with incompletely known transition probabilities

DOI:
10.6100/IR33253

Document status and date:
Published: 01/01/1968

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

DECISION RULES IN MARKOVIAN DECISION PROCESSES WITH INCOMPLETELY KNOWN TRANSITION PROBABILITIES

J. WESSELS

DECISION RULES IN MARKOVIAN DECISION PROCESSES WITH INCOMPLETELY KNOWN TRANSITION PROBABILITIES

DECISION RULES IN MARKOVIAN DECISION PROCESSES WITH INCOMPLETELY KNOWN TRANSITION PROBABILITIES

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL TE EINDHOVEN, OP GEZAG VAN DE RECTOR MAGNIFICUS, DR. K. POSTHUMUS, HOOGLERAAR IN DE AFDELING DER SCHEIKUNDIGE TECHNOLOGIE, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP DINSDAG 30 JANUARI 1968 DES NAMIDDAGS TE 4 UUR

DOOR

JACOBUS WESSELS

GEBOREN TE AMSTERDAM

TECHNISCHE HOGESCHOOL EINDHOVEN

Dit proefschrift is goedgekeurd door de promotor
PROF. DR. J. F. BENDERS

voor Inez

voor mijn ouders

CONTENTS

PREFACE

In recent years considerable effort has been spent on the investigation of stochastic decision processes. A stochastic decision process may be described roughly as a stochastic process, which can be influenced from the outside. The investigations in this field have the common purpose to provide the surveyor of the process with a recipe, which defines a rule for influencing the process in an optimal way.
The optimum criterion is mostly a function of costs. Both costs due to actions of the surveyor and costs due to the autonomous steps of the stochastic process. Examples of such functions are: expected total costs during a specified time interval, expected total discounted costs during a time interval, expected costs per unit of time.
For studies on such decision processes, especially on the situation where the underlying stochastic process is Markovian, one is referred to [1960, R. Howard; 1962, 1965, D. Blackwell; 1965, G. de Leve].

A practical draw-back to the application of results obtained in this field, is the commonly occurring lack of knowledge on the probabilistic behavior of the underlying stochastic process. In this study some observations will be presented on stochastic decision processes incorporating incomplete knowledge of the probability distributions. A new aspect - compared with common stochastic decision processes - is constituted by the possibility of gathering information on the unknown distributions during the

progress of the process. Information thus gathered may be of help in reaching further decisions. The research will be restricted to the situation where the underlying stochastic process is a Markov chain with a finite number of states. This situation will be studied because of the simple character of the probability distributions involved together with the surveyability of the information gathering and the obvious meaning of the information with respect to unknown distributions. The probability distribution of a Markov chain is characterized by its initial distribution and its matrix of transition probabilities. In this study it will be assumed that the transition probabilities do not depend on time. However the numerical values of the transition probabilities are not completely known by the surveyor of the process.
About the influencing possibilities it will be supposed that between any two autonomous transitions of the process, the surveyor is allowed to transfer the system from one state to another. Furthermore it is presumed that the surveyor knows at any time of decision the complete history of the process until that time.

The central difficulty in this type of problem, as in the theory of statistical inference, is the question of which criterion will be applied in order to discriminate between different feasible decision rules.

This difficulty in assigning an optimum criterion may be outlined as follows. A risk function may be developed in a natural way. Such a risk function presents the surveyor's evaluation for any decision rule combined with any parameter value (in this case: allowed matrix of transition probabilities). For any feasible decision rule, the risk function provides an evaluation in the form of a function of the parameter values. The object of the introduction of an optimum criterion is to present a means for comparing these evaluation functions for the decision rules. The final object - of course - is to provide the possibility to design a "best" decision rule.

Some criteria, which were proposed earlier for other problems (game theory, theory of statistical inference), will be considered: maximum risk, maximum regret, weighed risk (Bayes).
However the first point to arrive at is a clear statement of the problem. This includes the introduction of a class of feasible decision rules.

In [1956, R.N. Bradt e.a.; 1966, D. Sworder] related topics are studied. The problem of the first publication is a very special case of the problem in this study. An important result of the work by R.N. Bradt e.a. is their proof of the active role in (Bayes optimal) decision making played by the gathering of information (their theorem 3.1). This means: decisions are influenced both by information obtained in the past and by the possibility to gather information in the future.
The problem in D. Sworder's monograph is somewhat different from the problem in this study. However in some instances there is a certain similarity in the method of investigation.

In order to prevent misunderstandings over the use of intuitive notions, a rigorous distinction will be maintained between the formal structure of the mathematical theory and the elaborations meant as comments on or justifications of formal steps. Especially in the first few sections these comments serve the purpose of facilitating the mutual translation of the mathematical theory and the terminology of a practical problem.
The distinction will be obtained by developing the mathematical theory completely in formal assumptions, definitions, lemmas, etc., which are all identifiable as such. Inserted verbal elucidation is marked by **. Consequently there is no need for typographical identification of ends of proofs etc.

INTRODUCTION AND SUMMARY

** The formulation of a mathematical model describing decision processes based on time independent Markov chains with incompletely known transition probabilities will be initiated in this section. This formulation begins with the description of a Markovian decision process.

A system is given, which is - at any time of observation - in one state of a set S of n states. The possible states (elements of S) are called s_i $(1 \leq i \leq n)$. For example, the system may be a storehouse for certain product and the states different numbers of stock. Or the system may be a machine and the states different maintenance positions.

Assumption 1.1: n is a given natural number $(\neq 1)$;
$N := \{i \mid i \text{ natural}, \ i \leq n\}$;
S is a given set: $S = \{s_i \mid i \in N\}$.

** The system is observed at discrete points of time, say $t = 0,1,2,\ldots$. Immediately after any observation, the surveyor of the process may take action. He is allowed to transfer the system from the observed state - say s_k - to another one - say s_i - which is preferred by him. It is supposed that the selection and execution of an action require no time. Thus the observation of the system and the reaction of the surveyor take place in the same instant of time.

The autonomous transitions of the process - which are supposed to take place between two subsequent points of time - are governed by the Markov transition probabilities p_{ij} $(i,j \in N)$. These probabilities form the Markov transition matrix P. In case the system is in state s_i at time t (as the result of an action by the surveyor), the probability of observing the system to be in state s_j at time $t+1$ equals p_{ij}. Hence: $\sum_{j=1}^{n} p_{ij} = 1$ $(i \in N)$ and $p_{ij} \geq 0$ $(i,j \in N)$.

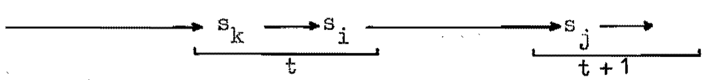
Thus it is supposed, that the transition probabilities of the basic Markov chain do not depend on time.

<u>Definition 1.1</u>: $\mathcal{P} := \{P \mid P = (p_{ij})_{i,j \in N}, \sum_{j=1}^{n} p_{ij} = 1 \ (i \in N), p_{ij} \geq 0 \ (i,j \in N)\}$;

$\mathcal{P}$ is the <u>set of allowed Markov transition matrices</u>.

** In the following sections specific assumptions on the knowledge concerning the Markov transition matrix governing the basic Markov chain of the decision process will be presented. Namely, partial numerical knowledge in section 4 and knowledge of a weight function on $\mathcal{P}$ in section 7.

One step of the Markovian decision process may be represented as follows:



In this representation, s_k is the state of the system observed at time t. s_i is the state resulting from the surveyor's action, right after the observation of s_k. This part of the process is supposed to be concentrated at time t. Then the Markov mechanism produces a transfer to state s_j. This state is observed at time $t+1$. It is further supposed, that state transitions of both types are evaluated by real numbers. In our informal language, these evaluations will be called costs. However it seems obvious that the evaluations are not necessary measured in units of money. In the informal terminology, it will be said, that action $s_k \rightarrow s_i$ costs d_{ki} (decision costs, e.g. costs for buying stock) and Markov transition $s_i \longrightarrow s_j$ costs c_{ij} (process costs, e.g. operating costs).

Without restriction, it may be assumed, that any action $s_k \rightarrow s_i$ is permitted when s_k has been observed at certain time. Actions which are practically forbidden, can be taxed heavily. Later on this situation will be studied more specifically (section 5). At this moment, it is simply assumed that any action is permitted, possibly with very high costs.

Assumption 1.2: $D = (d_{ki})_{k,i \in N}$ and $C = (c_{j\ell})_{j,\ell \in N}$ are given $n \times n$-matrices with real elements.

** With regard to the running period of the decision process, both finite and infinite numbers of steps will be investigated. In either case the total costs of a realizable state history is calculated with discounting. For finite running period, the discount factor is arbitrarily positive. For infinite running period, the discount factor is supposed to be less than 1. This condition guarantees, that for each possible state history the present time value of the total costs is finite.

Assumption 1.3: T represents a given natural number or the symbol ∞ ; β is a given real positive number; when $T = \infty$, then $\beta < 1$; T is both the total number of steps of the decision process and its running period;

β is the discount factor.

** The costs of action $s_k \rightarrow s_i$ at time t are supposed to have the present time value $\beta^t d_{ki}$. The costs of Markov transition

$s_i \longrightarrow s_j$ from time t to time t + 1 are supposed to have the present time value $\beta^t c_{ij}$.

[1956, R.N. Bradt e.a.] investigates - applying the Bayesian approach - a situation which coincides with our case:

$n = 2,\ T < \infty,\ \beta = 1,\ d_{ki} = 0\ (k,i \in N),\ c_{11} = c_{22} = -1,\ c_{12} = c_{21} = 0.$

Before the decision processes can be investigated properly, it is necessary, that a concept of decision rule has been introduced. It is supposed, that at any time t the state history of the process until that time (the observed state at time t included) is known by the surveyor of the process. Hence it is supposed, that at the time of the first decision (t = 0), the surveyor does know the initial state of the process. This implies the superfluity of introducing general initial distributions for the underlying Markov chain: there is only interest in the resulting processes for given initial states.

Then, any possible initial state $s_j \in S$, any allowed Markov transition matrix $P \in \mathcal{P}$, and any feasible decision rule determine together a stochastic process. In sections 2 and 3 this will be proved for two different concepts of decision rule.

Given the knowledge at the time of decision of the state history realized until that time, a decision rule should prescribe an action for any thinkable state history until any time t. In fact, a decision rule maps $\bigcup_{t=1}^{T-1} S^{2t+1}$ into S according to this concept.

A generalization of this concept would allow mixing of decision rules of the first type. This outline will not be followed. It will be proved however, that the decision rules which will be introduced in section 2 are in fact equivalent to the mixed decision rules just mentioned (section 3).

The decision rules introduced in section 2 allow mixing at any moment of decision. Hence a decision rule maps $\bigcup_{t=1}^{T-1} S^{2t+1}$ into the

set of all probability distributions on S. Those decision rules are called accordingly: "decision rules applying mixed strategies". However, the addition "applying mixed strategies" will be commonly omitted, since decision rules of this type will be the common ones in this study. With the same terminology, the decision rules mapping $\bigcup_{t=1}^{T-1} S^{2t+1}$ into S may be called: "decision rules applying pure strategies".
In section 3 "mixed decision rules applying mixed strategies" are introduced. Furthermore it is demonstrated, that these mixings do not form an essential extension to decision rules applying mixed strategies. And it is demonstrated, that mixed decision rules applying pure strategies and decision rules applying mixed strategies are equivalent in a sense. Working with decision rules applying mixed strategies is preferred, since they give better chances to detailed investigation of the resulting stochastic processes. However, in some instances the results of section 3 are profitably applied. Any initial state, any Markov transition matrix, and any defined decision rule determine a stochastic process. Hence expected total costs of the decision process may be calculated as a function of initial state, decision rule, and Markov matrix. This function, which serves as a risk function, and some of its properties are presented in sections 2 and 3 for both types of decision rules.

The decision rules as introduced in sections 2 and 3 base their actual decisions at any time on the complete state history realized so far. However, it seems likely, that some possible state histories until certain time bear the same information with respect to the unknown Markov transition probabilities. This information may be condensed in a so called "information matrix". The information matrix of a state history until certain time is a $n \times n$-matrix with for its (i,j)-element the number of Markov transitions $s_i \longrightarrow s_j$ occurring in that state history.

In section 4, it is proved, that any decision rule (applying mixed strategies) is equivalent with regard to the expected total discounted costs as a function of P (for fixed initial state) to a decision rule always prescribing the same decision for two realized state histories with the same information matrix and the same state observed at the time of decision. If some elements of P are known numerically, the corresponding elements of the information matrices may be neglected. If all elements of D are equal, the observed state at the time of decision is not needed explicitly for decision making.

Section 5 is devoted to the partial ordering of the decision rules, induced by the risk functions as functions of P. The notions of admissibility of decision rules (having non-dominated risk functions) and completeness of subsets of decision rules (every decision rule is dominated by one of the subset) are investigated. Special attention is devoted to the question whether sets of admissible decision rules are complete.

The partial ordering of decision rules according to their risk functions gives no possibility to select a best decision rule. For that, other criteria are needed. In sections 6 and 7 a few criteria are considered. Namely maximum risk and maximum regret (both with respect to P) in section 6; weighed risk in section 7.

The existence of best decision rules according to these criteria is proved. For maximum risk and maximum regret, there exist best decision rules only taking into account: initial state, (sub)-information matrix, and observed state (the latter may be skipped in the case of equal decision costs). For weighed risk, there exists a best decision rule applying pure strategies, only taking into account: (sub)information matrix and observed state. In the case $T = \infty$, there is moreover a certain time independence. The same holds in the case of equal decision costs for all possible T when the known elements of P fill complete rows.

The property: min max risk = max min risk, which does not hold generally, appears to be true in the case of equal decision costs and (in section 8) when strategies of "Nature" are extended to weighings over $\mathcal{P}$.

In section 8 it will be proved that each decision rule, which is admissible for certain initial state, is best for certain weighing over $\mathcal{P}$. This provides a characterization of admissibility.

The appendix collects some examples with properties mentioned in the main text.

DECISION RULES

Definition 2.1: a) the elements of the $(2t+1)$-fold Cartesian product $S \times S \times \ldots S = S^{2t+1}$ are called: allowed (state) histories until time t $(t = 0,1,2,\ldots)$;
the mapping from S^{2t+1} into N^{2t+1}, which maps $(s_{i_1}, s_{i_2}, \ldots, s_{i_{2t+1}})$ on $(i_1, i_2, \ldots i_{2t+1})$ is 1 - 1 and onto, therefore:
b) the elements of the $(2t+1)$-fold Cartesian product $N \times N \times \ldots N = N^{2t+1}$ are called: allowed (index) histories until time t $(t = 0,1,2,\ldots)$.

** In $(s_{k_0}, s_{i_0}, s_{k_1}, s_{i_1}, \ldots, s_{i_{t-1}}, s_{k_t})$ - an allowed state history until time t - the component s_{k_τ} $(\tau = 0,\ldots,t)$ denotes the observed state at time τ; the component s_{i_τ} $(\tau = 0,\ldots,t-1)$ denotes the state resulting from the surveyor's action at time τ. The one-to-one correspondence between allowed state histories and allowed index histories, presents the opportunity of applying in the mathematical theory the latter instead of the former. The application of allowed index histories yields notational profit.

Definition 2.2: R is the set of real numbers;
$U := \{x \in R \mid 0 \leq x \leq 1\}$;
U^n is the n-fold Cartesian product $U \times U \times \ldots \times U$;
$\mathcal{V} := \{(v_1,\ldots,v_n) \in U^n \mid \sum_{i=1}^{n} v_i = 1\}$.

** Since it is assumed, that the surveyor of the process knows at any time t which allowed history until time t has been realized, a decision rule has to give a recipe to find an action for any allowed history until time t ($0 \leq t < T$). It will be permitted to draw lots in order to decide on an action. Hence a recipe is permitted which prescribes the surveyor at the moments of decision to execute chance experiments with n elementary events and to select the action with the same number as the occurring event. The probability distribution of such a chance experiment is characterized by an element of the set $\mathcal{V}$.

Definition 2.3: a decision rule (applying mixed strategies) B is a sequence of mappings

$b^t : N^{2t+1} \to \mathcal{V}$ $(0 \leq t < T)$;

the set of all decision rules is denoted by $\mathcal{B}$

Convention 2.1: $b^t(h)$ and $b^t(k_0, i_0, \ldots, k_t)$ denote the image with respect to b^t of $h = (k_0, i_0, \ldots, k_t) \in N^{2t+1}$ $(t = 0, 1, 2, \ldots)$; components of the image are denoted by $b_i^t(h)$ and $b_i^t(k_0, i_0, \ldots, k_t)$ $(i \in N)$; $b^t(h) \in \mathcal{V}$, $b_i^t(h) \in U$; $b^t(h)$ is sometimes called a decision vector.

Elements of $\mathcal{B}$ are denoted by B, possibly indexed: B_0, B_r; the mappings constituting these decision rules will be denoted by: b^t, ${}^0b^t$, ${}^rb^t$ $(0 \leq t < T)$.

The Cartesian product notation is used in such a way, that:

$\{j\} \times N^{2t} \subset N^{2t+1}$ $(j \in N)$;

$(j,h) \in N^{2t+1}$, when $j \in N$, $h \in N^{2t}$ $(0 \leq t < T)$; $\{j,h\} := \{(j,h)\}$.

** The letter h - short for history - always denotes an element in N^m $(0 \leq m)$. Later on, decision rules will be introduced, which do not base their decisions on the full histories until the moments of decision (see section 4).

Definition 2.4: $B \in \mathcal{B}$ is a decision rule applying pure strategies if and only if

$$\forall_{t(0 \leq t < T)} \forall_{h \in N^{2t+1}} \forall_{i \in N} [b_i^t(h) \in \{0,1\}] ;$$

the set of all decision rules applying pure strategies is denoted by $\mathcal{A}$.

** For convenience n classes will be defined, which are strongly related to $\mathcal{B}$. The j-th class contains the parts of the decision rules related to histories with initial state s_j.

Definition 2.5a: $(j \in N)$. A j-decision rule (applying mixed strategies) ${}_jB$ is a sequence of mappings

$${}_jb^t : \{j\} \times N^{2t} \to \mathcal{V} \quad (0 \leq t < T) ;$$

the set of all j-decision rules is denoted by ${}_j\mathcal{B}$.

Lemma 2.1: a) to any $B \in \mathcal{B}$ there corresponds a ${}_jB \in {}_j\mathcal{B}$ $(j \in N)$, such that the mappings ${}_jb^t$, constituting ${}_jB$, are the restrictions of the mappings b^t to $\{j\} \times N^{2t}$;

b) to any n-tuple ${}_1B \in {}_1\mathcal{B}, \ldots, {}_nB \in {}_n\mathcal{B}$ there corresponds exactly one $B \in \mathcal{B}$, such that the ${}_jB$ are the restrictions of B in the sense of assertion a).

Convention 2.2: Elements of ${}_j\mathcal{B}$ $(j \in N)$ are denoted by ${}_jB$, possibly indexed: ${}_jB_o$, ${}_jB_r$; whenever an element of ${}_j\mathcal{B}$ and one of $\mathcal{B}$ with the same index (or no index) are mentioned together, they have the relation of lemma 2.1.

For the mappings constituting ${}_jB$, ${}_jB_0$, ${}_jB_r$ the index j will be skipped, thus the same notations will be applied as for the corresponding mappings constituting B, B_0, B_r; hence no different notations will be applied for a mapping and certain restrictions.

Definition 2.5b: If $\mathcal{B}_0 \subset \mathcal{B}$, then ${}_j\mathcal{B}_0$ ($j \in N$) denotes $\{{}_jB \in {}_j\mathcal{B} | B \in \mathcal{B}_0\}$;

for each n-tuple of sets ${}_1\mathcal{B}_0 \subset {}_1\mathcal{B}, \ldots, {}_n\mathcal{B}_0 \subset {}_n\mathcal{B}$ the set $\{B \in \mathcal{B} | \forall_{j \in N} \; {}_jB \in {}_j\mathcal{B}_0\}$ is denoted by $\overline{\mathcal{B}}_0$.

Remark: The notation $\mathcal{B}$, ${}_j\mathcal{B}$ ($j \in N$) is consistent with the convention 2.2;

$\overline{\mathcal{B}} = \mathcal{B}$, $\overline{\mathcal{A}} = \mathcal{A}$;

If $\mathcal{B}_0 \subset \mathcal{B}$, then $\overline{\mathcal{B}}_0 \supset \mathcal{B}_0$.

** In the following part of this section it will be demonstrated that an initial state, a Markov transition matrix and a decision rule together determine a stochastic process.

Definition 2.6: Σ is the set consisting of all subsets of N (Σ is the power set of N).

Definition 2.7: Let X be a set and let Ψ be a σ-algebra of subsets of X, then

a) X^∞ is the countably infinite-fold Cartesian product $X \times X \times \ldots$;

b) Ψ_m (natural m) is the σ-algebra of subsets of X^m, which is generated by Ψ^m;

c) Ψ_∞ is the σ-algebra of subsets of X^∞, which is generated by

$$\bigcup_{m=1}^{\infty} \{Y \times X^\infty | Y \in \Psi^m\} ;$$

d) Ψ_m^∞ (natural m) is the σ-algebra of subsets of X^∞:

$\{Y \times X^\infty \mid Y \in \Psi_m\}$.

** Lemma 2.2 mentions some elementary and well-known assertions on the sets of definition 2.7 (see e.g. [1955, M. Loève]).

Lemma 2.2: If Ψ is a σ-algebra of subsets of a set X, then:

a) $\bigcup_{m=1}^{\infty} \Psi_m^\infty$ is an algebra;

b) the σ-algebra generated by $\bigcup_{m=1}^{\infty} \Psi_m^\infty$ equals Ψ_∞ .

** For Σ one obtains some elementary results:

Lemma 2.3: a) Σ^m (m natural) contains all subsets of N^m which consist of one element;

b) Σ is a σ-algebra of subsets of N;

c) Σ_m (natural m) is the power set of N^m.

Theorem 2.1: To each $j \in N$, ${}_jB \in {}_j\mathcal{B}$, $P \in \mathcal{P}$ there corresponds exactly one probability measure $\mu^j_{({}_jB,P)}$ on the measurable space $(N^{2T+1}, \Sigma_{2T+1})$, such that

(2.1) a) $\mu^j_{({}_jB,P)}(\{j\} \times N^{2T}) = 1$;

(2.2)

b) $\forall_{t(o \leq t < T)} \forall_{h \in N^{2t+1}} \forall_{i \in N} [\mu^j_{({}_jB,P)}\left(\{h\} \times N^{2(T-t)}\right) \neq 0 \Rightarrow$

$\mu^j_{({}_jB,P)}\left(\{h,i\} \times N^{2(T-t)-1} \middle| \{h\} \times N^{2(T-t)}\right) = b_i^t(h)]$;

c) $\forall_{t(o \leq t < T)} \forall_{h \in N^{2t+1}} \forall_{i \in N} \forall_{k \in N}$

$$(2.3)\qquad [\mu^j_{({}_jB,P)}\left(\{h,i\} \times N^{2(T-t)-1}\right) \neq 0 \Rightarrow$$

$$\mu^j_{({}_jB,P)}\left(\{h,i,k\} \times N^{2(T-t-1)} \,\middle|\, \{h,i\} \times N^{2(T-t)-1}\right) = p_{ik}] \,.$$

Remark: $\{h,i,k\} \times N^0 := \{h,i,k\}$.

** This theorem - which will be proved below - shows the existence of exactly one stochastic process on the assumed set of states satisfying the following conditions: The process has a given initial state and state transitions alternately with the Markov property (given Markov transition matrix P) and the gambling device of the given decision rule. It also appears that the distribution of the resulting stochastic process is already determined by the j-restriction of a decision rule.
With this theorem in mind an obvious formulation of a Markovian decision problem with unknown Markov transition matrix could be the following:
We consider the set of stochastic processes

$$\{(N^{2T+1}, \Sigma_{2T+1}, \mu^j_{({}_jB,P)}) \,\big|\, j \in N,\ {}_jB \in {}_j\mathcal{B},\ P \in \mathcal{P}\} \,,$$

of which one will be assigned by the determination of j, ${}_jB$, P. The surveyor of the process is entitled to choose ${}_jB$ after the observation of the initial state s_j.

Definition 2.8: For any $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$ $\mu_{(j,B,P)}$ is defined to be the same probability measure on $(N^{2T+1}, \Sigma_{2T+1})$ as $\mu^j_{({}_jB,P)}$.

** With this definition another formulation becomes possible: we consider the set of stochastic processes

$$\{(N^{2T+1}, \Sigma_{2T+1}, \mu_{(j,B,P)}) \,\big|\, j \in N,\ B \in \mathcal{B},\ P \in \mathcal{P}\} \,,$$

of which one will be assigned by the determination of j, B, P. The surveyor of the process is entitled to choose B.

Proof of theorem 2.1:

A. For each probability measure $\mu^{j}_{(_jB,P)}$ - which satisfies the conditions a, b, c - one proves by induction with respect to t:

$$\forall t(0 \leq t < T)\ \forall i_0,\ldots,i_t \in N\ \forall k_0,\ldots,k_{t+1} \in N$$

$$(2.4)\quad \mu^{j}_{(_jB,P)}\left(\{k_0,i_0,\ldots,k_t,i_t\} \times N^{2(T-t)-1}\right) =$$

$$= \begin{cases} \left(\prod_{\tau=0}^{t-1} p_{i_\tau k_{\tau+1}}\right)\left(\prod_{\tau=0}^{t} b^{\tau}_{i_\tau}(k_0,i_0,\ldots,k_\tau)\right), & \text{when } k_0 = j \\ 0 & \text{, when } k_0 \neq j \end{cases}$$

$$(2.5)\quad \mu^{j}_{(_jB,P)}\left(\{k_0,i_0,\ldots,k_t,i_t,k_{t+1}\} \times N^{2(T-t-1)}\right) =$$

$$= \begin{cases} \left(\prod_{\tau=0}^{t} p_{i_\tau k_{\tau+1}}\right)\left(\prod_{\tau=0}^{t} b^{\tau}_{i_\tau}(k_0,i_0,\ldots,k_\tau)\right), & \text{when } k_0 = j \\ 0 & \text{, when } k_0 \neq j\ . \end{cases}$$

In case t = 0, the product $\prod_{\tau=0}^{t-1} p_{i_\tau k_{\tau+1}}$ is equal to 1, by definition.

B. If $T < \infty$, each element of Σ_{2T+1} contains a finite number of allowed histories until time T. Hence formula (2.5) $(t = T-1)$ already determines the probabilities of the elements of Σ_{2T+1}. It only remains to be checked whether this measure determines a probability measure with properties a, b, c. By recursion (starting with $t = T-1$) one proves (2.4) and (2.5) for each t $(0 \leq t < T)$ and those formulae prove a, b, c and the equality to 1 of the measure of N^{2T+1}.

C. The case $T = \infty$. The same reasoning as in part B leads to a uniquely defined probability measure on Σ_m^∞ (natural m), which satisfies a, b, c for $t < \frac{m-1}{2}$. These measures define an additive set function on the algebra $\bigcup_{m=1}^{\infty} \Sigma_m^\infty$

This algebra generates the σ-algebra Σ_∞ (lemma 2.2), hence there exists exactly one extension of this additive set function to a measure on $(N^\infty, \Sigma_\infty)$ (a well-known theorem of measure theory, see for example [1955, M. Loève]). This measure is necessarily a probability measure, since $N^\infty \in \Sigma_m^\infty$ (each natural m).

** The following definitions and lemmas introduce the risk function and some of its properties. The risk function evaluates the stochastic process resulting from the fixing of P and B.

Definition 2.9: a) For each t $(0 \leq t < T)$ a mapping v_t is defined, which maps N^{2T+1} into R (the set of real numbers) by

$$\forall_{h=(k_0,i_0,k_1,i_1,\ldots)\in N^{2T+1}} \; [v_t(h) := \beta^t(d_{k_t i_t} + c_{i_t k_{t+1}})] \; ;$$

b) A mapping v is defined, which maps N^{2T+1} into R by

$$\forall_{h \in N^{2T+1}} \; [v(h) := \sum_{t=0}^{T-1} v_t(h)] \; .$$

** For measure theoretic concepts applied in formulation and proof of the following lemma, reference is made to [1955, M. Loève].

Lemma 2.4: v_t $(0 \leq t < T)$ and v map $(N^{2T+1}, \Sigma_{2T+1})$ into $(R, \mathcal{B})$ measurably ($\mathcal{B}$ is the σ-algebra of Borel sets in R);
moreover these mappings are integrable with respect to every probability measure on $(N^{2T+1}, \Sigma_{2T+1})$.

Proof: v_t is a step function with n^3 steps (called simple function by Loève) and hence measurable.

v is the sum of a finite or countably infinite number of step functions each with n^3 steps. $\sum_{\tau=0}^{t} v_\tau$ converges to v for $t \to T-1$ (pointwise, but even uniform, since $\beta < 1$ in case $T = \infty$). This proves its measurability. The integrability of v_t and v with respect to any probability measure on $(N^{2T+1}, \Sigma_{2T+1})$ follows easily: v_t and v are both bounded:

$$\forall_{h \in N^{2T+1}} \left[|v(h)| \leq \sum_{t=0}^{T-1} |v_t(h)| \leq \left(\sum_{t=0}^{T-1} \beta^t \right) \cdot \max_{k,i,\ell \in N} |d_{ki} + c_{i\ell}| \right] .$$

Definition 2.10: For each $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$:

$$V(j,B,P) := \int_{N^{2T+1}} v d\mu_{(j,B,P)} .$$

** V(j,B,P) may be interpreted as the expected total discounted costs of the process $(N^{2T+1}, \Sigma_{2T+1}, \mu_{(j,B,P)})$. One easily verifies, that the expected total discounted costs are equal to the total expected discounted costs (an application of the dominated convergence theorem):

Lemma 2.5: For each $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$:

$$V(j,B,P) = \sum_{t=0}^{T-1} \int_{N^{2T+1}} v_t d\mu_{(j,B,P)} .$$

Lemma 2.6: For each $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$ the following assertions hold:

a) For any t $(0 \leq t < T)$:

$$\left| \int_{N^{2T+1}} v_t d\mu_{(j,B,P)} \right| \leq \beta^t \cdot \max_{k,i,\ell \in N} |d_{ki} + c_{i\ell}| \quad ;$$

b) $|V(j,B,P)| \leq \left(\sum_{t=0}^{T-1} \beta^t\right) \cdot \max_{k,i,\ell \in N} |d_{ki} + c_{i\ell}|$.

Proof: a) $v_t(h) \leq \beta^t \cdot \max_{k,i,\ell \in N} |d_{ki} + c_{i\ell}|$ (definition 2.9a).

b) Combination of result a) with lemma 2.5.

Lemma 2.7: All $j \in N$, $B_1, B_2 \in \mathcal{B}$ satisfy:

$${}_jB_1 = {}_jB_2 \Longrightarrow \forall_{P \in \mathcal{P}} [V(j,B_1,P) = V(j,B_2,P)] \ .$$

Proof: The measure $\mu_{(j,B_1,P)}$ coincides with the measure $\mu^j_{({}_jB_1,P)}$ (definition 2.8), just like the measure $\mu_{(j,B_2,P)}$ in case ${}_jB_1 = {}_jB_2$.

** For application in other sections limit concepts and hence topologies will be introduced in $\mathcal{P}$, $\mathcal{B}$, ${}_j\mathcal{B}$ (a limit concept in a set induces in a natural way a closure operation for subsets, which defines a topological space according to the Kuratovski definition; see e.g. [1955, J.L. Kelley]).
In $\mathcal{P}$ the common $n \times n$-matrix topology is introduced. All topological assertions involving $\mathcal{P}$ refer to this topology:

Definition 2.11: Let $P_\ell \in \mathcal{P}$ $(\ell = 0,1,2,\ldots)$, with elements $p_{ik}^{(\ell)}$, then $\lim_{\ell \to \infty} P_\ell = P_0$ if and only if $\lim_{\ell \to \infty} p_{ik}^{(\ell)} = p_{ik}^{(0)}$ (all $i,k \in N$).
The topology in $\mathcal{P}$ is the one induced by this limit concept.

Lemma 2.8: $\mathcal{P}$ is compact.

Proof: $\mathcal{P}$ is homeomorfic with $\mathcal{V}^n \subset R^{n^2}$ (with the natural topology in R^{n^2} and the relative topology in $\mathcal{V}^n$). $\mathcal{V}$ is compact, hence $\mathcal{V}^n$ according to Tychonov's theorem.

Lemma 2.9: For each $j \in N$, $B \in \mathcal{B}$ the mapping $V(j,B,\cdot)$ from $\mathcal{P}$ into $\mathbb{R}$ is continuous.
Even: if $P_\ell \in \mathcal{P}$ and $\lim_{\ell\to\infty} P_\ell = P_0$, then

$$\lim_{\ell\to\infty} V(j,B,P_\ell) = V(j,B,P_0), \text{ uniformly in } j,\ B.$$

Proof: $\varepsilon > 0$.

$$|V(j,B,P_\ell) - V(j,B,P_0)| \leq$$

$$\leq \sum_{\tau=0}^{T-1} \left| \int_{N^{2T+1}} v_\tau d\mu(j,B,P_\ell) - \int_{N^{2T+1}} v_\tau d\mu(j,B,P_0) \right| \qquad \text{(lemma 2.5)}$$

$$\leq \sum_{\tau=0}^{t} \left| \int_{N^{2T+1}} v_\tau d\mu(j,B,P_\ell) - \int_{N^{2T+1}} v_\tau d\mu(j,B,P_0) \right| + \frac{\varepsilon}{2}$$

(for t sufficiently large, $t < T$, lemma 2.6)

$$\leq \max_{k,i,r\in N} |d_{ki} + c_{ir}| \sum_{\tau=0}^{t} \beta^\tau$$

$$\sum_{(i_0,k_1,\dots,i_\tau,k_{\tau+1})\in N^{2T+2}} \left| \prod_{\rho=0}^{\tau} p^{(\ell)}_{i_\rho k_{\rho+1}} - \prod_{\rho=0}^{\tau} p^{(0)}_{i_\rho k_{\rho+1}} \right| + \frac{\varepsilon}{2}$$

(formula (2.5), $0 \leq b^\rho_{i_\rho}(h) \leq 1$)

$\leq \varepsilon$, for ℓ sufficiently large (with no dependence on j,B).

Definition 2.12: Let $B_\ell \in \mathcal{B}$ ($\ell = 0,1,2,\dots$), then $\lim_{\ell\to\infty} B_\ell = B_0$ if and only if

$$\forall_{t(0\leq t<T)} \forall_{h\in N^{2t+1}} \forall_{i\in N} \left[\lim_{\ell\to\infty} {}^\ell b^t_i(h) = {}^0 b^t_i(h)\right].$$

 The topology in $\mathcal{B}$ is the one induced by this limit concept.

Definition 2.13: In case $T < \infty$, $N(T)$ is a natural number equal to $\sum_{t=0}^{T-1} n^{2t+1} = n \frac{n^{2T} - 1}{n^2 - 1}$;
in case $T = \infty$, $N(T)$ represents the symbol ∞.

Lemma 2.10: The topology in $\mathcal{V}^{N(T)}$ induced by the limit concept of componentwise convergence is the same as the product topology in $\mathcal{V}^{N(T)}$ generated by the relative topology in $\mathcal{V}$ with respect to the natural topology in R^n.

Proof: In the topological product of an arbitrary set of topological spaces holds: the limit concepts of componentwise convergence and product topological convergence coincide (e.g. [1955, J.L. Kelley]).

Lemma 2.11: $\mathcal{B}$ is homeomorfic with $\mathcal{V}^{N(T)}$ (topology of lemma 2.10).

Proof: $N(T)$ is the total number of allowed histories until any time t $(0 \leq t < T)$ (when $T < \infty$, otherwise the number is countably infinite). Let a numbering of the allowed histories until any time t $(0 \leq t < T)$ be given. Then a 1 - 1 correspondence between $\mathcal{B}$ and $\mathcal{V}^{N(T)}$ is obtained by the linking of the decision vector belonging to the m-the allowed history with the m-th component of an element of $\mathcal{V}^{N(T)}$. Since the topology in $\mathcal{V}^{N(T)}$ is induced by the limit concept of componentwise convergence, the homeomorfy is obvious.

Lemma 2.12: $\mathcal{B}$ is compact; $\mathcal{A}$ is a compact subset of $\mathcal{B}$.

Proof: When focussing on the topology in $\mathcal{V}^{N(T)}$ as a product topology (lemma 2.10), the compactness of $\mathcal{V}^{N(T)}$ appears as a consequence of Tychonov's theorem, since $\mathcal{V}$ is compact. Lemma 2.11 implies the compactness of $\mathcal{B}$. $\mathcal{A} \subset \mathcal{B}$ is closed.

Lemma 2.13: For any $j \in N$, $P \in \mathcal{P}$ the mapping $V(j,\cdot,P)$ from $\mathcal{B}$ into $\mathbb{R}$ is continuous.

Even: if $B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\ldots)$ and $\lim\limits_{\ell\to\infty} B_\ell = B_0$, then

$$\lim_{\ell\to\infty} V(j,B_\ell,P) = V(j,B_0,P), \text{ uniformly in } j,P.$$

Proof: $\varepsilon > 0$.

$$|V(j,B_\ell,P) - V(j,B_0,P)| \leqslant$$

$$\leqslant \sum_{\tau=0}^{T-1} \left| \int_{N^{2T+1}} v_\tau d\mu(j,B_\ell,P) - \int_{N^{2T+1}} v_\tau d\mu(j,B_0,P) \right| \qquad \text{(lemma 2.5)}$$

$$\leqslant \sum_{\tau=0}^{t} \left| \int_{N^{2T+1}} v_\tau d\mu(j,B_\ell,P) - \int_{N^{2T+1}} v_\tau d\mu(j,B_0,P) \right| + \frac{\varepsilon}{2}$$

(for t sufficiently large, $t < T$, lemma 2.6)

$$\leqslant \max_{k,i,r\in N} |d_{ki} + c_{ir}| \sum_{\tau=0}^{t} \beta^\tau \sum_{(i_0,k_1,\ldots,k_\tau,i_\tau)\in N^{2\tau+1}}$$

$$\left| \prod_{\rho=0}^{\tau} {}^\ell b_{i_\rho}^\rho (j,i_0,\ldots,k_\rho) - \prod_{\rho=0}^{\tau} {}^0 b_{i_\rho}^\rho (j,i_0,\ldots,k_\rho) \right| + \frac{\varepsilon}{2}$$

(formula (2.5), $0 \leqslant p_{i_\rho k_{\rho+1}} \leqslant 1$)

$\leqslant \varepsilon$, for ℓ sufficiently large (with no dependence on j,P, since there is only a finite number of j's).

Lemma 2.14: Let $P_\ell \in \mathcal{P}$, $B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\ldots)$ and $\lim\limits_{\ell\to\infty} P_\ell = P_0$, $\lim\limits_{\ell\to\infty} B_\ell = B_0$, then
$\lim\limits_{\ell\to\infty} V(j,B_\ell,P_\ell) = V(j,B_0,P_0)$ (uniformly in j).

Proof:

$$|V(j,B_\ell,P_\ell) - V(j,B_0,P_0)| \leq |V(j,B_\ell,P_\ell) - V(j,B_0,P_\ell)| +$$

$$+ |V(j, B_0,P_\ell) - V(j,B_0,P_0)| \quad .$$

Both terms in the right hand part of the inequality are less than $\frac{\varepsilon}{2}$ for ℓ sufficiently large (lemma 2.13 and lemma 2.9 respectively).

Definition 2.14: $j \in N$; let ${}_jB_\ell \in {}_j\mathcal{B}$ $(\ell = 0,1,2,\ldots)$, then $\lim\limits_{\ell\to\infty} {}_jB_\ell = {}_jB_0$, if and only if

$$\forall_{t(0 \leq t < T)} \forall_{h \in N^{2t}} \forall_{i \in N} \left[\lim_{\ell\to\infty} {}^\ell b_i^t(j,h) = {}^0b_i^t(j,h) \right] \quad .$$

The topology in ${}_j\mathcal{B}$ is the one induced by this limit concept.

Lemma 2.15: $B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\ldots)$, then

$$\lim_{\ell\to\infty} B_\ell = B_0 \Longleftrightarrow \forall_{j \in N} \lim_{\ell\to\infty} {}_jB_\ell = {}_jB_0 \quad .$$

Lemma 2.16: $\mathcal{B}_0 \subset \mathcal{B}$, then

a) $\overline{\mathcal{B}}_0$ closed $\Longleftrightarrow \forall_{j \in N}$ ${}_j\mathcal{B}_0$ closed;

b) $\overline{\mathcal{B}}_0$ open $\Longleftrightarrow \forall_{j \in N}$ ${}_j\mathcal{B}_0$ open;

c) $\mathcal{B}_0$ open $\Longrightarrow \forall_{j \in N}$ (${}_j\mathcal{B}_0$ open) and (hence) $\overline{\mathcal{B}}_0$ open;

d) $\mathcal{B}_0$ compact $\Longrightarrow \forall_{j \in N}$ (${}_j\mathcal{B}_0$ compact) and (hence) $\overline{\mathcal{B}}_0$ compact;

e) specifically: $\forall_{j \in N}$ (${}_j\mathcal{B}$ and ${}_j\mathcal{A}$ compact).

Proof: a) A direct consequence of lemma 2.15.

b) Define n subsets of ${}_j\mathcal{B}$ for any $j \in N$ by:

$${}_j\mathcal{B}_j := {}_j\mathcal{B} \setminus {}_j\mathcal{B}_0 \text{ and } {}_j\mathcal{B}_k := {}_j\mathcal{B} \quad \text{for } k \in N,\ k \neq j\ .$$

For fixed k the sets ${}_j\mathcal{B}_k$ define a subset $\overline{\mathcal{B}}_k \subset \mathcal{B}$.
Then $\mathcal{B}\setminus \overline{\mathcal{B}}_0 = \bigcup_{k \in N} \overline{\mathcal{B}}_k$.

Let all ${}_j\mathcal{B}_0$ be open, then $\overline{\mathcal{B}}_k$ closed ($k \in N$, assertion a)); hence $\overline{\mathcal{B}}_0$ is open.
Let $\overline{\mathcal{B}}_0$ be open and non-empty (when empty ${}_j\mathcal{B}_0 = \emptyset$ and open); say that ${}_1\mathcal{B}_0$ is not open, or ${}_1\mathcal{B}_1$ is not closed: there exists a sequence $\{{}_1B_\ell\}_{\ell=1}^{\infty} \subset {}_1\mathcal{B}_1$ with $\lim_{\ell\to\infty} {}_1B_\ell = {}_1B_0 \in {}_1\mathcal{B}_0$; define for $j \neq 1$ ${}_jB_\ell$ $(\ell = 0,1,2,\ldots)$ such that for each j all the ${}_jB_\ell$ are equal and ${}_jB_\ell \in {}_j\mathcal{B}_0$; hence $B_\ell \notin \overline{\mathcal{B}}_0$ $(\ell \geq 1)$ and $\lim_{\ell\to\infty} B_\ell = B_0 \in \overline{\mathcal{B}}_0$, which is contradictory with $\overline{\mathcal{B}}_0$ open.

c) In a similar way as the second part of b).

d) Let $\{{}_j\mathcal{B}_\alpha\}_\alpha$ be an open covering of ${}_j\mathcal{B}_0$ then $\{\overline{\mathcal{B}}_\alpha\}_\alpha$ with ${}_\ell\mathcal{B}_\alpha := {}_\ell\mathcal{B}$ for $\ell \neq j$ constitutes an open covering of $\mathcal{B}_0$ (assertion b)). A finite subcovering $\{\overline{\mathcal{B}}_{\alpha_i}\}_{\alpha_i}$ of $\mathcal{B}_0$ exists and hence $\{{}_j\mathcal{B}_{\alpha_i}\}_{\alpha_i}$ constitutes a finite subcovering of ${}_j\mathcal{B}_0$.

<u>Lemma 2.17</u>: Let $B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\ldots)$ and $\lim_{\ell\to\infty} {}_jB_\ell = {}_jB_0$ for certain $j \in N$, then

$$\lim_{\ell\to\infty} V(j,B_\ell,P) = V(j,B_0,P) \text{ uniformly in P.}$$

<u>Proof</u>: Exactly like the proof of lemma 2.13.

MIXED DECISION RULES

** This section is devoted to the introduction of a new type of decision rule. The new decision rules may be interpreted as mixings of decision rules applying mixed strategies. The new decision rules proceed by drawing one element from $\mathcal{B}$ with prescribed probabilities. Then the obtained decision rule is applied during the process. In fact a mixed decision rule is defined as a probability measure on $\mathcal{B}$. It will be proved that the stochastic processes defined by any "mixed decision rule applying mixed strategies" (for different $j \in N$, $P \in \mathcal{P}$) essentially agree with the stochastic processes defined by certain "(pure) decision rule applying mixed strategies" (theorem 3.2) and those defined by certain "mixed decision rule applying pure strategies" (theorem 3.3).

It is necessary to define a collection of measurable subsets of $\mathcal{B}$, in order to be able to define probability measures on $\mathcal{B}$.

The relation between $\mathcal{B}$ and $\mathcal{V}^{N(T)}$ (see lemma 2.12) provides the possibility of introducing a σ-algebra of subsets of $\mathcal{B}$ with an abundance of opportunities for the definition of probability measures.

Definition 3.1: $\Phi := \{X \in \mathcal{B}_n | X \subset \mathcal{V}\}$, hence Φ is the σ-algebra of the n-dimensional Borel sets contained in $\mathcal{V}$ ($\mathcal{V}$ is a Borel set).

Lemma 3.1: a) Let f be the 1 - 1 mapping from $\mathcal{B}$ onto $\mathcal{V}^{N(T)}$ induced by a given numbering of $\bigcup_{t=0}^{T-1} N^{2t+1}$ as defined in the proof of lemma 2.11. Then f induces a σ-algebra of subsets of $\mathcal{B}$:

$$\{\mathcal{B}_0 \subset \mathcal{B} \mid f(\mathcal{B}_0) \in \Phi_{N(T)}\} \quad .$$

b) All numberings of $\bigcup_{t=0}^{T-1} N^{2t+1}$ - the set of allowed histories until any time t $(0 \leq t < T)$ - induce the same σ-algebra of subsets of $\mathcal{B}$ (in the sense of a)).

Proof: a) Obvious, since f is 1 - 1 and onto.

b) Each two permitted numberings of $\bigcup_{t=0}^{T-1} N^{2t+1}$ are permutations of each other. Hence if $f_1(\mathcal{B}_0)$ with $\mathcal{B}_0 \subset \mathcal{B}$ is generated by elements of

$$\bigcup_{m=1}^{N(T)} \{Y \times \vartheta^{N(T)-m} \mid Y \in \Phi^m\}$$

(see definition 2.7), then $f_2(\mathcal{B}_0)$ is generated in the same manner by elements with permuted indices.

Definition 3.2: Let Γ denote the σ-algebra of subsets of $\mathcal{B}$ introduced in lemma 3.1. A mixed decision rule (applying mixed strategies) is a probability measure on the measure space $(\mathcal{B}, \Gamma)$. The set of all mixed decision rules is denoted by $\mathcal{B}^*$. Elements of $\mathcal{B}^*$ are denoted by B*, possibly indexed B^*_0, B^*_r.

Lemma 3.2: $j \in N$, $P \in \mathcal{P}$, $1 \leq m < 2T + 2$, $h \in N^m$; then: $\mu_{(j,\cdot,P)}(\{h\} \times N^{2T+1-m})$ maps $(\mathcal{B}, \Gamma)$ into $(R, \mathcal{L})$ measurably, moreover the mapping is integrable with respect to any probability measure on $(\mathcal{B}, \Gamma)$.

Proof: Formulae (2.4) and (2.5) present explicit expressions for this mapping: a constant multiplied by a finite product of com-
 ponents of decision vectors. The induced mapping from $(\vartheta^{N(T)}, \Phi_{N(T)})$

into $(R, \mathcal{B})$ (given a numbering of allowed histories) is measurable, hence the mapping considered is measurable.
The integrability is implied by the boundedness of the mapping.

Definition 3.3: If ${}_1\Psi$ and ${}_2\Psi$ are σ-algebras of subsets of ${}_1X$ and ${}_2X$, then ${}_1\Psi * {}_2\Psi$ denotes the σ-algebra of subsets of ${}_1X \times {}_2X$ generated by ${}_1\Psi \times {}_2\Psi$.

Remark: Definition 3.3 combined with definition 2.7 implies:

$$\Psi \times \Psi = \Psi^2 \quad ,$$

$$\Psi * \Psi = \Psi_2 \quad .$$

Theorem 3.1: To any $j \in N$, $B^* \in \mathcal{B}^*$, $P \in \mathcal{P}$ there corresponds exactly one probability measure $\mu^*_{(j,B^*,P)}$ on $(\mathcal{B} \times N^{2T+1}, \Gamma * \Sigma_{2T+1})$, such that

$$\forall m(1 \leq m < 2T+2) \ \forall h \in N^m \ \forall \mathcal{B}_0 \in \Gamma \quad :$$

$$[\mu^*_{(j,B^*,P)}\left(\mathcal{B}_0 \times \{h\} \times N^{2T+1-m}\right) = \int_{\mathcal{B}_0} \mu_{(j,B,P)}\left(\{h\} \times N^{2T+1-m}\right) dB^*].$$

Proof: Let $\mu^*_{(j,B^*,P)}$ be the set function defined by the condition of the theorem. It is the purpose of this proof to show that this set function can be extended in exactly one way to a function on $\Gamma * \Sigma_{2T+1}$ satisfying the conditions of a probability measure on $(\mathcal{B} \times N^{2T+1}, \Gamma * \Sigma_{2T+1})$.
The extension in a unique way to a function

on $\quad \Gamma \times \Sigma_{2T+1} \quad$ in case $T < \infty$,

or on $\quad \Gamma \times \left(\bigcup_{m=1}^{\infty} \Sigma_m^{\infty}\right) \quad$ in case $T = \infty$

is obvious (each set considered is the union of a finite number of disjunct sets of the type $\mathcal{B}_0 \times \{h\} \times N^{2T+1-m}$). A similar reasoning proves the unique extension of $\mu^*_{(j,B^*,P)}$ to an additive set function on the algebra consisting of finite unions of sets with a function value already defined. This algebra generates the σ-algebra $\Gamma * \Sigma_{2T+1}$, hence, according to a well-known theorem of measure theory (see e.g. [1955, M. Loève]) the extension to a probability measure on $(\mathcal{B} \times N^{2T+1}, \Gamma * \Sigma_{2T+1})$ is uniquely determined.

** Theorem 3.1 shows that an obvious formulation of a Markovian decision problem with unknown Markov transition matrix could be as follows:

we consider the set of stochastic processes

$$\left\{\left(\mathcal{B} \times N^{2T+1}, \Gamma * \Sigma_{2T+1}, \mu^*_{(j,B^*,P)}\right) \middle| \, j \in N,\ B^* \in \mathcal{B}^*, P \in \mathcal{P}\right\} ,$$

of which one will be assigned by the determination of j,B*,P. The surveyor of the process is entitled to choose B*.
The next problem is to determine whether the mixed decision rules provide an essential extension to the already introduced decision rules. In view of theorem 3.2 the answer is: the extension is not essential.
Theorem 3.2 shows, that for the set of stochastic processes just described the set of restricted processes (restricted to the histories - that is the only part we are interested in)

$$\left\{\left(\mathcal{B} \times N^{2T+1}, \{\mathcal{B}\} \times \Sigma_{2T+1}, \mu^*_{(j,B^*,P)}\right) \middle| \, j \in N, B^* \in \mathcal{B}^*, P \in \mathcal{P}\right\}$$

possesses the following property. For each $B^* \in \mathcal{B}^*$, there exists a $B \in \mathcal{B}$, such that all restricted B*-processes ($j \in N, P \in \mathcal{P}$) have exactly similar probability properties as the corresponding B-processes.

Theorem 3.2:

$$\forall_{B^*\in\mathcal{B}^*}\ \exists_{B_o\in\mathcal{B}}\ \forall_{j\in N}\ \forall_{H\in\Sigma_{2T+1}}\ \forall_{P\in\mathcal{P}}$$

$$\left[\mu^*_{(j,B^*,P)}\left(\mathcal{B}\times H\right)=\mu_{(j,B_o,P)}\left(H\right)\right] .$$

Proof: In view of the construction of Σ_{2T+1}, it suffices to prove the assertion for sets H of the type:

$$\{h\}\times N^{2T+1-m} \quad (h\in N^m,\ 1\leq m<2T+2) .$$

Hence (theorem 3.1) it suffices to prove the existence of a $B_o\in\mathcal{B}$ for each $B^*\in\mathcal{B}^*$, such that

$$\forall_{j\in N}\ \forall_{P\in\mathcal{P}}\ \forall_{m(1\leq m<2T+2)}\ \forall_{h\in N^m}$$

$$\left[\int_{\mathcal{B}}\mu_{(j,B,P)}\left(\{h\}\times N^{2T+1-m}\right)dB^*=\mu_{(j,B_o,P)}\left(\{h\}\times N^{2T+1-m}\right)\right] .$$

In view of formulae (2.4) and (2.5), it is required that:

$$\forall_{t(o\leq t<T)}\ \forall_{k_o,\dots,k_t\in N}\ \forall_{i_o,\dots,i_t\in N}$$

$$\left[\int_{\mathcal{B}}\prod_{\tau=0}^{t}b^{\tau}_{i_\tau}(k_o,i_o,\dots,k_\tau)dB^*=\prod_{\tau=0}^{t}{}^ob^{\tau}_{i_\tau}(k_o,i_o,\dots,k_\tau)\right] .$$

If in an integrand the factor $b^{\tau}_{i_\tau}(k_o,i_o,\dots,k_\tau)$ occurs, then the factors $b^{\rho}_{i_\rho}(k_o,i_o,\dots,k_\rho)$ $(0\leq\rho<\tau)$ occur also. This fact presents the possibility to define ${}^ob^{\tau}$ inductively.

$$\forall_{i_o\in N}\ \forall_{k_o\in N}\left[{}^ob^o_{i_o}(k_o):=\int_{\mathcal{B}}b^o_{i_o}(k_o)dB^*\right] .$$

One verifies: ${}^ob^o(k_o)\in\mathcal{V}$ $(k_o\in N)$.

Say that the ${}^{o}b^{\tau}$ are defined for $0 \leq \tau \leq t$ $(t < T-1)$, such that

$$\prod_{\tau=0}^{t} {}^{o}b^{\tau}_{i_\tau}(k_o,i_o,\ldots,k_\tau) = \int_{\mathcal{B}} \prod_{\tau=0}^{t} b^{\tau}_{i_\tau}(k_o,i_o,\ldots,k_\tau)dB^*$$

for all $k_o,\ldots,k_t \in N$ and $i_o,\ldots,i_t \in N$.

If $\prod_{\tau=0}^{t} {}^{o}b^{\tau}_{i_\tau}(k_o,i_o,\ldots,k_\tau) = 0$ all ${}^{o}b^{t+1}_{i_{t+1}}(k_o,i_o,\ldots,k_t,i_t,k_{t+1})$ may be defined freely, if only ${}^{o}b^{t+1}(k_o,\ldots,k_{t+1}) \in \mathcal{V}$. Otherwise define:

$${}^{o}b^{t+1}_{i_{t+1}}(k_o,\ldots,k_t,i_t,k_{t+1}) := \frac{\int_{\mathcal{B}} \prod_{\tau=0}^{t+1} b^{\tau}_{i_\tau}(k_o,\ldots,k_\tau)dB^*}{\prod_{\tau=0}^{t} {}^{o}b^{\tau}_{i_\tau}(k_o,\ldots,k_\tau)}$$

$(k_{t+1} \in N,\ i_{t+1} \in N)$.

One easily verifies that ${}^{o}b^{t+1}(k_o,\ldots,k_t,i_t,k_{t+1}) \in \mathcal{V}$.

** In the sequel of this section it is proved that there is no essential difference between mixings on $\mathcal{B}$ and mixings on $\mathcal{A}$. In fact theorem 3.3 shows that for each $B^* \in \mathcal{B}^*$, there exists a mixing on $\mathcal{A}$, say B^*_1, such that all restricted B^*-processes $(j \in N, P \in \mathcal{P})$ have exactly similar probability properties as the corresponding restricted B^*_1-processes.

Lemma 3.3: $\mathcal{A} \in \Gamma$.

Proof:

$$\mathcal{A} = \bigcap_{t=0}^{T-1} \{B \in \mathcal{B} \mid \forall_{h \in N^{2t+1}} \forall_{i \in N} [b^t_i(h) \in \{0,1\}]\} \ .$$

$\mathcal{A}$ is the intersection of an at most countably infinite number of measurable subsets of $\mathcal{B}$, hence $\mathcal{A}$ is measurable.

Theorem 3.3: $\mathcal{A}^* := \{B^* \in \mathcal{B}^* \mid B^*(\mathcal{A}) = 1\}$, then

$$\forall_{B^* \in \mathcal{B}^*} \exists_{B_1^* \in \mathcal{A}^*} \forall_{j \in N} \forall_{P \in \mathcal{P}} \forall_{H \in \Sigma_{2T+1}}$$

$$\mu^*_{(j,B^*,P)}\left(\mathcal{B} \times H\right) = \mu^*_{(j,B_1^*,P)}\left(\mathcal{B} \times H\right) \quad .$$

Proof: According to theorem 3.2 and a similar reasoning as in the proof of theorem 3.2, it suffices to prove the existence of a $B_1^* \in \mathcal{A}^*$ for each $B_0 \in \mathcal{B}$, such that

$$\forall_{t(0 \le t < T)} \forall_{k_0,\ldots,k_t \in N} \forall_{i_0,\ldots,i_t \in N}$$

$$\prod_{\tau=0}^{t} {}^{0}b^{\tau}_{i_\tau}(k_0,\ldots,k_\tau) = \int_{\mathcal{A}} \prod_{\tau=0}^{t} b^{\tau}_{i_\tau}(k_0,\ldots,k_\tau)dB_1^* \quad .$$

Define: $B_1^*(\mathcal{B} \setminus \mathcal{A}) := 0$ and hence $B_1^*(\mathcal{B}_0) := 0$, when $\mathcal{B}_0 \in \Gamma$, $\mathcal{B}_0 \subset \mathcal{B} \setminus \mathcal{A}$.

$$B_1^*(\{B \in \mathcal{A} \mid \prod_{\tau=0}^{t} b^{\tau}_{i_\tau}(k_0,\ldots,k_\tau) = 1\}) := \prod_{\tau=0}^{t} {}^{0}b^{\tau}_{i_\tau}(k_0,\ldots,k_\tau)$$

$(0 \le t < T;\ k_0,\ldots,k_t \in N,\ i_0,\ldots,i_t \in N)$.

One easily verifies that the probabilities as defined are mutually consistent. Furthermore the collection of subsets of $\mathcal{B}$ with defined probability generates the σ-algebra Γ, which proves the existence of a probability measure on $(\mathcal{B},\Gamma)$ with the defined probabilities. On the other hand a probability measure with the defined probabilities satisfies the conditions which have been put forward at the beginning of the proof.

** Theorem 3.4 states a result on the risk function with regard to mixed decision rules. The two integrals in the assertion both present a reasonable generalization of the concept of a risk function to the case of mixed decision rules. The equality of both integrals is a consequence of the way of introducing mixed decision rules. Theorem 3.2 implies the equality with the risk function for the corresponding $B_0 \in \mathcal{B}$. This result may be applied in some proofs in subsequent sections of this study.

<u>Theorem 3.4</u>: $B^* \in \mathcal{B}^*$, then for any $j \in N$, $P \in \mathcal{P}$:

$$\int_{\mathcal{B}} V(j,B,P)dB^* = \int_{\mathcal{B}\times N^{2T+1}} w d\mu^*_{(j,B^*,P)} = V(j,B_0,P) \quad ,$$

with $w(B,h) := v(h)$ $(B \in \mathcal{B},\ h \in N^{2T+1})$, B_0 is a decision rule which corresponds to B^* according to theorem 3.2.

<u>Proof</u>: Note that the integrability of $V(j,\cdot,P)$ and w has not been proved up till now.

Introduce $w_t(B,h) := v_t(h)$ $(0 \leq t < T,\ B \in \mathcal{B},\ h \in N^{2T+1})$. w_t $(0 \leq t < T)$ and w map $(\mathcal{B} \times N^{2T+1},\ \Gamma * \Sigma_{2T+1})$ into $(R, \mathcal{L})$ measurably and are integrable with respect to any probability measure on the first mentioned measurable space (lemma 2.4; inverse images of Borel sets with respect to w_t or w are the Cartesian products of $\mathcal{B}$ and the inverse images with respect to v_t or v respectively). Furthermore

$$\int_{\mathcal{B}} w d\mu^*_{(j,B^*,P)} = \sum_{t=0}^{T-1} \int_{\mathcal{B}\times N^{2T+1}} w_t d\mu^*_{(j,B^*,P)} =$$

$$= \sum_{t=0}^{T-1} \sum_{h \in N^{2t}} \sum_{k,i,\ell \in N} \beta^t(d_{ki} + c_{i\ell}) \mu^*_{(j,B^*,P)}\left(\mathcal{B} \times \{h,k,i,\ell\} \times N^{2(T-t-1)}\right) \quad .$$

This sum will be transformed in two different ways:

1. $$= \sum_{t=0}^{T-1} \sum_{h \in N^{2t}} \sum_{k,i,\ell \in N} \beta^t (d_{ki} + c_{i\ell}) \; \mu_{(j,B_0,P)}\left(\{h,k,i,\ell\} \times N^{2(T-t-1)}\right) \qquad \text{(theorem 3.2)}$$

$$= \sum_{t=0}^{T-1} \int_{N^{2T+1}} v_t \, d\mu_{(j,B_0,P)} = V(j,B_0,P) \qquad \text{(lemma 2.5)} .$$

2. $$= \sum_{t=0}^{T-1} \sum_{h \in N^{2t}} \sum_{k,i,\ell \in N} \beta^t (d_{ki} + c_{i\ell}) \int_{\mathcal{B}} \mu_{(j,B,P)}\left(\{h,k,i,\ell\} \times N^{2(T-t-1)}\right) dB^* \qquad \text{(theorem 3.1)}$$

$$= \sum_{t=0}^{T-1} \int \left[\int_{N^{2T+1}} v_t \, d\mu_{(j,B,P)} \right] dB^* \qquad \text{(transposing finite summation and integration)}$$

$$= \int_{\mathcal{B}} \left[\sum_{t=0}^{T-1} \int_{N^{2T+1}} v_t \, d\mu_{(j,B,P)} \right] dB^* \qquad \text{(Lebesgue's theorem, applying lemma 2.6)}$$

$$= \int_{\mathcal{B}} V(j,B,P) \, dB^* .$$

SUFFICIENT INFORMATION

** In view of the results in section 3, attention will be restricted to $\mathcal{B}$, the set of decision rules applying mixed strategies. Actually this section is devoted to the investigation of the possibility to restrict attention to a subset of $\mathcal{B}$. This investigation concentrates on the possibility of refraining from discriminating between each two different allowed histories until time t. The purpose of this section is to prove that any decision rule is equivalent (in terms of risk) to a decision rule, which identifies allowed histories presenting the same information (in some sense) with regard to further decisions. In a subsequent part of this section, a generalization to the situation with some elements of the Markov transition matrix known to the surveyor and others unknown, is treated.

This section begins with the development of some tools, which will be used in constructing the main results.
Definition 4.1 introduces the expected total costs of the process from time t onwards, given the history of the process until the decision at time t.

<u>Definition 4.1</u>: $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$, $0 \leq t < T$, $h \in N^{2t+2}$.

a) When $\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)-1}\right) \neq 0$, the probability measure on the measurable space $(N^{2T+1}, \Sigma_{2T+1})$ defined by $\mu_{(j,B,P)}\left(H \,\middle|\, \{h\} \times \right.$ $\left. \times N^{2(T-t)-1}\right)$ for all $H \in \Sigma_{2T+1}$ is denoted by $\mu^{(h)}_{(j,B,P)}$;

b) $V_t(j,B,P|h) := \begin{cases} \int_{N^{2T+1}} \sum_{\tau=t}^{T-1} v_t d\mu^{(h)}_{(j,B,P)} , \\ \qquad \text{when } \mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)-1}\right) \neq 0 \\ 0 \text{ , otherwise.} \end{cases}$

** The existence of the integral in part b) of definition 4.1 is a consequence of lemma 2.4.

<u>Lemma 4.1</u>: $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$, t $(0 \leq t < T)$, then:

$$V(j,B,P) = \sum_{\tau=0}^{t-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} +$$

$$+ \sum_{h \in N^{2t+2}} V_t\left(j,B,P \,\middle|\, h\right) \mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)-1}\right) \quad .$$

<u>Proof</u>: In view of definitions 2.9 and 2.10, it suffices to prove:

$$\int_{N^{2T+1}} \sum_{\tau=t}^{T-1} v_\tau d\mu_{(j,B,P)} =$$

$$= \sum_{h \in N^{2t+1}} V_t(j,B,P \mid h)\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)-1}\right) \quad .$$

Summation and integration may be transposed in the left part (compare lemma 2.5).

$$\sum_{\tau=t}^{T-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} = \sum_{\tau=t}^{T-1} \sum_{(k_0,i_0,\ldots,k_{\tau+1}) \in N^{2\tau+3}} \beta^\tau\left(d_{k_\tau i_\tau} + c_{i_\tau k_{\tau+1}}\right)$$

$$\mu_{(j,B,P)}\left(\{k_0,i_0,\ldots,k_{\tau+1}\} \times N^{2(T-\tau-1)}\right)$$

$$= \sum_{h \in N^{2t+2}} \mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)-1}\right) \cdot V_t(j,B,P \mid h) \quad .$$

Lemma 4.2: $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$, $0 \leq t < T$, $h \in N^{2t+2}$, then:

a) $V_t(j,B,P \mid h)$ does not depend on the choice of b^τ $(0 \leq \tau \leq t)$ as long as $\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)-1}\right) \neq 0$;

b) $V_t(j,B,P \mid h)$ does not depend on the choice of the $b^\tau(h_1,h_2)$ for all τ $(t+1 \leq \tau < T)$, $h_1 \in N^{2t+2}$ $(h_1 \neq h)$, $h_2 \in N^{2(\tau-t)-1}$.

Proof: a) The assertion follows easily, since $\mu^{(h)}_{(j,B,P)}$ does not depend on b^τ $(0 \leq \tau \leq t)$ (definition 4.1 and formula (2.4)).

b) This assertion is implied directly by definition 4.1: $\mu^{(h)}_{(j,B,P)}$ does not depend on the $b^\tau(h_1,h_2)$ mentioned.

Lemma 4.3: $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$, t $(0 \leq t < T)$, then:

a) $\int\limits_{N^{2T+1}} v_t d\mu_{(j,B,P)}$ does not depend on the choice of b^τ $(t+1 \leq \tau < T)$.

b) For any $h \in N^{2t+1}$ with $\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)}\right) = 0$, $V(j,B,P)$ does not depend on the choice of $b^t(h)$.

Proof:

a) $\int\limits_{N^{2T+1}} v_t d\mu_{(j,B,P)} =$

$$= \sum_{h \in N^{2t}} \sum_{k,i,\ell \in N} \beta^t\left(d_{ki} + c_{i\ell}\right) \mu_{(j,B,P)}\left(\{h,k,i,\ell\} \times N^{2(T-t-1)}\right) .$$

The $\mu_{(j,B,P)}$ - factors in the terms of this finite sum do not depend on the choice of the b^τ $(t+1 \leq \tau < T)$ (formula (2.5)).

b) $\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)}\right)$ does not depend on b^t; apply lemma

 4.1: the first sum does not depend on b^t according to assertion a); the second sum does not depend on b^t according to the supposition (one term) and lemma 4.2 a).

** Coming to the main topic of this section, the first task is the formal introduction of an equivalence concept in the set of decision rules.

Definition 4.2: $B_1, B_2 \in \mathcal{B}$.

a) $(j \in N)$ ${}_jB_1$ is said to be equivalent to ${}_jB_2$, when

$$V(j,B_1,P) = V(j,B_2,P) \ , \quad \text{for each } P \in \mathcal{P};$$

notation: ${}_jB_1 \sim {}_jB_2$;

b) B_1 is said to be equivalent to B_2, when

$${}_jB_1 \sim {}_jB_2 \ , \quad \text{for each } j \in N \ ;$$

notation: $B_1 \sim B_2$.

** The $\sim$-concept defines relations in the sets ${}_j\mathcal{B}$, since all $B \in \mathcal{B}$ with ${}_jB = {}_jB_1$ possess the same $V(j,B,P)$ for each $P \in \mathcal{P}$ (lemma 2.7) and any ${}_jB$ is the j-restriction of at least one decision rule (lemma 2.1 b)).

Lemma 4.4: The $\sim$-concepts of definition 4.2 define equivalence relations in the sets ${}_j\mathcal{B}$ $(j \in N)$ and $\mathcal{B}$ respectively.

** When applying a decision rule, the decision at any time is based on the realized allowed history until that time. It seems reasonable to investigate those decision rules, which base their decisions at time t on the numbers of the different Markov transitions in the realized allowed history until that time.
Definition 4.3 formalizes this concept of information offered by allowed histories until certain time.

Definition 4.3: for each $h = (k_0, i_0, \ldots, k_t) \in N^{2t+1}$ $(0 \leq t < T+1)$, $K(h)$ denotes the $n \times n$-matrix of nonnegative integers, with element labelled (i,k) - $i,k \in N$ - equal to the number of τ-values $(0 \leq \tau < t)$ such that $(i_\tau, k_{\tau+1}) = (i,k)$;
$K(h)$ is called the information matrix of h.

** The subset of $\mathcal{B}$ consisting of the decision rules, which base actual decisions on the momentary information matrix and the observed state at the time of decision, is introduced in the following definition:

Definition 4.4: $B \in \mathcal{B}$ is called an information decision rule (applying mixed strategies), when the following condition is satisfied:

$$\forall_{t(0 \leq t < T)} \forall_{h \in N^{2t}} \forall_{h' \in N^{2t}} \forall_{k \in N}$$

$$[K(h,k) = K(h',k) \Longrightarrow b^t(h,k) = b^t(h',k)] \quad .$$

The subset of $\mathcal{B}$ containing all information decision rules will be denoted by $\mathcal{R}$.

Lemma 4.5: $\mathcal{R}$ is compact (hence ${}_j\mathcal{K}$ $(j \in N)$ and $\overline{\mathcal{K}}$ are compact (lemma 2.16 a))).
$\mathcal{R}$ is a proper subset of $\overline{\mathcal{K}}$.

Proof: The assertions follow directly from the definitions.

Lemma 4.6: $0 \leq r < T$, $B \in \mathcal{B}$ and B satisfies the condition of definition 4.4 for all t with $r < t < T$, then:

$$\forall_{P \in \mathcal{P}} \forall_{j,j' \in N} \forall_{h \in N^{2r}} \forall_{h' \in N^{2r}} \forall_{k,i \in N}$$

$$\Big[\mu_{(j,B,P)}\Big(\{h,k,i\} \times N^{2(T-r)-1}\Big)\, \mu_{(j',B,P)}\Big(\{h',k,i\} \times N^{2(T-r)-1}\Big) \neq 0,$$

$$K(h,k) = K(h',k) \Longrightarrow V_r(j,B,P \mid h,k,i) = V_r(j',B,P \mid h',k,i)\Big] \quad .$$

Proof: Be $k = k_r$, $i = i_r$

$$V_r(j,B,P \mid h,k_r,i_r) = \sum_{\tau=r}^{T-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)}^{(h,k_r,i_r)}$$

(dominated convergence)

$$= \sum_{\tau=r}^{T-1} \sum_{(k_{r+1},\ldots,i_\tau,k_{\tau+1}) \in N^{2(\tau-r)+1}} \beta^\tau\left(d_{k_\tau i_\tau} + c_{i_\tau k_{\tau+1}}\right)$$

$$\mu_{(j,B,P)}^{(h,k_r,i_r)}\left(\{h,k_r,i_r,\ldots,k_{\tau+1}\} \times N^{2(T-\tau-1)}\right) \quad .$$

The probabilities involved in this sum do not depend on b^t with $0 \leq t \leq r$. They do depend on b^t with $r < t < T$. However the b^t with $r < t < T$ depend only on the information matrix and the observed state and they do not depend on the complete allowed history until time t. This proves the assertion.

** Theorem 4.1 proves that each historical j-decision rule is equivalent to an information j-decision rule.

Theorem 4.1: If $j \in N$, then $\forall_{B \in \mathcal{B}} \exists_{B_0 \in \mathcal{R}} \left[{}_jB \sim {}_jB_0\right]$.

Proof: The proof consists of two parts. In part A an induction step will be proved. In part B it will be demonstrated that the induction step may be applied to establish the theorem.

A. In this part of the proof it will be shown, that for any decision rule $B_{r+1} \in \mathcal{B}$ (given r: $0 \leq r < T$) - which satisfies the condition of definition 4.4 for all t with $r+1 \leq t < T$ - there exists a decision rule $B_r \in \mathcal{B}$, which satisfies the condition of definition 4.4 for all t with $r \leq t < T$ and furthermore ${}_jB_r \sim {}_jB_{r+1}$.

It will appear, that a B_r suffices with ${}^rb^\tau = {}^{r+1}b^\tau$ for $\tau \neq r$. Hence this proof consists mainly of the construction of ${}^rb^r$.

Lemma 4.1 implies:

$$(4.1)\quad V(j,B_{r+1},P) = \sum_{\tau=0}^{r-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B_{r+1},P)} +$$

$$+ \sum_{h \in N^{2r+2}} V_r(j,B_{r+1},P \mid h)\mu_{(j,B_{r+1},P)}\Big(\{h\} \times N^{2(T-r)-1}\Big) \quad .$$

The first sum in formula (4.1) does not depend on ${}^{r+1}b^r$ (lemma 4.3a)), hence this sum does not alter when B_{r+1} is replaced by the decision rule B_r, which will be constructed.
The second sum in formula (4.1) may be rewritten as a finite sum of finite subsums, such that any subsum collects all terms corresponding to allowed histories until time r, which possess a certain information matrix and a certain observed state at time r. To be explicit, regard the subsum belonging to information matrix K and state s_{k_r} observed at time r (K a given n × n matrix of nonnegative integers and s_{k_r} a given element of S):

$$(4.2)\quad \sum_{\substack{h \in N^{2r} \\ K(h,k_r) = K}} \sum_{i_r \in N} V_r(j,B_{r+1},P \mid h,k_r,i_r)$$

$$\mu_{(j,B_{r+1},P)}\Big(\{h,k_r,i_r\} \times N^{2(T-r)-1}\Big) \quad .$$

The quantities V_r in this expression do not differ with h (lemma 4.6), provided that the corresponding $\mu_{(j,B_{r+1},P)}$-factor does not equal zero. In this proof the V_r will be denoted henceforth by $V_r(P,K,k_r,i_r)$. The $\mu_{(j,B_{r+1},P)}$ - factors are determined by formula (2.4). Hence they all contain the same elements of P as subfactors. In this proof, the product of these subfactors will be denoted henceforth by $\Pi(P;K)$.

Expression (4.2) may be transformed into:

$$(4.3)\quad \Pi(P;K) \sum_{i_r \in N} V_r(P,K,k_r,i_r)$$

$$\sum_{\substack{(k_0,i_0,\dots,i_{r-1}) \in N^{2r} \\ K(k_0,i_0,\dots,k_r)=K}} \delta_{jk_0} \prod_{\tau=0}^{r} {}^{r+1}b^{\tau}_{i_\tau}(k_0,i_0,\dots,k_\tau) \quad .$$

It suffices to find a decision vector ${}^{r}b^{r}(h,k_r)$ - the same for each $h \in N^{2r}$ with $K(h,k_r) = K$ - which leaves the value of expression (4.3) unchanged. Since the $V_r(P,K,k_r,i_r)$ do not alter when ${}^{r+1}b^{r}$ is altered (lemma 4.2a)), the following choice suffices when the denominator involved is not equal to zero:

$$(4.4)\quad \forall_{h \in N^{2r}(K(h,k_r)=K)}\ \forall_{i_r \in N}\ {}^{r}b^{r}_{i_r}(h,k_r) :=$$

$$:= \frac{\displaystyle\sum_{\substack{(k_0,i_0,\dots,i_{r-1}) \in N^{2r} \\ K(k_0,i_0,\dots,k_r)=K}} \delta_{jk_0} \prod_{\tau=0}^{r} {}^{r+1}b^{\tau}_{i_\tau}(k_0,i_0,\dots,k_\tau)}{\displaystyle\sum_{\substack{(k_0,i_0,\dots,i_{r-1}) \in N^{2r} \\ K(k_0,i_0,\dots,k_r)=K}} \delta_{jk_0} \prod_{\tau=0}^{r-1} {}^{r+1}b^{\tau}_{i_\tau}(k_0,i_0,\dots,k_\tau)} \quad .$$

If the denominator in expression (4.4) equals zero, expression (4.3) is equal to zero and hence the choice of the ${}^{r}b^{r}_{i_r}(h,k_r)$ is arbitrary, except for the condition of nonnegativeness and summing to 1 for $i_r = 1,\dots,n$.

It is easily verified that ${}^{r}b^{r}(h,k_r)$ as defined by (4.4) is an element of $\mathcal{V}$.

B. When $T < \infty$, the assertion follows directly on application of the induction step derived in part A of this proof: The induction process may be started with $r = T-1$ and $B_{r+1} = B_T := B$; r decreases by 1 at each step of the induction process. The decision rule B_0, which results finally, is an element of $\mathcal{K}$ and ${}_jB_0 \sim {}_jB_T$. If $T = \infty$, the establishment of the assertion is somewhat more complicated:

$$(4.5)\quad V(j,B,P) = \sum_{\tau=0}^{\rho-1} \int_{N^\infty} v_\tau d\mu_{(j,B,P)} + \sum_{\tau=\rho}^{\infty} \int_{N^\infty} v_\tau d\mu_{(j,B,P)}$$

$(0 \leq \rho < \infty)$ (lemma 2.5) .

The first sum in the right hand part of equation (4.5) does not depend on b^t for $t \geq \rho$ (lemma 4.3a)). The second sum satisfies:

$$\left| \sum_{\tau=\rho}^{\infty} \int_{N^\infty} v_\tau d\mu_{(j,B,P)} \right| \leq \frac{\beta^\rho}{1-\beta} \max_{k,i,\ell \in N} |d_{ki} + c_{i\ell}|$$

independent of B,P (lemma 2.6).

The decision rule $B_{\rho\rho} \in \mathcal{B}$ is defined, such that

$${}^{\rho\rho}b^t = b^t \quad \text{for} \quad t < \rho ,$$

$B_{\rho\rho}$ satisfies the condition of definition 4.4 for $t \geq \rho$.

Then

$$(4.6)\quad |V(j,B,P) - V(j,B_{\rho\rho},P)| \leq \frac{2\beta^\rho}{1-\beta} \max_{k,i,\ell \in N} |d_{ki} + c_{i\ell}|$$

(uniformly in P) .

By a finite induction process - applying the induction step derived in part A of this proof - with r decreasing by 1 each step: from $r = \rho$ to $r = 0$, a decision rule $B_{\rho 0} \in \mathcal{K}$ is obtained, with

$$V(j,B_{\rho 0},P) = V(j,B_{\rho\rho},P) \quad \text{for all} \quad P \in \mathcal{P} .$$

 The compactness of $\mathcal{K}$ (lemma 4.5) implies the existence of a decision rule $B_0 \in \mathcal{K}$, which is the limit of a subsequence of the sequence $\{B_{\rho 0}\}_{\rho=0}^{\infty}$.

Formula (4.6) implies: $V(j,B,P) = V(j,B_0,P)$ (all $P \in \mathcal{P}$) (lemma 2.13), hence ${}_jB_0 \sim {}_jB$.

<u>Corollary 4.1</u>: $\forall_{B \in \mathcal{B}} \; \exists_{B_0 \in \overline{\mathcal{K}}} \; [B \sim B_0]$.

** Corollary 4.1 asserts, that the only information supplied by the realized allowed history until time t, which is relevant with regard to decision making at time t, consists of the initial state of the process, the momentary information matrix, and the observed state at time t. This result is analogous to the well-known fact, that, when estimating the probabilities for the elementary events of a trial, based on a number of independent repetitions of the trial, it is useless to pay attention to the order in which the elementary events occurred, only their numbers count: the numbers form a set of sufficient statistics.

In the next part of this section, the result of theorem 4.1 will be generalized. This generalization initiates the specification of the expression "incompletely known transition probabilities" in the title of this study. The situation will be investigated, with some Markov transition probabilities having values which are known by the surveyor of the process and others which are unknown. Theorem 4.2 demonstrates that if the Markov transition probability corresponding to Markov transition $s_i \longrightarrow s_k$ is known, the elements labelled (i,k) of the information matrices are not relevant in decision making.

<u>Assumption 4.1</u>: $I_\sigma \in N^2$ is a given set; to each element $(i,k) \in I_\sigma$ there corresponds a given real number π_{ik}, such that

$$\exists_{P \in \mathcal{P}} \; \forall_{(i,k) \in I_\sigma} \; [p_{ik} = \pi_{ik}] \; .$$

** Definition 4.5 presents some adaptations of notions which were introduced earlier.

<u>Definition 4.5</u>: a) $\mathcal{P}_\sigma := \{P \in \mathcal{P} \mid \forall_{(i,k)\in I_\sigma} [p_{ik} = \pi_{ik}]\}$.

b) $B_1, B_2 \in \mathcal{B}$; ${}_jB_1$ is said to be <u>sub-equivalent</u> to ${}_jB_2$, when

$$\forall_{P\in \mathcal{P}_\sigma} [V(j,B_1,P) = V(j,B_2,P)] \qquad (j \in N) ;$$

notation: ${}_jB_1 \overset{\sigma}{\sim} {}_jB_2$.

c) $B_1, B_2 \in \mathcal{B}$; B_1 is said to be <u>sub-equivalent</u> to B_2, when

$$\forall_{j\in N} [{}_jB_1 \overset{\sigma}{\sim} {}_jB_2] ;$$

notation: $B_1 \overset{\sigma}{\sim} B_2$.

d) Two $n \times n$ matrices - say K and L, with elements K_{ik} and L_{ik} respectively - are said to be <u>sub-equal</u>, when

$$\forall_{(i,k)\notin I_\sigma} [K_{ik} = L_{ik}] ;$$

notation: $K \overset{\sigma}{=} L$.

** The $\overset{\sigma}{\sim}$-concept defines relations in the sets ${}_j\mathcal{B}$, since all $B \in \mathcal{B}$ with ${}_jB = {}_jB_1$ possess the same $V(j,B,P)$ for each $P \in \mathcal{P}_\sigma$ (lemma 2.7) and any ${}_jB$ is the j-restriction of at least one decision rule (lemma 2.1b)). Some simple properties are enumerated in the following lemma:

<u>Lemma 4.7</u>: a) $\mathcal{P}_\sigma \neq \emptyset$ and $\mathcal{P}_\sigma$ is a compact subset of $\mathcal{P}$.

b) The $\overset{\sigma}{\sim}$-concepts establish equivalence relations in the sets ${}_j\mathcal{B}$ $(j \in N)$ and $\mathcal{B}$.

c) The matrix-valued function K on $\bigcup_{t=0}^{T} N^{2t+1}$ and the sub-equality concept establish an equivalence relation in $\bigcup_{t=0}^{T} N^{2t+1}$, the set of allowed histories until any time.

d) Each pair of n × n matrices K,L satisfies:

$$K = L \Longrightarrow K \overset{\sigma}{\cong} L \quad .$$

e) $\forall_{B_1,B_2 \in \mathcal{B}} [{}_jB_1 \sim {}_jB_2 \Longrightarrow {}_jB_1 \overset{\sigma}{\sim} {}_jB_2] \qquad (j \in N)$.

f) $\forall_{B_1,B_2 \in \mathcal{B}} [B_1 \sim B_2 \Longrightarrow B_1 \overset{\sigma}{\sim} B_2]$.

** Definition 4.6 introduces decision rules, which do not take into account the data, which were gathered in the past of the process, on Markov transitions with known probabilities. The decision vectors of these rules are exclusively based on the "reduced" information matrix (the information matrix with the elements labelled (i,k) suppressed if $(i,k) \in I_\sigma$), the time of decision and the observed state at that time.

Definition 4.6: $B \in \mathcal{B}$ is called a sub-information decision rule (applying mixed strategies), when the following condition is satisfied:

$$\forall_{t(o \le t < T)} \forall_{h,h' \in N^{2t}} \forall_{k \in N}$$

$$[K(h,k) \overset{\sigma}{\cong} K(h',k) \Longrightarrow b^t(h,k) = b^t(h',k)] \quad .$$

The subset of $\mathcal{B}$ containing all sub-information decision rules will be denoted by $\mathcal{R}_\sigma$.

Lemma 4.8: $\mathcal{R}_\sigma \subset \mathcal{R}$; $\mathcal{R}_\sigma$ is a compact subset of $\mathcal{B}$ (hence ${}_j\mathcal{R}_\sigma$ and $\overline{\mathcal{R}}_\sigma$ $(j \in N)$ are compact (lemma 2.16d)));

$\mathcal{K}_\sigma$ is a proper subset of $\mathcal{K}$, in case $I_\sigma \neq \emptyset$;

$\mathcal{K}_\sigma$ is a proper subset of $\overline{\mathcal{K}}_\sigma$.

** Lemma 4.9 presents a result of the same kind as lemma 4.6. The proof proceeds along the same lines as the proof of lemma 4.6 and will be omitted for that reason.

Lemma 4.9: $0 \leq r < T$; $B \in \mathcal{B}$ and B satisfies the condition of definition 4.6 for every t with $r < t < T$, then

$$\forall_{P \in \mathcal{P}_\sigma} \forall_{j,j' \in N} \forall_{h,h' \in N^{2r}} \forall_{k,i \in N}$$

$$\Big[\mu_{(j,B,P)}\Big(\{h,k,i\} \times N^{2(T-r)-1}\Big)\mu_{(j',B,P)}\Big(\{h',k,i\} \times N^{2(T-r)-1}\Big) \neq 0,$$

$$K(h,k) \overset{\sigma}{=} K(h',k) \Longrightarrow V_r(j,B,P \mid h,k,i) = V_r(j',B,P \mid h',k,i)\Big] \quad .$$

Theorem 4.2: If $j \in N$, then $\forall_{B \in \mathcal{B}} \exists_{B_0 \in \mathcal{K}_\sigma} \left[{}_jB \overset{\sigma}{\sim} {}_jB_0\right]$.

Proof: The proof shows a fair similarity with the proof of theorem 4.1. Only the proof of the induction step needs modification: Say that $B_{r+1} \in \mathcal{B}$ satisfies the condition of definition 4.6 for all t with $r+1 \leq t < T$ (given r, $0 \leq r < T$). A decision rule $B_r \in \mathcal{B}$ will be constructed, such that ${}_jB_r \overset{\sigma}{\sim} {}_jB_{r+1}$ and B_r satisfies the condition of definition 4.6 for all t with $r \leq t < T$. It again appears that a B_r suffices with ${}^rb^t = {}^{r+1}b^t$ for $0 \leq t < T$, $t \neq r$.

Again consider formula (4.1). The first sum on the right hand side is invariant for the replacement of B_{r+1} by B_r as planned. The second sum may be rewritten as a finite sum of finite subsums, such that any subsum collects all terms corresponding to allowed histories until time r, which have information matrices sub-equal to a certain matrix and which have the same observed state at time r. Formula (4.2) presents a typical subsum, if only $K(h,k_r) =$ $= K$ is replaced by $K(h,k_r) \overset{\sigma}{\leq} K$. The factors V_r in this expression do not differ with h (lemma 4.9), provided that the corresponding $\mu_{(j,B_{r+1},P)}$-factor does not equal zero. These factors will again be denoted by $V_r(P,K,k_r,i_r)$ in this proof. The factors $\mu_{(j,B_{r+1},P)}$ are determined by formula (2.4) and hence all contain the same numbers of factors p_{ik} for those labels $(i,k) \notin I_\sigma$. The product of these factors will be denoted in this proof by $\Pi(P,K)$; the product of the remaining π-factors is denoted by $\Pi(\{\pi\};\ h,k_r)$. Remind that the factors of $\Pi(\{\pi\};\ h,k_r)$ are given real numbers. The modified expression (4.2) may be rewritten as follows:

$$(4.7)\quad \Pi(P,K) \sum_{i_r \in N} V_r(P,K,k_r,i_r)$$

$$\sum_{\substack{(k_0,i_0,\ldots,i_{r-1}) \in N^{2r} \\ K(k_0,i_0,\ldots,k_r) \cong K}} \delta_{jk_0} \Pi\Big(\{\pi\};k_0,i_0,\ldots,k_r\Big) \prod_{\tau=0}^{t} {}^{r+1}b^{\tau}_{i_\tau}\Big(k_0,i_0,\ldots,k_\tau\Big).$$

Since the $V_r(P,K,k_r,i_r)$ do not alter when ${}^{r+1}b^r$ is altered, the following choice suffices, if the denominator involved does not equal zero. For all $h \in N^{2r}$ with $K(h,k_r) \cong K$ and all $i_r \in N$:

$$(4.8)\quad {}^{r}b^{r}_{i_r}(h,k_r) :=$$

$$\frac{\displaystyle\sum_{\substack{(k_0,i_0,\ldots,i_{r-1}) \in N^{2r} \\ K(k_0,i_0,\ldots,k_r) \cong K}} \delta_{jk_0} \Pi\Big(\{\pi\};k_0,i_0,\ldots,k_r\Big) \prod_{\tau=0}^{r} {}^{r+1}b^{\tau}_{i_\tau}(k_0,i_0,\ldots,k_\tau)}{\displaystyle\sum_{\substack{(k_0,i_0,\ldots,i_{r-1}) \in N^{2r} \\ K(k_0,i_0,\ldots,k_r) \cong K}} \delta_{jk_0} \Pi\Big(\{\pi\};k_0,i_0,\ldots,k_r\Big) \prod_{\tau=0}^{r-1} {}^{r+1}b^{\tau}_{i_\tau}(k_0,i_0,\ldots,k_\tau)}$$

If the denominator in expression (4.8) equals zero, expression

(4.7) is equal to zero and hence the choice of the ${}^{r}b^{r}(h,k_r)$ with $K(h,k_r) \overset{g}{=} K$ is arbitrary, if only ${}^{r}b^{r}(h,k_r) \in \mathcal{V}$.
It is easily verified that the ${}^{r}b^{r}(h,k_r)$ defined by formula (4.8) satisfies: ${}^{r}b^{r}(h,k_r) \in \mathcal{V}$.

Corollary 4.2: $\forall_{B \in \mathcal{B}} \exists_{B_0 \in \overline{\mathcal{R}}_\sigma} [B \overset{g}{\sim} B_0]$.

** It will be proved in the sequel, that in a special case it suffices to consider a more restricted subset of $\mathcal{B}$. If the decision costs d_{ki} all have the same value, one may restrict consideration to the class of decision rules, which only depend on the time of decision and the "reduced" information matrix of the realized allowed history until that time.
The case of equal (in fact vanishing) decision costs establishes an appropriate mathematical model for the following type of problem: a finite number of experiments is available, each with a finite number of possible outcomes (all expressed in the same unit, say dollar or success); the probability distributions for the different experiments are incompletely known; at discrete points of time an experiment has to be selected (compare [1956, R.N. Bradt, e.a.]).

Definition 4.7: $B \in \mathcal{B}$ is said to be a state-free sub-information decision rule (applying mixed strategies), when the following condition is satisfied:

$$\forall_{t(0 \leq t < T)} \forall_{h,h' \in N^{2t+1}} [K(h) = K(h') \Longrightarrow b^t(h) = b^t(h')] \quad .$$

The subset of $\mathcal{B}$ containing all state-free sub-information decision rules will be denoted by $\mathcal{L}_\sigma$.

** The difference between state-free sub-information decision rules and sub-information decision rules is found in the following feature. When applying a decision rule of the state-free type, decisions are not directly influenced by the actual state of the system, but only by the time of decision and the elements of the information matrix of the realized history corresponding to unknown Markov transition probabilities. Whereas sub-information decision rules may base decisions on the actual state of the system.

Lemma 4.10: $\mathcal{L}_\sigma \subset \mathcal{R}_\sigma$; $\mathcal{L}_\sigma$ is a compact subset of $\mathcal{B}$ (hence ${}_j\mathcal{L}_\sigma$ and $\overline{\mathcal{L}}_\sigma$ are compact $(j \in N)$ (lemma 2.17d))).
$\mathcal{L}_\sigma$ is a proper subset of $\mathcal{R}_\sigma$; $\mathcal{L}_\sigma$ is a proper subset of $\overline{\mathcal{L}}_\sigma$.

Lemma 4.11: $0 \leq r < T$; $\forall_{k,i \in N} [d_{ki} = d_{11}]$; $B \in \mathcal{R}_0$, then:

$$\forall_{P \in \mathcal{P}_\sigma} \forall_{j,j' \in N} \forall_{h,h' \in N^{2r+1}} \forall_{i \in N}$$

$$\Big[\mu_{(j,B,P)}\Big(\{h,i\} \times N^{2(T-r)-1}\Big)\mu_{(j',B,P)}\Big(\{h',i\} \times N^{2(T-r)-1}\Big) \neq 0 \quad ,$$

$$K(h) \cong K(h') \Longrightarrow V_r(j,B,P \mid h,i) = V_r(j',B,P \mid h',i)\Big] \quad .$$

Proof: Say $i = i_r$, then

$$V_r(j,B,P \mid h,i_r) = \sum_{\tau=r}^{T-1} \int_{N^{2T+1}} v_\tau d\mu^{(h,i_r)}_{(j,B,P)} \qquad \text{(compare lemma 2.5)}$$

$$= \sum_{\tau=r}^{T-1} \sum_{(k_{r+1},\dots,i_\tau,k_{\tau+1}) \in N^{2(\tau-r)+1}} \beta^\tau\Big(d_{11} + c_{i_\tau k_{\tau+1}}\Big)$$

$$\mu^{(h,i_r)}_{(j,B,P)}\Big(\{h,i_r,k_{r+1},\dots,k_{\tau+1}\} \times N^{2(T-\tau-1)}\Big)$$

$$= d_{11} \sum_{\tau=r}^{T-1} \beta^{\tau} + \sum_{\tau=r}^{T-1} \sum_{(k_{r+1},\ldots,i_{\tau},k_{\tau+1}) \in N^{2(\tau-r)+1}} \beta^{\tau} c_{i_{\tau} k_{\tau+1}}$$

$$\mu^{(h,i_r)}_{(j,B,P)}\left(\{h,i_r,k_{r+1},\ldots,k_{\tau+1}\} \times N^{2(T-\tau-1)}\right)$$

The $\mu^{(h,i_r)}_{(j,B,P)}$-factors do not depend on b^t $(0 \leq t \leq r)$; they do depend on the b^t $(r < t < T)$, however these b^t map $(h,i_r,k_{r+1},\ldots,k_{t+1})$ and $(h',i_r,k_{r+1},\ldots,k_{t+1})$ on the same decision vector in $\mathcal{V}$ when $K(h) \overset{\sigma}{=} K(h')$.

<u>Lemma 4.12</u>: $\forall_{k,i \in N} [d_{ki} = d_{11}]$, then

$$\forall_{j,j' \in N} \forall_{B \in \mathcal{L}_{\sigma}} \forall_{P \in \mathcal{P}} [V(j,B,P) = V(j',B,P)] \ .$$

<u>Proof</u>:

$$V(j,B,P) = d_{11} \sum_{t=0}^{T-1} \beta^{t} + \sum_{t=0}^{T-1} \sum_{(k_0,i_0,\ldots,k_{t+1}) \in N^{2t+3}} \beta^{t} c_{i_t k_{t+1}} \delta_{jk_0}$$

$$\left(\prod_{\tau=0}^{t} p_{i_{\tau} k_{\tau+1}}\right)\left(\prod_{\tau=0}^{t} b^{\tau}_{i_{\tau}}(k_0,i_0,\ldots,k_{\tau})\right) \quad \text{(lemma 2.5, formula (2.5))}$$

$b^{\tau}(k_0,i_0,\ldots,k_{\tau})$ does not differ with k_0 $(B \in \mathcal{L}_{\sigma})$.

<u>Theorem 4.3</u>: $(j \in N)$ $\forall_{k,i \in N} [d_{ki} = d_{11}]$, then

$$\forall_{B \in \mathcal{B}} \exists_{B_T \in \mathcal{L}_{\sigma}} [{}_jB \overset{\sigma}{\sim} {}_jB_T] \ .$$

<u>Proof</u>: Theorem 4.2 states the existence of a decision rule $B_0 \in \mathcal{K}_{\sigma}$, such that ${}_jB \overset{\sigma}{\sim} {}_jB_0$. An induction process will modify B_0 into a decision rule $B_T \in \mathcal{L}_{\sigma}$, such that ${}_jB_T \overset{\sigma}{\sim} {}_jB_0$.

 The induction process applies the following property: each $B_r \in \mathcal{K}_{\sigma}$

($0 \leq r < T$), may be modified - by only altering the ${}^r b^r$ - into a decision rule $B_{r+1} \in \mathcal{K}_\sigma$, which satisfies the condition of definition 4.7 for $t = r$ and ${}_j B_r \overset{\sigma}{\sim} {}_j B_{r+1}$. Before proving this property, it will be demonstrated that it furnishes a step-by-step procedure for modification of B_0 into $B_T \in \mathcal{L}_\sigma$ with ${}_j B_T \overset{\sigma}{\sim} {}_j B_0$.

$B_0 \in \mathcal{K}_\sigma$; say that $B_r \in \mathcal{K}_\sigma$, ${}_j B_r \overset{\sigma}{\sim} {}_j B_0$, B_r satisfies the condition of definition 4.7 for all t with $0 \leq t < r$, then there exists a decision rule $B_{r+1} \in \mathcal{K}_\sigma$ with ${}_j B_{r+1} \overset{\sigma}{\sim} {}_j B_0$ and B_{r+1} satisfies the condition of definition 4.7 for all t with $0 \leq t < r+1$ (${}^r b^t = {}^{r+1} b^t$ for $t \neq r$). If $T < \infty$, this procedure leads in a finite number of steps to the decision rule B_T, which satisfies the condition of the theorem. If $T = \infty$, the procedure generates a sequence of decision rules $\{B_r\}_{r=0}^{\infty}$. Since $\mathcal{K}_\sigma$ is compact (lemma 4.8), there exists a decision rule $B_\infty \in \mathcal{K}_\sigma$ which is the limit of a subsequence. For each r ($0 \leq r < T$) the set $\mathcal{K}_r := \{B \in \mathcal{K}_\sigma | B$ satisfies the condition of definition 4.7 for all t with $0 \leq t < r\}$ is compact and $B_\rho \in \mathcal{K}_r$ for $\rho \geq r$. Hence $B_\infty \in \bigcap_{r=0}^{\infty} \mathcal{K}_r = \mathcal{L}_\sigma$. ${}_j B_\infty \overset{\sigma}{\sim} {}_j B_0$ (lemma 2.13).

Now the step-property will be proved:
Consider the assertion in lemma 4.1 with $B = B_r \in \mathcal{K}_\sigma$ ($0 \leq r < T$) and $t = r$. The first sum on the right hand side of the equality does not depend on ${}^r b^r$ (lemma 4.3). The second sum may be rewritten as a finite sum of finite subsums, such that each subsum collects all terms corresponding to allowed histories until time r, which have information matrices sub-equal to a certain matrix.

For given matrix K, such a subsum is:

$$(4.9)\quad \sum_{\substack{h\in N^{2r+1}\\ K(h)\geqq K}} \; \sum_{i_r\in N} V_r(j,B_r,P\mid h,i_r)\; \mu_{(j,B_r,P)}\left(\{h,i_r\}\times N^{2(T-r)-1}\right) .$$

The factors V_r in expression (4.9) do not depend on h (lemma 4.11), provided that the corresponding $\mu_{(j,B_r,P)}$-factor does not equal zero. In this proof the factors V_r will be denoted henceforth by $V_r(P,K,i_r)$. The factors $\mu_{(j,B_r,P)}$ are determined by formula (2.4) and hence all contain the same subfactors p_{ki} with $(k,i)\notin I_\sigma$; the product of these subfactors is denoted by $\Pi(P,K)$; the products of the remaining (known) subfactors is denoted by $\Pi(\{\pi\};h)$. Expression (4.9) may be rewritten as:

$$(4.10)\quad \Pi(P,K)\sum_{i_r\in N} V_r(P,K,i_r)$$

$$\sum_{\substack{(k_0,i_0,\dots,k_r)\in N^{2r+1}\\ K(k_0,i_0,\dots,k_r)\geqq K}} \delta_{jk_0}\,\Pi\left(\{\pi\};k_0,\dots,k_r\right)\prod_{\tau=0}^{r} {}^{r}b_{i_\tau}^{\tau}(k_0,i_0,\dots,k_\tau) .$$

Since the $V_r(P,K,i_r)$ do not alter when ${}^{r}b^{r}$ is altered, the following choice suffices (if the denominator involved does not vanish). For all $h\in N^{2r+1}$ with $K(h)\geqq K$ and all $i_r\in N$:

$$(4.11)\quad {}^{r+1}b_{i_r}^{r}(h) := \frac{\displaystyle\sum_{\substack{(k_0,i_0,\dots,k_r)\in N^{2r+1}\\ K(k_0,i_0,\dots,k_r)\geqq K}} \delta_{jk_0}\,\Pi\left(\{\pi\};k_0,\dots,k_r\right)\prod_{\tau=0}^{r} {}^{r}b_{i_\tau}^{\tau}(k_0,i_0,\dots,k_\tau)}{\displaystyle\sum_{\substack{(k_0,i_0,\dots,k_r)\in N^{2r+1}\\ K(k_0,i_0,\dots,k_r)\geqq K}} \delta_{jk_0}\,\Pi\left(\{\pi\};k_0,\dots,k_r\right)\prod_{\tau=0}^{r-1} {}^{r}b_{i_\tau}^{\tau}(k_0,i_0,\dots,k_\tau)}$$

If the denominator in expression (4.11) does equal zero, expression (4.10) vanishes and hence ${}^{r+1}b^{r}(h)$ may be chosen arbitrarily from $\mathcal{V}$.

Corollary 4.3: $\forall_{k,i \in N}\,[d_{ki} = d_{11}]$, then

$$\forall_{B \in \mathcal{B}}\ \exists_{B_T \in \overline{\mathcal{L}_\sigma}}\ [B \overset{\sigma}{\sim} B_T] \quad .$$

** For application in other sections, the following lemma is needed:

Lemma 4.13: $\forall_{k,i \in N}\,[d_{ki} = d_{11}]$; for no $\ell, m \in N$:

$$\forall_{P \in \mathcal{P}_\sigma} \left[\sum_{i=1}^{n} c_{\ell i} p_{\ell i} > \sum_{i=1}^{n} c_{mi} p_{mi} \right] \quad ,$$

then there exists a non-empty subset $\mathcal{P}_1$ of $\mathcal{P}_\sigma$, such that

$$\forall_{P \in \mathcal{P}_1}\ \forall_{j,j' \in N}\ \forall_{B_1,B_2 \in \mathcal{B}}\ [V(j,B_1,P) = V(j',B_2,P)] \quad .$$

Proof: $R_\ell := \{x \in R \mid \exists_{P \in \mathcal{P}_\sigma} \left[\sum_{i=1}^{n} c_{\ell i} p_{\ell i} = x \right] \}$ $(\ell \in N)$. R_ℓ is a closed, bounded interval in R. Hence $R_0 = \bigcap_{\ell \in N} R_\ell \neq \emptyset$.

Choose: $\mathcal{P}_1 := \{P \in \mathcal{P}_\sigma \mid \exists_{x \in R_0}\ \forall_{\ell \in N} \left[\sum_{i=1}^{n} c_{\ell i} p_{\ell i} = x \right] \}$, then $\mathcal{P}_1 \neq \emptyset$ and for each $P \in \mathcal{P}_1$, $B \in \mathcal{B}$, $j \in N$ holds (x corresponding with P):

$$V(j,B,P) = \sum_{t=0}^{T-1} x\beta^t \quad .$$

** This may be interpreted as: If decision costs are equal and no action dominates any other, then there exist $P \in \mathcal{P}_\sigma$, such that for those P the risk function is constant in j,B.

PARTIALLY ORDERED SETS OF DECISION RULES

** This section will be devoted to the comparison of the effectiveness of different decision rules. This will be executed by comparing the $V(j,B,P)$ for different $B \in \mathcal{B}$. In fact, the introduction of the relations $\sim$ and $\mathcal{R}$ in $\mathcal{B}$ and ${}_j\mathcal{B}$ (section 4) was already an initiation of such a comparison. In this section, however, notions of the type "better than" and hence orderings will be introduced and investigated.
The material in this section is treated according to the methods which have been outlined in [1954, D. Blackwell, M.A. Girshick].

Definition 5.1: $B_1, B_2 \in \mathcal{B}$;

a) $(j \in N)$ ${}_jB_1$ is said to be at least as good as ${}_jB_2$, when

$$\forall_{P \in \mathcal{P}_\sigma} [V(j,B_1,P) \leq V(j,B_2,P)] \quad ;$$

notation: ${}_jB_1 \leqslant {}_jB_2$.

b) $(j \in N)$ ${}_jB_1$ is said to be better than ${}_jB_2$, when

$${}_jB_1 \leqslant {}_jB_2 \quad \text{and not} \quad {}_jB_1 \mathcal{R} {}_jB_2 \quad ;$$

notation: ${}_jB_1 < {}_jB_2$.

c) B_1 is said to be at least as good as B_2, when

$$\forall_{j \in N} [{}_jB_1 \leqslant {}_jB_2] \quad ;$$

notation: $B_1 \preccurlyeq B_2$.

d) B_1 is said to be <u>better than</u> B_2, when

$$B_1 \preccurlyeq B_2 \quad \text{and not} \quad B_1 \approx B_2 \quad ;$$

notation: $B_1 \prec B_2$.

** The $\preccurlyeq$- and the $\prec$-concept define relations in the sets ${}_j\mathcal{B}$, since all $B \in \mathcal{B}$ with ${}_jB = {}_jB_1$ possess the same $V(j,B,P)$ for each $P \in \mathcal{P}$ (lemma 2.7) and each ${}_jB$ is the j-restriction of at least one decision rule (lemma 2.1b)).

<u>Lemma 5.1</u>: a) The $\preccurlyeq$-concepts of definition 5.1a) and 5.1c) define weak, partial orderings in ${}_j\mathcal{B}$ ($j \in N$) and $\mathcal{B}$ respectively.

b) The $\prec$-concepts of definition 5.1b) and 5.1d) define partial orderings in ${}_j\mathcal{B}$ ($j \in N$) and $\mathcal{B}$ respectively.

** Some other useful notions for the investigation of the quality of decision rules follow:

<u>Definition 5.2</u>: $\mathcal{B}_0 \subset \mathcal{B}$, $B_0 \in \mathcal{B}_0$.

a) ($j \in N$) ${}_jB_0$ is said to be <u>admissible</u> in ${}_j\mathcal{B}_0$, when for no ${}_jB \in {}_j\mathcal{B}_0$

$${}_jB \prec {}_jB_0 \quad ;$$

the set of all j-decision rules, which are admissible in ${}_j\mathcal{B}_0$ is denoted by $\left({}_j\mathcal{B}_0\right)$;

b) B_0 is said to be <u>admissible</u> in $\mathcal{B}_0$, when for no $B \in \mathcal{B}_0$

$$B \prec B_0 \quad ;$$

the set of all decision rules, which are admissible in $\mathcal{B}_0$ is denoted by $\left(\mathcal{B}_0\right)$.

Lemma 5.2: $B \in \mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}$, then

a) $\forall_{j \in N} \left[{}_jB \in \left({}_j\mathcal{B}_1\right) \Longrightarrow {}_jB \in \left({}_j\mathcal{B}_0\right)\right]$;

b) $B \in \left(\mathcal{B}_1\right) \Longrightarrow B \in \left(\mathcal{B}_0\right)$;

c) $\left[\forall_{j \in N} \left({}_jB \in \left({}_j\mathcal{B}_0\right)\right)\right] \Longrightarrow B \in \left(\mathcal{B}_0\right)$.

Proof: The assertions are implied directly by the definitions. It should be emphasized, that none of the implications may be reversed.

Lemma 5.3: $\mathcal{B}_0 \subset \mathcal{B}$, then

$$B \in \left(\overline{\mathcal{B}_0}\right) \Longleftrightarrow \forall_{j \in N} \left[{}_jB \in \left({}_j\mathcal{B}_0\right)\right]$$

or

$$\forall_{j \in N} \left[{}_j\left(\overline{\mathcal{B}_0}\right) = \left({}_j\mathcal{B}_0\right)\right] .$$

Proof: "$\Longrightarrow$" say ${}_1B \notin \left({}_1\mathcal{B}_0\right)$: ${}_1B_1 \in {}_1\mathcal{B}_0$ and ${}_1B_1 \prec {}_1B$; then define $B_1 \in \overline{\mathcal{B}_0}$ by this ${}_1B_1$ and ${}_jB_1 := {}_jB$ $(j \in N,\ j \neq 1)$ then $B_1 \prec B$ (contradiction).
"$\Longleftarrow$" say $B \notin \left(\overline{\mathcal{B}_0}\right)$, then $B_1 \in \overline{\mathcal{B}}_0$, $B_1 \prec B$, hence for certain $j \in N$: ${}_jB_1 \prec {}_jB$ (contradiction).

Lemma 5.4:

a) $(j \in N)$ $\left({}_j\mathcal{K}_\sigma\right) \subset \left({}_j\mathcal{K}\right) \subset \left({}_j\mathcal{B}\right)$;

b) $\left(\overline{\mathcal{K}}_\sigma\right) \subset \left(\overline{\mathcal{K}}\right) \subset \left(\mathcal{B}\right)$;

$\forall_{k,i \in N} \left[d_{ki} = d_{11}\right]$, then

 c) $(j \in N)$ $\left({}_j\mathcal{L}_\sigma\right) \subset \left({}_j\mathcal{K}_\sigma\right)$;

d) $\left(\overline{\mathcal{L}_\sigma}\right) \subset \left(\overline{\mathcal{K}_\sigma}\right)$;

e) $\left(\mathcal{L}_\sigma\right) \subset \left(\overline{\mathcal{L}_\sigma}\right)$ and $(j \in N)$ $_j\left(\mathcal{L}_\sigma\right) = \left({}_j\mathcal{L}_\sigma\right)$.

<u>Proof</u>: a) is implied by theorems 4.1 and 4.2;
b) is implied by corollaries 4.1 and 4.2;
c) and d) are consequences of theorem 4.3 and corollary 4.3 respectively;
e) is implied by lemma 4.12.

** The concept of admissibility only becomes satisfactory in case the sets of admissible decision rules possess some completeness property:

<u>Definition 5.3</u>: $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}$, then

a) $(j \in N)$ $_j\mathcal{B}_0$ is said to be <u>complete</u> in $_j\mathcal{B}_1$, when

$$\forall B \in \mathcal{B}_1 \setminus \mathcal{B}_0 \; \exists B_0 \in \mathcal{B}_0 \; [{}_jB_0 < {}_jB] \;;$$

b) $\mathcal{B}_0$ is said to be <u>complete</u> in $\mathcal{B}_1$, when

$$\forall B \in \mathcal{B}_1 \setminus \mathcal{B}_0 \; \exists B_0 \in \mathcal{B}_0 \; [B_0 < B] \;;$$

c) $(j \in N)$ $_j\mathcal{B}_0$ is said to be <u>essentially complete</u> in $_j\mathcal{B}_1$, when

$$\forall B \in \mathcal{B}_1 \; \exists B_0 \in \mathcal{B}_0 \; [{}_jB_0 \leqslant {}_jB] \;;$$

d) $\mathcal{B}_0$ is said to be <u>essentially complete</u> in $\mathcal{B}_1$, when

$$\forall B \in \mathcal{B}_1 \; \exists B_0 \in \mathcal{B}_0 \; [B_0 \leqslant B] \;.$$

** Lemma 5.5 enumerates some properties, which are derived easily. They are presented without proof for that reason.

Lemma 5.5: $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}$, $j \in N$, then

a) if $\mathcal{B}_0$ (or ${}_j\mathcal{B}_0$) is complete in $\mathcal{B}_1$ (or ${}_j\mathcal{B}_1$ respectively), then $\mathcal{B}_0$ (or ${}_j\mathcal{B}_0$) is essentially complete in $\mathcal{B}_1$ (or ${}_j\mathcal{B}_1$ respectively);

b) if $\mathcal{B}_0$ (or ${}_j\mathcal{B}_0$) is (essentially) complete in $\mathcal{B}_2$ (or ${}_j\mathcal{B}_2$ respectively), then the same is true in $\mathcal{B}_1$ (or ${}_j\mathcal{B}_1$ respectively);

c) if $\mathcal{B}_0$ (or ${}_j\mathcal{B}_0$) is (essentially) complete in $\mathcal{B}_2$ (or ${}_j\mathcal{B}_2$ respectively), then $\mathcal{B}_1$ (or ${}_j\mathcal{B}_1$) is (essentially) complete in $\mathcal{B}_2$ (or ${}_j\mathcal{B}_2$ respectively);

d) if $\mathcal{B}_0$ is essentially complete in $\mathcal{B}_1$, then ${}_j\mathcal{B}_0$ is essentially complete in ${}_j\mathcal{B}_1$;

e) $\mathcal{B}_0$ is complete in $\mathcal{B}_0$; ${}_j\mathcal{B}_0$ is complete in ${}_j\mathcal{B}_0$;

f) if $\mathcal{B}_0$ is complete in $\mathcal{B}_1$, then $\left(\mathcal{B}_1\right) \subset \mathcal{B}_0$;
   if ${}_j\mathcal{B}_0$ is complete in ${}_j\mathcal{B}_1$, then $\left({}_j\mathcal{B}_1\right) \subset {}_j\mathcal{B}_0$;

g) if $\mathcal{B}_0$ (or ${}_j\mathcal{B}_0$) is complete in $\mathcal{B}_1$ (or ${}_j\mathcal{B}_1$ respectively) and this holds for no subset of $\mathcal{B}_0$, then $\mathcal{B}_0 = \left(\mathcal{B}_1\right)$ (or ${}_j\mathcal{B}_0 = \left({}_j\mathcal{B}_1\right)$ respectively);

h) $\left(\mathcal{B}_1\right)$ (or $\left({}_j\mathcal{B}_1\right)$) is complete in $\mathcal{B}_1$ (or ${}_j\mathcal{B}_1$ respectively), if and only if $\left(\mathcal{B}_1\right)$ (or $\left({}_j\mathcal{B}_1\right)$) is essentially complete in $\mathcal{B}_1$ (or ${}_j\mathcal{B}_1$ respectively).

i) the completeness relations between sets of (j-) decision rules are transitive; the same holds for essential completeness.

** Item g) of lemma 5.5 indicates the relevance of an investigation of the possibility, that the subset of admissible decision rules in a set of decision rules is complete in that set.

Lemma 5.6: $\mathcal{B}_0 \subset \mathcal{B}$, then, if $\overline{\mathcal{B}}_0 \subset \mathcal{B}_1 \subset \mathcal{B}$:

a) $[\forall_{j \in N} ({}_j\mathcal{B}_0$ essentially complete in ${}_j\mathcal{B}_1)] \Longrightarrow \overline{\mathcal{B}}_0$ essentially complete in $\mathcal{B}_1$;

b) $\left.\begin{array}{l}\exists_{j\in N}\,({}_j\mathcal{B}_0 \text{ complete in } {}_j\mathcal{B}_1) \text{ and} \\ \forall_{j\in N}\,({}_j\mathcal{B}_0 \text{ essentially complete in } {}_j\mathcal{B}_1)\end{array}\right\} \Longrightarrow \mathcal{B}_0$ complete in $\mathcal{B}_1$.

<u>Proof</u>: a) For any $B \in \mathcal{B}_1$ and $j \in N$, there exists a ${}_jB_0 \in {}_j\mathcal{B}_0$, such that ${}_jB_0 \leqslant {}_jB$; hence $B_0 \in \overline{\mathcal{B}_0}$ and $B_0 \leqslant B$.

b) Say ${}_1\mathcal{B}_0$ is complete in ${}_1\mathcal{B}_1$.
For any $B \in \mathcal{B}_1 \setminus \overline{\mathcal{B}_0}$ and $j \in N$, there exists a ${}_jB_0 \in {}_j\mathcal{B}_0$, such that ${}_jB_0 \leqslant {}_jB$ (for $j \neq 1$)

${}_jB_0 < {}_jB$ (for $j = 1$).

Hence $B_0 \in \overline{\mathcal{B}_0}$ and $B_0 < B$.

<u>Lemma 5.7</u>: $\mathcal{B}_0 \subset \mathcal{B}$, $j \in N$ and for every sequence $\{B_\ell\}_{\ell=1}^{\infty}$ with

$$\forall_{\ell(1\leqslant\ell<\infty)}\,[B_\ell \in \mathcal{B}_0,\ {}_jB_{\ell+1} < {}_jB_\ell]$$

there exists a decision rule $B_\infty \in \mathcal{B}_0$, with $\forall_{\ell(1\leqslant\ell<\infty)}[{}_jB_\infty < {}_jB_\ell]$.
Then $\left({}_j\mathcal{B}_0\right)$ is complete in ${}_j\mathcal{B}_0$.

<u>Proof</u>: Be $\{P_r\}_{r=1}^{\infty}$ a countable subset of $\mathcal{P}_\sigma$, dense in $\mathcal{P}_\sigma$.
Define for $B \in \mathcal{B}$: $U(j,B) := \sum_{r=1}^{\infty} 2^{-r}\, V(j,B,P_r)$. The sum exists for each $B \in \mathcal{B}$ and is bounded as a function of B, since the V-values are uniformly bounded in j,B and P (lemma 2.6b)).
For any $B_a, B_b \in \mathcal{B}$: ${}_jB_a \leqslant {}_jB_b \Longrightarrow U(j,B_a) \leqslant U(j,B_b)$, with equality only in the case ${}_jB_a \overset{g}{\sim} {}_jB_b$ (lemma 2.9).
Assume that $\left({}_j\mathcal{B}_0\right)$ is not complete in ${}_j\mathcal{B}_0$:
let $B_0 \in \mathcal{B}_0$ with ${}_jB_0 \in {}_j\mathcal{B}_0 \setminus \left({}_j\mathcal{B}_0\right)$ and $\mathcal{B}_1 := \{B \in \mathcal{B}_0 \mid {}_jB < {}_jB_0\}$ (whence $\mathcal{B}_1 \neq \emptyset$), such that ${}_j\mathcal{B}_1 \cap \left({}_j\mathcal{B}_0\right) = \emptyset$.

An induction process will present a decision rule $B_\infty \in \mathcal{B}_0$, such that ${}_jB_\infty \in {}_j\mathcal{B}_0 \setminus \left({}_j\mathcal{B}_0\right)$, but with no $B \in \mathcal{B}_0$ such that ${}_jB$ is better than ${}_jB_\infty$ (a contradiction). The induction process produces a sequence of sets $\mathcal{B}_\ell$ and decision rules B_ℓ ($\ell = 1,2,3,\ldots$), such that

$$\mathcal{B}_{\ell+1} := \{B \in \mathcal{B}_1 \mid {}_jB \prec {}_jB_\ell\} \quad ,$$

$$B_\ell \in \mathcal{B}_\ell \text{ , with } U(j,B_\ell) \leq \inf_{B \in \mathcal{B}_\ell} U(j,B) + \frac{1}{\ell} \quad (\ell = 1,2,..)$$

This is possible, since $\mathcal{B}_\ell \neq \emptyset$, for otherwise ${}_jB_{\ell-1} \in \left({}_j\mathcal{B}_0\right)$; furthermore ${}_j\mathcal{B}_\ell \cap \left({}_j\mathcal{B}_0\right) = \emptyset$.

The sequence $\{B_\ell\}_{\ell=1}^\infty$ possesses the monotonicity property of the assumption: ${}_jB_{\ell+1} \prec {}_jB_\ell$, hence there exists a $B_\infty \in \mathcal{B}_0$, with ${}_jB_\infty \prec {}_jB_\ell$ ($\ell = 1,2,\ldots$). Hence ${}_jB_\infty \notin ({}_j\mathcal{B}_0)$. On the other hand, there is no decision rule in $\mathcal{B}_0$ with a j-restriction, which is better than ${}_jB_\infty$, for otherwise it would possess a U-value less than $U(j,B_\infty)$ and it would satisfy the same inequalities:

$$\inf_{B \in \mathcal{B}_\ell} U(j,B) \leq U(j,B_\infty) \leq \inf_{B \in \mathcal{B}_\ell} U(j,B) + \frac{1}{\ell} \qquad (\ell = 1,2,\ldots) .$$

This establish the contradiction.

<u>Lemma 5.8</u>: $\mathcal{B}_0 \subset \mathcal{B}$, $j \in N$, ${}_j\mathcal{B}_0$ is closed, then $\left({}_j\mathcal{B}_0\right)$ is complete in ${}_j\mathcal{B}_0$.

<u>Proof</u>: ${}_j\mathcal{B}_0$ is closed, hence compact, which implies the monotonicity condition of lemma 5.7: each sequence $\{B_\ell\}_{\ell=1}^\infty$ with $B_\ell \in \mathcal{B}_0$ possesses a subsequence $\left\{B_{\ell_m}\right\}_{m=1}^\infty$, with $\lim_{m\to\infty} {}_jB_m = {}_jB_\infty \in {}_j\mathcal{B}_0$. The result of lemma 2.17 implies:

$$\forall_{P \in \mathcal{P}_\sigma} \left[\lim_{m\to\infty} V(j,B_{\ell_m},P) = V(j,B_\infty,P)\right] \quad ,$$

 hence ${}_jB_\infty \prec {}_jB_\ell$ ($\ell = 1,2,\ldots$).

Theorem 5.1: $j \in N$, then

a) $\left({}_j\mathcal{R}_\sigma\right)$ is essentially complete in ${}_j\mathcal{B}$;

b) if $\forall_{k,i \in N}\ [d_{ki} = d_{11}]$, then $\left({}_j\mathcal{L}_\sigma\right)$ is essentially complete in ${}_j\mathcal{B}$.

Proof: a) ${}_j\mathcal{R}_\sigma$ is closed (lemma 4.8), hence (lemma 5.8) $\left({}_j\mathcal{R}_\sigma\right)$ is complete in ${}_j\mathcal{R}_\sigma$ (and therefore essentially complete, lemma 5.5a)). ${}_j\mathcal{R}_\sigma$ is essentially complete in ${}_j\mathcal{B}$ (theorem 4.2, since ${}_jB \gtrsim {}_jB_0$ implies ${}_jB \preccurlyeq {}_jB_0$). Hence (lemma 5.5i)) $\left({}_j\mathcal{R}_\sigma\right)$ is essentially complete in ${}_j\mathcal{B}$.

b) The proof proceeds analogously: ${}_j\mathcal{L}_\sigma$ is closed (lemma 4.10) and ${}_j\mathcal{L}_\sigma$ is essentially complete in ${}_j\mathcal{B}$ as a result of theorem 4.3.

Lemma 5.9: $\mathcal{B}_0 \subset \mathcal{B}$, then:

$$\left[\forall_{j \in N}\left(\left({}_j\mathcal{B}_0\right) \text{ complete in } {}_j\mathcal{B}_0\right)\right] \Longrightarrow \left[\left(\overline{\mathcal{B}}_0\right) \text{ complete in } \overline{\mathcal{B}}_0\right] \quad .$$

Proof: $\left({}_j\mathcal{B}_0\right) = {}_j\left(\overline{\mathcal{B}}_0\right)$ (lemma 5.3) $(j \in N)$. ${}_j\left(\overline{\mathcal{B}}_0\right)$ complete for any $j \in N$ in ${}_j\mathcal{B}_0$ implies (lemma 5.6b)) the completeness of $\left(\overline{\mathcal{B}}_0\right)$ in $\overline{\mathcal{B}}_0$.

Lemma 5.10: $\mathcal{B}_0 \subset \mathcal{B}$ and for any sequence $\{B_\ell\}_{\ell=1}^{\infty}$ with

$$\forall_{\ell(1 \leq \ell < \infty)}\ [B_\ell \in \mathcal{B}_0\, ,\, B_{\ell+1} \prec B_\ell] \quad ,$$

there exists a decision rule $B_\infty \in \mathcal{B}_0$ with $\forall_{\ell(1 \leq \ell < \infty)}\ [B_\infty \prec B_\ell]$. Then $\left(\mathcal{B}_0\right)$ is complete in $\mathcal{B}_0$.

Proof: This proof possesses some features in common with the proof of lemma 5.7, but it is somewhat more complicated and will therefore be presented in full.

Again, let $\{P_r\}_{r=1}^{\infty}$ be a countable subset which separates $\mathcal{P}_o$. For every $j \in N$, $B \in \mathcal{B}$ the value $U(j,B)$ is defined as in the proof of lemma 5.7.

When $B_a, B_b \in \mathcal{B}$ and $B_a \not< B_b$, then $\forall_{j \in N} [U(j,B_a) \leq U(j,B_b)]$, with n equalities only if $B_a \gtrsim B_b$ (lemma 2.9).

Assume that $(\mathcal{B}_o)$ is not complete in $\mathcal{B}_o$:

let $B_o \in \mathcal{B}_o \setminus (\mathcal{B}_o)$ and $\mathcal{B}_1 := \{B \in \mathcal{B}_o \mid B < B_o\}$ (whence $\mathcal{B}_1 \neq \emptyset$), such that $\mathcal{B}_1 \cap (\mathcal{B}_o) = \emptyset$.

A finite induction process will provide sets $\mathcal{B}_j \subset \mathcal{B}_o$ and decision rules B_j $(j = 1,2,\ldots,n)$, such that

$$\mathcal{B}_j := \{B \in \mathcal{B}_o \mid B < B_{j-1}\} \quad ,$$

$$B_j \in \mathcal{B}_j \, , \; {}_jB_j \in \left({}_j\mathcal{B}_j\right) \qquad (j = 1,2,\ldots,n) \quad .$$

$\mathcal{B}_j \neq \emptyset$, since otherwise $B_{j-1} \in \mathcal{B}_1 \cap (\mathcal{B}_o)$, which produces a contradiction.

There is no decision rule in $\mathcal{B}_o$ which is better than B_n, hence $B_n \in \mathcal{B}_1 \cap (\mathcal{B}_o)$, which provides the desired contradiction.

It remains to be verified that there exists for any $j \in N$ a decision rule B_j satisfying the conditions assuming $B_o,\ldots,B_{j-1}$ to be given. B_j is constructed by an induction process, which provides sets $\mathcal{B}_{jm}$ and decision rules B_{jm} $(m = 1,2,\ldots)$, such that

$$\mathcal{B}_{j1} := \mathcal{B}_j$$

$$\mathcal{B}_{jm+1} := \{B \in \mathcal{B}_o \mid B < B_{jm}\}$$

$$B_{jm} \in \mathcal{B}_{jm} \, , \; U(j,B_{jm}) \leq \inf_{B \in \mathcal{B}_{jm}} U(j,B) + \frac{1}{m} \qquad (m = 1,2,\ldots) \quad .$$

$\mathcal{B}_{jm} \neq \emptyset$, since otherwise $\qquad B_{jm-1} \in \mathcal{B}_1 \cap (\mathcal{B}_o)$,

which provides a contradiction.

The sequence $\{B_{jm}\}_{m=1}^{\infty}$ satisfies the condition of the lemma, hence there exists a decision rule $B_j \in \mathcal{B}_0$ with $B_j \prec B_{jm}$ $(m=1,2,\dots)$. There is no decision rule in $\mathcal{B}_0$, which is better than B_{j-1} and possesses a j-restriction better than ${}_jB_j$, since this decision rule would be an element of $\mathcal{B}_{jm}$ $(m = 1,2,\dots)$ and B_j satisfies the inequalities

$$\inf_{B\in \mathcal{B}_{jm}} U(j,B) \leq U(j,B_j) \leq \inf_{B\in \mathcal{B}_{jm}} U(j,B) + \frac{1}{m} \quad (m = 1,2,\dots) \quad .$$

Lemma 5.11: $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{B}_0$ is closed, then $\left(\mathcal{B}_0\right)$ is complete in $\mathcal{B}_0$.

Proof: $\mathcal{B}_0$ is closed, hence compact. This implies the monotonicity condition of lemma 5.10: any sequence $\{B_\ell\}_{\ell=1}^{\infty}$ with $B_\ell \in \mathcal{B}_0$ possesses a convergent subsequence $\left\{B_{\ell_m}\right\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} B_{\ell_m} = B_\infty \in \mathcal{B}_0$. Lemma 2.13 implies:

$$\forall_{j\in N}\ \forall_{P\in\mathcal{P}_\sigma} \left[\lim_{m\to\infty} V(j,B_{\ell_m},P) = V(j,B_\infty,P)\right] \quad ,$$

hence $B_\infty \prec B_\ell$ $(\ell = 1,2,\dots)$.

** Hence for all special subsets of $\mathcal{B}$, which were introduced earlier ($\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}_\sigma, \overline{\mathcal{K}_\sigma}, \mathcal{L}_\sigma, \overline{\mathcal{L}_\sigma}$), the subsubset of admissible decision rules is complete in the subset itself. For the subsets with bars this is implied already by lemma 5.9.

Theorem 5.2: a) $\left(\overline{\mathcal{K}_\sigma}\right)$ is essentially complete in $\mathcal{B}$.
b) If $\forall_{k,i\in N}\ (d_{ki} = d_{11})$, then $\left(\overline{\mathcal{L}_\sigma}\right)$ is essentially complete in $\mathcal{B}$.

Proof: a) $\left(\overline{\mathcal{K}_\sigma}\right)$ is complete in $\overline{\mathcal{K}_\sigma}$ (via lemma 5.9 and lemma 5.8, since ${}_j\mathcal{K}_\sigma$ is closed (lemma 4.8), or via lemma 5.11, since $\overline{\mathcal{K}_\sigma}$ is closed (lemma 4.8)).

$\overline{\mathcal{K}}_\sigma$ is essentially complete in $\mathcal{B}$ (corollary 4.2, since $B \precsim B_0$ implies $B \leqslant B_0$). Hence (lemma 5.5i)) $\left(\overline{\mathcal{K}}_\sigma\right)$ is essentially complete in $\mathcal{B}$.

b) The proof proceeds analogously: ${}_j\mathcal{L}_\sigma$ and $\overline{\mathcal{L}}_\sigma$ are compact (lemma 4.10) and $\left(\overline{\mathcal{L}}_\sigma\right)$ is essentially complete in $\mathcal{B}$ (corollary 4.3).

** A specific feature of the concept of admissibility just introduced is the following: decision rules may exist, which are admissible, but possess for no $j \in N$, $P \in \mathcal{P}_\sigma$ a risk value equal to

$$\min_{B \in \mathcal{B}} V(j,B,P) \quad .$$

See example A.4 in the appendix. This seems to be rather obvious: such admissible decision rules will be relatively good for all $P \in \mathcal{P}_\sigma$, while decision rules with minimum risk value for certain j,P are as good as possible for some j,P, but perhaps rather bad for others. This agrees with the fact that any j-decision rule, which is admissible in ${}_j\mathcal{B}$, is Bayes-optimal for initial state j and at least one prior distribution on $\mathcal{P}_\sigma$ (this will be proved in section 8). Hence some admissible decision rules seem to meet our desire for a compromise. The property, that any admissible decision rule is optimal for certain $j \in N$, $P \in \mathcal{P}_\sigma$ is present in some special cases. For example in the case of equal decision costs with no action uniformly (in P) dominated by another one (lemma 4.13). For some further remarks on such special cases the reader is referred to the following sections and the appendix. It should be remarked that in the case of equal decision costs an admissible decision rule may prescribe the application of an action, which is uniformly dominated by another one (see example A.2). Hence, if one prefers to refrain from applying uniformly dominated actions, they should be removed before one studies the decision rules.

Reviewing these problems, it appears to be necessary to study the determination of fictitious costs, which may be introduced in order to fit the mathematical model of a practical situation so that the theory of this study may be applied.
Two different cases will be considered.
In the first case a finite number of experiments each with a finite number of elementary outcomes is available (equal decision costs). The numbers of elementary outcomes may be equalized by adding elementary outcomes with given probability zero and arbitrary costs. A condition imposed by the theory, is the equality of the number of elementary outcomes for each experiment with the number of experiments available. If the first number happens to be exceeded by the second, more elementary outcomes with given probability zero can be added. If the first number exceeds the second, an obvious trick is the addition of fictitious experiments with fictitious costs. However, one wants to design these dummy experiments so that they do not play an active role in decision making. Example A.2 shows that this should be done carefully. Theorem 5.3 demonstrates, how such fictitious experiments may be designed.
Secondly, the case of forbidden actions is investigated. Say that, in the system considered, the action $1 \rightarrow 3$ is physically impossible. An obvious way-out is the designation of a fictitious decision cost d_{13}. The question remains, which value for d_{13} guarantees, that the action $1 \rightarrow 3$ does not play an active role in decision making.

Before the assertions on these topics are stated, some auxiliary theory is developed, which is anyhow desirable for application in section 7.

Those decision vectors of a decision rule, corresponding to allowed histories until time t $(t = T_0, T_0 + 1, \ldots)$ which coincide until T_0 with a given allowed history establish a decision rule for the decision problem with $T-T_0$ steps. This notion of the restriction of a decision rule is a generalization of the concept "j-decision rule". A new aspect, however, is the fact that such a restriction induces a decision rule for a problem with an other number of steps. All elaboration up till now have assumed a total number of steps equal to T. T has been considered as given but arbitrarily natural or (countably) infinite. The reason for assuming T as given in section 1, is to prevent the necessity of explicitly stating T as a parameter in all notions.

Definition 5.4: m natural, $1 \leq m < 2T+1$, $h \in N^m$.
A h-decision rule (applying mixed strategies) ${}_hB$ is a sequence of mappings

$$ {}_hb^t : \{h\} \times N^{2t+1-m} \longrightarrow \mathcal{V} \qquad (\tfrac{m-1}{2} \leq t < T) \quad ; $$

the set of all h-decision rules is denoted by ${}_h\mathcal{B}$.

Lemma 5.12: $1 \leq m < 2T+1$.

a) To any $B \in \mathcal{B}$ there correspond ${}_hB \in {}_h\mathcal{B}$ $(h \in N^m)$, such that the mappings ${}_hb^t$, consituting ${}_hB$, are the restrictions of b^t to $\{h\} \times N^{2T+1-m}$.

b) $h \in N^m$; to any ${}_hB \in {}_h\mathcal{B}$ there corresponds at least one $B \in \mathcal{B}$, such that ${}_hB$ is the restriction of B in the sense of a).

Convention 5.1: $(1 \leq m < 2T+1)$ elements of ${}_h\mathcal{B}$ $(h \in N^m)$ are denoted by ${}_hB$, possibly indexed: ${}_hB_o$, ${}_hB_r$; whenever an element of ${}_h\mathcal{B}$ and one of $\mathcal{B}$ with the same index (or no index) are mentioned together, they satisfy the relation of lemma 5.12.
If $\mathcal{B}_o \subset \mathcal{B}$, then ${}_h\mathcal{B}_o$ $(h \in N^m)$ denotes: $\{{}_hB \in {}_h\mathcal{B} \mid B \in \mathcal{B}_o\}$.
$B, B_o, B_r \in \mathcal{B}$, $h \in N^m$; for the mappings constituting ${}_hB$, ${}_hB_o$, ${}_hB_r$

 the index h will be skipped, thus the same notations will be applied as for the mappings constituting B, B_0, B_r; hence no different notations will be applied for a mapping and certain restrictions.

In all cases where is referred to a situation with a total number of steps unequal to the given T, this is stated explicitly. For those cases, the same assumptions, definitions, (and hence) lemmas, theorems hold, with only T replaced by the explicitly stated value.

Remark: The notation $\mathcal{B}$, ${}_h\mathcal{B}$ $(h \in N^m)$ (definition 5.4) is consistent with convention 5.1.

Lemma 5.13: $0 < T_0 < T$, $j \in N$, $h \in N^{2T_0}$, $B \in \mathcal{B}$.

a) ${}_hB$ induces a decision rule for the problem with $T - T_0$ steps in a natural way, by defining as decision vector for $h_1 \in N^{2t+1}$ $(0 \leq t < T - T_0)$: $b^{T_0+t}(h, h_1)$.
Reversely, any decision rule for the problem with $T - T_0$ steps induces a h-decision rule.

b) ${}_{(h,j)}B$ induces a j-decision rule for the problem with $T - T_0$ steps in a natural way, by defining as decision vector for (j, h_1) with $h_1 \in N^{2t}$ $(0 \leq t < T - T_0)$: $b^{T_0+t}(h, j, h_1)$.
Reversely any j-decision rule for the problem with $T - T_0$ steps induces a (h,j)-decision rule.

Convention 5.2: $0 < T_0 < T$, $j \in N$, $h \in N^{2T_0}$.
The (j-)decision rules for the problem with $T - T_0$ steps (lemma 5.13), which correspond to ${}_hB \in {}_h\mathcal{B}$ and ${}_{(h,j)}B \in {}_{(h,j)}\mathcal{B}$, will be denoted by the same symbols: ${}_hB$ and ${}_{(h,j)}B$ respectively. The same convention will be maintained with regard to the sets ${}_{(h,j)}\mathcal{B}$ and ${}_h\mathcal{B}$.

Lemma 5.14: $0 < T_0 < T$, $h \in N^{2T_0}$, $B \in \mathcal{B}$.
$B \in \mathcal{K}_\sigma$ implies: ${}_hB$ is a sub-information decision rule for the problem with $T - T_0$ steps (if $B \in \mathcal{L}_\sigma$, ${}_hB$ is moreover state-free).

Proof: Say $h_1 \in N^{2t+1}$, $h_2 \in N^{2t+1}$ $(0 \leq t < T-T_0)$ and $K(h_1) \leqq K(h_2)$, then $K(h,h_1) \leqq K(h,h_2)$.

Lemma 5.15: $0 < T_0 < T$; $h \in N^{2T_0}$; $j,k \in N$; $P \in \mathcal{P}$; $B \in \mathcal{B}$ and $\mu_{(j,B,P)}\left(\{h,k\} \times N^{2(T-T_0)}\right) \neq 0$, then: the expected total costs for the problem with $T-T_0$ steps $V(k,{}_hB,P)$ are equal to

$$\beta^{-T_0} \sum_{i \in N} b_i^{T_0}(h,k) V_{T_0}(j,B,P \mid h,k,i) \quad .$$

Proof: For $0 \leq t < T-T_0$, $i \in N$, $h_1 \in N^{2t+1}$, $b_i^{T_0}(h,k) \neq 0$, the following equality holds:

$$\mu_{(k,{}_hB,P)}\left(\{k,i,h_1\} \times N^{2(T-T_0-t-1)}\right) =$$

$$= b_i^{T_0}(h,k)\mu_{(j,B,P)}^{(h,k,i)}\left(\{h,k,i,h_1\} \times N^{2(T-T_0-t-1)}\right) \quad .$$

The probability on the left hand side refers to the problem with $T - T_0$ steps.
Application of definition 4.1 then delivers the desired result,

Lemma 5.16: $0 \leq t < T$, $j \in N$, $B \in \mathcal{B}$, $P \in \mathcal{P}$, then:

$$V(j,B,P) = \sum_{\tau=0}^{t-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} +$$

$$+ \beta^t \sum_{h \in N^{2t}} \sum_{k \in N} V(k,{}_hB,P)\mu_{(j,B,P)}\left(\{h,k\} \times N^{2(T-t)}\right) \quad ,$$

where the V-values on the right hand side denote total expected costs for the problem with $T-t$ steps.

Proof: (lemma 4.1)

$$V(j,B,P) = \sum_{\tau=0}^{t-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} +$$

$$+ \sum_{h \in N^{2t}} \sum_{i,k \in N} \mu_{(j,B,P)}\left(\{h,k,i\} \times N^{2(T-t)-1}\right) V_t(j,B,P \mid h,k,i)$$

$$= \sum_{\tau=0}^{t-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} +$$

$$+ \sum_{h \in N^{2t}} \sum_{k \in N} \mu_{(j,B,P)}\left(\{h,k\} \times N^{2(T-t)}\right) \sum_{i \in N} b_i^t(h,k) V_t(j,B,P|h,k,i).$$

This is equal to the stated formula (lemma 5.15).

Theorem 5.3: $\forall_{k,i \in N} [d_{ki} = d_{11}]$; $r,\ell \in N$, $\{r\} \times N \subset I_\sigma$, $\forall_{P \in \mathcal{P}_\sigma} \left(\sum_{i \in N} c_{ri}\pi_{ri} > \sum_{i \in N} c_{\ell i} p_{\ell i}\right)$, $B \in (\mathcal{B})$, then

$$\forall_{t(0 \leq t < T)} \forall_{h \in N^{2t+1}} \left[b_r^t(h) > 0 \implies \forall_{j \in N} \forall_{P \in \mathcal{P}_\sigma} \left[\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)}\right) = 0\right]\right] .$$

Proof: Say $h \in N^{2t+1}$ and $b_r^t(h) > 0$.

Define $B_0 \in \mathcal{B}$:

$${}^0b^\tau := b^\tau \quad (0 \leq \tau < t)$$

$${}^0b^\tau(h_1,h_2) := b^\tau(h_1,h_2) \quad (t \leq \tau < T,\ h_1 \in N^{2t+1},\ h_1 \neq h,\ h_2 \in N^{2(\tau-t)})$$

$${}^0b_r^t(h) := 0$$

$^{o}b_{\ell}^{t}(h) := b_{\ell}^{t}(h) + b_{r}^{t}(h)$

$^{o}b_{i}^{t}(h) := b_{i}^{t}(h) \quad (i \in N,\ i \neq r,\ i \neq \ell)$

$^{o}b^{\tau}(h,i,h_1) := b^{\tau}(h,i,h_1) \quad (t < \tau < T,\ i \neq \ell,\ h_1 \in N^{2(\tau-t)-1})$

only $_{(h,\ell)}B_o$ remains to be defined:

for $_{(h,\ell)}B_o$ a decision rule for the problem with $T-t-1$ steps is chosen, corresponding (in the sense of theorem 3.2) to the mixed decision rule for that problem, which chooses $_{(h,\ell)}B$ with probability $\dfrac{b_{\ell}^{t}(h)}{b_{\ell}^{t}(h) + b_{r}^{t}(h)}$ and $_{(h,r)}B_i$ with probability $\dfrac{b_{r}^{t}(h)\pi_{ri}}{b_{\ell}^{t}(h) + b_{r}^{t}(h)}$ $(i \in N)$, with $_{(h,r)}B_i$ defined by: $^{i}b^{\tau}(h,r,k,h_1) := b^{\tau}(h,r,i,h_1)$ $(t < \tau < T,\ k \in N,\ h_1 \in N^{2(\tau-t-1)})$.

Note that $V\left(k, {}_{(h,r)}B_i, P\right) = V\left(i, {}_{(h,r)}B, P\right) \quad (k,i \in N,\ P \in \mathcal{P})$.

Now apply lemma 5.16:

$$(5.1)\quad V(j,B_0,P) = \sum_{\tau=0}^{t} \int_{N^{2T+1}} v_\tau d\mu_{(j,B_0,P)} +$$

$$+ \beta^{t+1} \sum_{h_1 \in N^{2t+1}} \sum_{i,k \in N} V\left(k, {}_{(h_1,i)}B_o, P\right) \mu_{(j,B_0,P)}\left(\{h_1, i, k\} \times N^{2(T-t-1)}\right)$$

$$= \sum_{\tau=0}^{t-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} + \int_{N^{2T+1}} v_t d\mu_{(j,B_0,P)} +$$

$$+ \beta^{t+1} \sum_{\substack{h_1 \in N^{2t+1} \\ h_1 \neq h}} \sum_{i,k \in N} V\left(k, {}_{(h_1,i)}B, P\right) \mu_{(j,B,P)}\left(\{h_1, i, k\} \times N^{2(T-t-1)}\right) +$$

$$+ \beta^{t+1} \mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)}\right) \sum_{i,k \in N} V\left(k, {}_{(h,i)}B_o, P\right) \overset{o}{b}{}_i^t(h) p_{ik}$$

(lemma 4.3a), definition of B_o, formula (2.5)).
The last sum may be rewritten as:

$$(5.2) \quad \left(b_\ell^t(h) + b_r^t(h)\right) \sum_{k \in N} V\left(k, {}_{(h,\ell)}B_o, P\right) p_{\ell k} +$$

$$+ \sum_{\substack{i \in N \\ i \neq \ell, r}} b_i^t(h) \sum_{k \in N} V\left(k, {}_{(h,i)}B, P\right) p_{ik} .$$

The first part of this sum equals (theorem 3.4):

$$\sum_{k \in N} \left(b_\ell^t(h) V\left(k, {}_{(h,\ell)}B, P\right) + b_r^t(h) \sum_{i \in N} \pi_{ri} V\left(k, {}_{(h,r)}B_i; P\right)\right) p_{\ell k} =$$

$$= b_\ell^t(h) \sum_{k \in N} V\left(k, {}_{(h,\ell)}B, P\right) p_{\ell k} + b_r^t(h) \sum_{i \in N} V\left(i, {}_{(h,r)}B, P\right) \pi_{ri} .$$

Hence (5.2) equals: $\sum_{i \in N} b_i^t(h) \sum_{k \in N} V\left(k, {}_{(h,i)}B, P\right) p_{ik} \quad (P \in \mathcal{P}_\sigma)$.

Then (5.1) and lemma 5.16 for B imply:

$$V(j,B,P) = V(j,B_o,P) + \int_{N^{2T+1}} v_t d\mu_{(j,B,P)} - \int_{N^{2T+1}} v_t d\mu_{(j,B_o,P)} .$$

If $\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)}\right) > 0$ for some $P \in \mathcal{P}_\sigma$ and $j \in N$, then for those P and j $\int_{N^{2T+1}} v_t d\mu_{(j,B,P)} > \int_{N^{2T+1}} v_t d\mu_{(j,B,P)}$; for the other P and j equality holds.

Hence $B_o < B$ if $\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)}\right) > 0$ for some $P \in \mathcal{P}_\sigma$, $j \in N$. This contradicts $B \in (\mathcal{B})$.

<u>Corollary 5.3</u>: $\forall_{k,i \in N} [d_{ki} = d_{11}]$; $I \subset N$, $\ell \in N$, $\ell \notin I$, $I \times N \subset I_\sigma$,
$\forall_{P \in \mathcal{P}_\sigma} \left(\min_{r \in I} \sum_{i \in N} c_{ri}\pi_{ri} \geq \sum_{i \in N} c_{\ell i}p_{\ell i} \right)$.
Then $(\overline{\mathcal{L}_1})$ is essentially complete in $\mathcal{B}$, when

$$\mathcal{L}_1 := \{B \in \mathcal{L}_\sigma \mid \forall_{r \in I} \forall_{t(0 \leq t < T)} \forall_{h \in N^{2t+1}} (b_r^t(h) = 0)\} \quad .$$

<u>Proof</u>: It will be proved that

$$\mathcal{B}_t := \{B \in \mathcal{B} \mid \forall_{r \in I} \forall_{\tau(0 \leq \tau < t)} \forall_{h \in N^{2\tau+1}} (b_r^\tau(h) = 0)\}$$

$(0 \leq t \leq T)$

is essentially complete in $\mathcal{B}$ and furthermore, that $(\overline{\mathcal{L}_1})$ is essentially complete in $\mathcal{B}_T$. Then lemma 5.5i) implies the assertion.
$\mathcal{B}_0 = \mathcal{B}$, hence $\mathcal{B}_0$ is essentially complete in $\mathcal{B}$.
Say $\mathcal{B}_t$ for certain t with $0 \leq t < T$ is essentially complete in $\mathcal{B}$. It will be proved, that $\mathcal{B}_{t+1} \subset \mathcal{B}_t$ is essentially complete in $\mathcal{B}$:
$B \in \mathcal{B}$, $B_t \in \mathcal{B}_t$, $B_t \preccurlyeq B$. $B_{t+1} \in \mathcal{B}_{t+1}$ with $B_{t+1} \preccurlyeq B_t$ and hence $B_{t+1} \preccurlyeq B$ may be constructed by applying the construction method of theorem 5.3 successively for all $r \in I, h \in N^{2t+1}$ with ${}^t b_r^t(h) > 0$.
This induction method implies the essential completeness of $\mathcal{B}_T$ in $\mathcal{B}$ in case $T < \infty$.
Say $T = \infty$, then the induction process delivers a sequence $\{B_t\}_{t=0}^{\infty}$ with $B_t \in \mathcal{B}_t$ and $B_t \preccurlyeq B$. This sequence contains a convergent subsequence (lemma 2.12) with limit $B_T \in \mathcal{B}_T$ and $B_T \preccurlyeq B$ (lemma 2.13).
$\mathcal{L}_1 = \mathcal{L}_0 \cap \mathcal{B}_T$; $\forall_{B_T \in \mathcal{B}_T} \exists_{B_{TT} \in \overline{\mathcal{L}_1}} \left[B_T \gtrsim B_{TT} \right]$. The proof proceeds exactly as the proof of theorem 4.3 and corollary 4.3, since the construction method in the proof of theorem 4.3 guarantees

${}^{T}T b_r^t(h) = 0$ for all $h \in N^{2t+1}$ in case ${}^{T}b_r^t(h) = 0$ for all $h \in N^{2t+1}$ (see formula (4.11)).
$\left(\overline{\mathcal{L}}_1\right)$ is essentially complete in $\mathcal{L}_1$ ($\overline{\mathcal{L}}_1$ is closed, lemma 5.11), hence $\left(\overline{\mathcal{R}}_1\right)$ is essentially complete in $\mathcal{B}_T$ (lemma 5.5i)).

** Theorem 5.4 and corollary 5.4 present a situation where certain actions do not play an active role in decision-making (e.g. "forbidden" actions).

<u>Theorem 5.4</u>: $m, r \in N$;

$$d_{mr} + \min_{P \in \mathcal{P}_\sigma} \sum_{\ell \in N} c_{r\ell} p_{r\ell} > \left(\sum_{t=0}^{T-1} \beta^t\right) \max_{k \in N} \min_{i \in N} \left[d_{ki} + \max_{P \in \mathcal{P}_\sigma} \sum_{\ell \in N} c_{i\ell} p_{i\ell}\right]$$

$B \in (\mathcal{B})$, then: $\forall_{t (0 \leq t < T)}\ \forall_{h \in N^{2t}}$

$$\left[b_r^t(h,m) > 0 \Longrightarrow \forall_{j \in N}\ \forall_{P \in \mathcal{P}_\sigma} \left[\mu_{(j,B,P)}\left(\{h,m\} \times N^{2(T-t)}\right) = 0\right]\right] .$$

<u>Proof</u>: Choose a $i_k \in N$ for each $k \in N$, such that

$$d_{ki_k} + \max_{P \in \mathcal{P}_\sigma} \sum_{\ell \in N} c_{i_k \ell} p_{i_k \ell} = \min_{i \in N} \left[d_{ki} + \max_{P \in \mathcal{P}_\sigma} \sum_{\ell \in N} c_{i\ell} p_{i\ell}\right] .$$

Then: $r \neq i_m$.

Define $B_1 \in \mathcal{A}$, by ${}^1b_{i_k}^t(h_1,k) = 1$, for every t $(0 \leq t < T)$, $h_1 \in N^{2t}$, $k \in N$. Say $h \in N^{2t}$ and $b_r^t(h,m) > 0$.

Define $B_0 \in \mathcal{B}$, by:

${}^0b^\tau := b^\tau \quad (0 \leq \tau < t)$

$$^{o}b^{\tau}(h_1,k,h_2) := b^{\tau}(h_1,k,h_2) \qquad (t \leq \tau < T,\ h_1 \in N^{2t},\ k \in N,$$
$$(h_1,k) \neq (h,m),\ h_2 \in N^{2(\tau-t)})$$

$$^{o}b^{t}_{r}(h,m) := 0$$

$$^{o}b^{t}_{i_m}(h,m) := b^{t}_{i_m}(h,m) + b^{t}_{r}(h,m)$$

$$^{o}b^{t}_{i}(h,m) := b^{t}_{i}(h,m) \quad (i \in N,\ i \neq r,\ i \neq i_m)$$

$$^{o}b^{\tau}(h,m,i,h_3) := b^{\tau}(h,m,i,h_3) \quad (t < \tau < T,\ i \in N,$$
$$i \neq i_m,\ h_3 \in N^{2(\tau-t)-1})$$

only $_{(h,m,i_m)}B_0$ remains to be defined:

Choose a decision rule for the problem with T - t - 1 steps, corresponding (in the sense of theorem 3.2) to the mixed decision rule for that problem, which selects

$$_{(h,m,i_m)}B \text{ with probability } \frac{b^{t}_{i_m}(h,m)}{b^{t}_{i_m}(h,m) + b^{t}_{r}(h,m)} \text{ and}$$

$$_{(h,m,i_m)}B_1 \text{ with probability } \frac{b^{t}_{r}(h,m)}{b^{t}_{i_m}(h,m) + b^{t}_{r}(h,m)} .$$

Now apply lemma 5.16:

$$V(j,B_0,P) = \sum_{\tau=0}^{t} \int_{N^{2T+1}} v_{\tau} d\mu_{(j,B_0,P)} +$$

$$+ \beta^{t+1} \sum_{h_1 \in N^{2t+2}} \sum_{k \in N} V\left(h,{}_{h_1}B_0,P\right)\mu_{(j,B_0,P)}\left(\{h_1,k\} \times N^{2(T-t-1)}\right)$$

$$(5.3) \quad = \sum_{\tau=0}^{t-1} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} + \int_{N^{2T+1}} v_t d\mu_{(j,B_0,P)} +$$

$$\beta^{t+1} \sum_{\substack{h_1 \in N^{2t+2} \\ h_1 \neq (h,m,i_m), \neq (h,m,r)}} \sum_{k \in N} V\left(k, {}_{h_1}B, P\right) \mu_{(j,B,P)}\left(\{h_1,k\} \times N^{2(T-t-1)}\right)$$

$$+ \beta^{t+1} \mu_{(j,B,P)}\left(\{h,m\} \times N^{2(T-t)}\right)\left(b^t_{i_m}(h,m) + b^t_r(h,m)\right)$$

$$\sum_{k \in N} V\left(k, {}_{(h,m,i_m)}B_0, P\right) p_{i_m k}$$

(lemma 4.3a), definition of B_0, formula (2.5)).

According to theorem 3.4:

$$V\left(k, {}_{(h,m,i_m)}B_0, P\right) = \left(b^t_{i_m}(h,m) + b^t_r(h,m)\right)^{-1}$$

$$\left\{b^t_{i_m}(h,m) V\left(k, {}_{(h,m,i_m)}B, P\right) + b^t_r(h,m) V\left(k, {}_{(h,m,i_m)}B_1, P\right)\right\} .$$

Hence (formula (5.3) and lemma 5.16 for B):

$$V(j,B,P) = V(j,B_0,P) + \int_{N^{2T+1}} v_t d\mu_{(j,B,P)} - \int_{N^{2T+1}} v_t d\mu_{(j,B_0,P)} +$$

$$+ \beta^{t+1} \mu_{(j,B,P)}\left(\{h,m\} \times N^{2(T-t)}\right) b^t_r(h,m)$$

$$\left\{\sum_{k \in N} V\left(k, {}_{(h,m,r)}B, P\right) p_{rk} - \sum_{k \in N} V\left(k, {}_{(h,m,i_m)}B_1, P\right) p_{i_m k}\right\} .$$

$$\int_{N^{2T+1}} v_t d\mu_{(j,B_0,P)} = \sum_{h_1 \in N^{2t}} \sum_{k \in N} \mu_{(j,B,P)}\left(\{h_1,k\} \times N^{2(T-t)}\right)$$

$$\sum_{i,\ell \in N} \beta^t (d_{ki} + c_{i\ell})^o b_i^t(h_1,k) p_{i\ell} \quad .$$

Hence $V(j,B,P) - V(j,B_0,P) =$

$$(5.4) \quad \beta^t \mu_{(j,B,P)}\left(\{h,m\} \times N^{2(T-t)}\right) b_r^t(h,m)$$

$$\left[\beta \sum_{\ell \in N} V\left(\ell, (h,m,r)^{B,P}\right) p_{r\ell} + d_{mr} + \sum_{\ell \in N} c_{r\ell} p_{r\ell} + \right.$$

$$\left. - \left\{\beta \sum_{\ell \in N} V\left(\ell, (h,m,i_m)^{B_1,P}\right) p_{i_m\ell} + d_{mi_m} + \sum_{\ell \in N} c_{i_m\ell} p_{i_m\ell}.\right\}\right] \quad .$$

$$V\left(\ell, (h,m,i_m)^{B_1,P}\right) \leq \left(\sum_{\tau=0}^{T-t-2} \beta^\tau\right) \max_{k \in N} \min_{i \in N} \left[d_{ki} + \max_{P \in \mathcal{P}_\sigma} \sum_{\ell \in N} c_{i\ell} p_{i\ell}\right] \quad .$$

Hence the form between square brackets in (5.4) exceeds $\beta \sum_{\ell \in N} V\left(\ell, (h,m,r)^{B,P}\right) p_{r\ell}$. If all d_{ki} and $c_{i\ell} > 0$ this means $B_0 \prec B$ when $\mu_{(j,B,P)}\left(\{h,m\} \times N^{2(T-t)}\right) > 0$ for some $j \in N$, $P \in \mathcal{P}_\sigma$, which delivers a contradiction. The difference of $V(j,B,P)$ and $V(j,B_0,P)$ does not alter when the same number is added to all d_{ki} and $c_{i\ell}$, which lifts the positiveness restriction on the cost-coefficients.

<u>Corollary 5.4</u>: $I \subset N^2$;

$$\forall_{(m,r) \in I} \left[d_{mr} + \min_{P \in \mathcal{P}_\sigma} \sum_{\ell \in N} c_{r\ell} p_{r\ell} > \right.$$

$$\left. > \left(\sum_{t=0}^{T-1} \beta^t\right) \max_{k \in N} \min_{i \in N} \left\{d_{ki} + \max_{P \in \mathcal{P}_\sigma} \sum_{\ell \in N} c_{i\ell} p_{i\ell}\right\}\right] \quad ;$$

 then $\left(\mathcal{R}_1\right)$ is essentially complete in $\mathcal{B}$, when

$$\mathcal{K}_1 := \{B \in \mathcal{K}_\sigma \mid \forall_{(m,r)\in I}\ \forall_{t(0\leq t<T)}\ \forall_{h\in N^{2t}}\ (b_r^t(h,m) = 0)\} \quad .$$

Proof: The proof of this corollary proceeds along the same lines as the proof of corollary 5.3: application of the construction method of the proof of theorem 5.4 in order to prove that

$$\mathcal{B}_t := \{B \in \mathcal{B} \mid \forall_{(m,r)\in I}\ \forall_{\tau(0\leq\tau<t)}\ \forall_{h\in N^{2\tau}}\ (b_r^\tau(h,m) = 0)\}$$

$$(0 \leq t \leq T)$$

is essentially complete in $\mathcal{B}$; and application of the proof of theorem 4.2 and corollary 4.2 in order to prove that $(\overline{\mathcal{K}_1})$ is essentially complete in $\mathcal{B}_T$.

OPTIMAL DECISION RULES: MIN-MAX RISK AND MIN-MAX REGRET

** Generally, a partially ordered set does not contain an element which is at least as good as all other elements of the set. The examples of the appendix demonstrate that, in the situation considered, a decision rule which is at least as good as all others may be absent.

The trouble is caused by the fact that the risk function $V(j,B,P)$ for fixed j and B depends on P, the (partially) unknown matrix of transition probabilities of the basic Markov chain. A possible way out is presented by the introduction of a new criterion function, which does not depend on P. Examples of such a new criterion function are

$$\sup_{P \in \mathcal{P}_\sigma} V(j,B,P)$$

and

$$\sup_{P \in \mathcal{P}_\sigma} [V(j,B,P) - \inf_{B \in \mathcal{B}} V(j,B,P)] \quad .$$

Another example is obtained by introducing a weight function on $\mathcal{P}_\sigma$ and computing the average of $V(j,B,P)$ with respect to that weight function

These examples lead to min-max risk, min-max regret and Bayes' theory respectively. In this section some implications of both min-max procedures will be investigated. The Bayesian approach is postponed to the next section.

<u>Lemma 6.1</u>:

a) $\forall_{B \in \mathcal{B}} \forall_{j \in N} \exists_{P_0 \in \mathcal{P}_\sigma} [V(j,B,P_0) = \sup\{V(j,B,P) \mid P \in \mathcal{P}_\sigma\}]$;

b) $\mathcal{B}_0 \subset \mathcal{B}$, $j \in N$, ${}_j\mathcal{B}_0$ is closed, then:

$$\forall_{P \in \mathcal{P}_\sigma} \exists_{B_0 \in \mathcal{B}_0} [V(j,B_0,P) = \inf\{V(j,B,P) \mid B \in \mathcal{B}_0\}] \quad .$$

Proof: a) For any $j \in N$, $B \in \mathcal{B}$ $V(j,B,\cdot)$ maps $\mathcal{P}_\sigma$ into R continuously (lemma 2.9) furthermore $\mathcal{P}_\sigma$ is compact (lemma 4.7a).

b) Let $\{B_\ell\}_{\ell=1}^\infty$ with $B_\ell \in \mathcal{B}_0$ be a sequence defining the infimum:

$$\lim_{\ell \to \infty} V(j,B_\ell,P) = \inf\{V(j,B,P) \mid B \in \mathcal{B}_0\} \quad .$$

${}_j\mathcal{B}_0$ is compact, hence a desicion rule $B_0 \in \mathcal{B}_0$ exists, such that ${}_j\mathcal{B}_0$ is the limit of a subsequence $\{{}_jB_{\ell_r}\}_{r=1}^\infty$.

Lemma 2.17 states: $\lim_{r \to \infty} V(j,B_{\ell_r},P) = V(j,B_0,P)$.

Definition 6.1: a) $\forall_{j \in N} \forall_{B \in \mathcal{B}} [\overline{V}(j,B) := \max_{P \in \mathcal{P}_\sigma} V(j,B,P)]$;

b) $\forall_{j \in N} \forall_{\mathcal{B}_0 \subset \mathcal{B}} \forall_{P \in \mathcal{P}_\sigma}$

$$\left[{}_j\mathcal{B}_0 \text{ closed} \Rightarrow \underline{V}_{\mathcal{B}_0}(j,P) := \min_{B \in \mathcal{B}_0} V(j,B,P)\right] \quad .$$

Lemma 6.2: $j \in N$; $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$; ${}_j\mathcal{B}_1$, ${}_j\mathcal{B}_2$ closed, then each of the conditions:

$${}_j\mathcal{B}_1, {}_j\mathcal{B}_2 \supset ({}_j\mathcal{R}_\sigma)$$

$${}_j\mathcal{B}_1, {}_j\mathcal{B}_2 \supset ({}_j\mathcal{L}_\sigma) \quad , \quad \forall_{k,i \in N} [d_{ki} = d_{11}]$$

implies

$$\forall_{P \in \mathcal{P}_\sigma} [\underline{V}_{\mathcal{B}_1}(j,P) = \underline{V}_{\mathcal{B}_2}(j,P)] \quad .$$

<u>Proof</u>: The sufficiency of the first condition is implied by theorem 5.1a); item b) of that theorem implies the sufficiency of the second condition.

<u>Lemma 6.3</u>: $j \in N$, $\mathcal{B}_0 \subset \mathcal{B}$, ${}_j\mathcal{B}_0$ is closed, then:

a) $\overline{V}(j,\cdot)$ maps $\mathcal{B}$ into R continuously;

even: $\lim_{\ell\to\infty} {}_jB_\ell = {}_jB_0$ for $B_\ell \in \mathcal{B}$ $(\ell = 0,1,\ldots)$ implies:

$\lim_{\ell\to\infty} \overline{V}(j,B_\ell) = \overline{V}(j,B_0)$;

b) $\underline{V}_{\mathcal{B}_0}(j,\cdot)$ maps $\mathcal{P}_\sigma$ into R continuously;

c) $$\forall_{B\in\mathcal{B}}\ \exists_{P_0\in\mathcal{P}_\sigma}[V(j,B,P_0) - \underline{V}_{\mathcal{B}_0}(j,P_0) = \sup\{V(j,B,P) - \underline{V}_{\mathcal{B}_0}(j,P) \mid P \in \mathcal{P}_\sigma\}] .$$

<u>Proof</u>: a) Let $\lim_{\ell\to\infty} B_\ell = B_0$, then $\lim_{\ell\to\infty} {}_jB_\ell = {}_jB_0$ (lemma 2.15).

Hence it suffices to prove that

$B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\ldots)$ and $\lim_{\ell\to\infty} {}_jB_\ell = {}_jB_0$ imply

$$\lim_{\ell\to\infty} \overline{V}(j,B_\ell) = \overline{V}(j,B_0) .$$

$$|\overline{V}(j,B_\ell) - \overline{V}(j,B_0)| \leq \sup_{P\in\mathcal{P}_\sigma} |V(j,B_\ell,P) - V(j,B_0,P)| .$$

According to lemma 2.13, this expression possesses a value less than any selected $\varepsilon > 0$, provided that ℓ is sufficiently large.

b) Let $P_\ell \in \mathcal{P}_\sigma$ $(\ell = 0,1,2,\ldots)$ and $\lim_{\ell\to\infty} P_\ell = P_0$.

$$|\underline{V}_{\mathcal{B}_0}(j,P_\ell) - \underline{V}_{\mathcal{B}_0}(j,P_0)| \leq \sup_{B\in\mathcal{B}} |V(j,B,P_\ell) - V(j,B,P_0)| .$$

According to lemma 2.9, this expression possesses a value less than any selected $\varepsilon > 0$, provided that ℓ is sufficiently large.

c) For any $B \in \mathcal{B}$ $V(j,B,\cdot) - \underline{V}_{\mathcal{B}_0}(j,\cdot)$ maps $\mathcal{P}_\sigma$ into R continuously (lemma 2.9 and assertion b) of this lemma), furthermore: $\mathcal{P}_\sigma$ is compact (lemma 4.7a)).

Definition 6.2: $j \in N$, $\mathcal{B}_0 \subset \mathcal{B}$, ${}_j\mathcal{B}_0$ closed, then:

$$\forall_{B \in \mathcal{B}} \left[\overline{V}^{\mathcal{B}_0}(j,B) := \max_{P \in \mathcal{P}_\sigma} \left(V(j,B,P) - \underline{V}_{\mathcal{B}_0}(j,P) \right) \right] .$$

** $\overline{V}(j,B)$ denotes the maximum risk for the decision rule B: the risk value for the most unfavourable $P \in \mathcal{P}_\sigma$ with regard to the decision rule B.
$V(j,B,P) - \underline{V}_{\mathcal{B}}(j,P)$ is the "regret" function: the difference of the actual risk and the minimum risk for that P. $\overline{V}^{\mathcal{B}}(j,B)$ denotes the maximum regret for the decision rule B.

Lemma 6.4: $j \in N$, $\mathcal{B}_0 \subset \mathcal{B}$, ${}_j\mathcal{B}_0$ is closed, then:

$\overline{V}^{\mathcal{B}_0}(j,\cdot)$ maps $\mathcal{B}$ into R continuously;

even: $\lim_{\ell \to \infty} {}_jB_\ell = {}_jB_0$ for $B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\ldots)$ implies:

$$\lim_{\ell \to \infty} \overline{V}^{\mathcal{B}_0}(j,B_\ell) = \overline{V}^{\mathcal{B}_0}(j,B_0) .$$

Proof: As in the proof of lemma 6.3a), it suffices to prove that $B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\ldots)$ and $\lim_{\ell \to \infty} {}_jB_\ell = {}_jB_0$ imply

$$\lim_{\ell \to \infty} \overline{V}^{\mathcal{B}_0}(j,B_\ell) = \overline{V}^{\mathcal{B}_0}(j,B_0) .$$

$$|\overline{V}^{\mathcal{B}_0}(j,B_\ell) - \overline{V}^{\mathcal{B}_0}(j,B_0)| \leq \sup_{P \in \mathcal{P}_\sigma} |V(j,B_\ell,P) - \underline{V}_{\mathcal{B}_0}(j,P) -$$

$$- V(j,B_0,P) + \underline{V}_{\mathcal{B}_0}(j,P)| = \sup_{P \in \mathcal{P}_\sigma} |V(j,B_\ell,P) - V(j,B_0,P)| .$$

Hence - as in the proof of lemma 6.3a) - lemma 2.13 implies the assertion.

<u>Definition 6.3</u>: $B_1, B_2 \in \mathcal{B}$, $\mathcal{B}_0 \subset \mathcal{B}$, then:

a) $(j \in N)$ $\quad {}_jB_1 \overline{\leqslant} {}_jB_2 \quad :\Longleftrightarrow \overline{V}(j,B_1) \leqslant \overline{V}(j,B_2)$;

b) $\quad B_1 \overline{\leqslant} B_2 \quad :\Longleftrightarrow \forall_{j \in N}({}_jB_1 \overline{\leqslant} {}_jB_2)$;

c) $(j \in N,\ {}_j\mathcal{B}_0$ is closed)

$$ {}_jB_1 \overline{\leqslant}^{\mathcal{B}_0} {}_jB_2 :\Longleftrightarrow \overline{V}^{\mathcal{B}_0}(j,B_1) \leqslant \overline{V}^{\mathcal{B}_0}(j,B_2) ; $$

d) $({}_j\mathcal{B}_0$ is closed for all $j \in N)$

$$ B_1 \overline{\leqslant}^{\mathcal{B}_0} B_2 :\Longleftrightarrow \forall_{j \in N}({}_jB_1 \overline{\leqslant}^{\mathcal{B}_0} {}_jB_2) . $$

** The $\overline{\leqslant}$- and $\overline{\leqslant}^{\mathcal{B}_0}$ - concepts define relations in the sets ${}_j\mathcal{B}$, since all $B \in \mathcal{B}$ with ${}_jB = {}_jB_1$ possess the same $V(j,B,P)$ for each $P \in \mathcal{P}$ (lemma 2.8) and each ${}_jB$ is the j-restriction of at least one decision rule (lemma 2.1b)).

<u>Lemma 6.5</u>: $\mathcal{B}_0 \subset \mathcal{B}$, then:

a) the relation $\overline{\leqslant}$ defines a weak ordering in ${}_j\mathcal{B}$ $(j \in N)$;

b) the relation $\overline{\leqslant}$ defines a partial, weak ordering in $\mathcal{B}$;

c) $(j \in N,\ {}_j\mathcal{B}_0$ closed) the relation $\overline{\leqslant}^{\mathcal{B}_0}$ defines a weak ordering in ${}_j\mathcal{B}$;

d) $({}_j\mathcal{B}_0$ is closed for all $j \in N)$ the relation $\overline{\leqslant}^{\mathcal{B}_0}$ defines a partial, weak ordering in $\mathcal{B}$;

e) $(j \in N,\ {}_j\mathcal{B}_0$ closed$)$

$$\forall_{B_1,B_2 \in \mathcal{B}} [{}_jB_1 \lessgtr {}_jB_2 \Rightarrow ({}_jB_1 \overline{\lessgtr}\ {}_jB_2 \text{ and } {}_jB_1 \overline{\lessgtr}^{\mathcal{B}_0}\ {}_jB_2)] \quad ;$$

f) $({}_j\mathcal{B}_0$ is closed for all $j \in N)$

$$\forall_{B_1,B_2 \in \mathcal{B}} [B_1 \lessgtr B_2 \Rightarrow (B_1 \overline{\lessgtr} B_2 \text{ and } B_1 \overline{\lessgtr}^{\mathcal{B}_0} B_2)] \quad .$$

Remark: ${}_jB_1 \overline{\lessgtr}\ {}_jB_2$ does not necessarily imply ${}_jB_1 \overline{\lessgtr}^{\mathcal{B}}\ {}_jB_2$ $(j \in N,$ $B_1,\ B_2 \in \mathcal{B})$ see example A.1b) in the appendix.

Lemma 6.6: $\mathcal{B}_1,\ \mathcal{B}_2 \subset \mathcal{B}$, then:

a) $(j \in N,\ {}_j\mathcal{B}_1$ and ${}_j\mathcal{B}_2$ are closed$)$ each of the conditions

$${}_j\mathcal{B}_1,\ {}_j\mathcal{B}_2 \supset \left({}_j\mathcal{K}_\sigma\right)$$

$${}_j\mathcal{B}_1,\ {}_j\mathcal{B}_2 \supset \left({}_j\mathcal{L}_\sigma\right),\ \forall_{k,i \in N}(d_{ki} = d_{11})$$

implies

$$\forall_{B_1,B_2 \in \mathcal{B}} [{}_jB_1 \overline{\lessgtr}^{\mathcal{B}_1}\ {}_jB_2 \Longleftrightarrow {}_jB_1 \overline{\lessgtr}^{\mathcal{B}_2}\ {}_jB_2] \quad ;$$

b) $({}_j\mathcal{B}_1$ and ${}_j\mathcal{B}_2$ are closed for each $j \in N)$ each of the conditions

$$\mathcal{B}_1,\ \mathcal{B}_2 \supset \left(\overline{\mathcal{K}}_\sigma\right)$$

$$\mathcal{B}_1,\ \mathcal{B}_2 \supset \left(\mathcal{L}_\sigma\right),\ \forall_{k,i \in N}(d_{ki} = d_{11})$$

implies

$$\forall_{B_1,B_2 \in \mathcal{B}} [B_1 \overline{\lessgtr}^{\mathcal{B}_1} B_2 \Longleftrightarrow B_1 \overline{\lessgtr}^{\mathcal{B}_2} B_2] \quad .$$

Proof: Assertion a) is a consequence of lemma 6.2.

b) $B_1 \supset \left(\overline{\mathcal{K}}_\sigma\right)$ implies ${}_j\mathcal{B}_1 \supset \left({}_j\mathcal{K}_\sigma\right)$ $(j \in N)$ (lemma 5.3);

$\mathcal{B}_1 \supset (\mathcal{L}_\sigma)$ implies ${}_j\mathcal{B}_1 \supset ({}_j\mathcal{L}_\sigma)$ $(j \in N)$ (lemma 5.4e)); hence b) is implied by definition 6.3d) and assertion a) of this lemma.

<u>Lemma 6.7</u>: $j \in N$, $\mathcal{B}_0 \subset \mathcal{B}$, ${}_j\mathcal{B}_0$ is closed, then:

a) $\exists_{{}_jB_0 \in ({}_j\mathcal{B}_0)} \forall_{B \in \mathcal{B}_0} [{}_jB_0 \lesseqgtr {}_jB]$;

b) $\exists_{{}_jB_0 \in ({}_j\mathcal{B}_0)} \forall_{B \in \mathcal{B}_0} [{}_jB_0 \lesseqgtr^{\mathcal{B}_0} {}_jB]$;

c) $\exists_{P_0 \in \mathcal{P}_\sigma} \exists_{B_0 \in \mathcal{B}_0} [V(j,B_0,P_0) = \sup\{\underline{V}_{\mathcal{B}_0}(j,P) \mid P \in \mathcal{P}_\sigma\}]$.

<u>Proof</u>: a) Let $\{B_\ell\}_{\ell=1}^\infty$ with $B_\ell \in \mathcal{B}_0$ be a sequence with $\lim_{\ell\to\infty} \overline{V}(j,B_\ell) = \inf\{\overline{V}(j,B) \mid B \in \mathcal{B}_\sigma\}$. Then a subsequence $\{B_{\ell_r}\}_{r=1}^\infty$ and a decision rule $B_0 \in \mathcal{B}_0$ exist with $\lim_{r\to\infty} {}_jB_{\ell_r} = {}_jB_0$ (lemma 2.16). Hence

$$\lim_{\ell\to\infty} \overline{V}(j,B_{\ell_r}) = \overline{V}(j,B_0) \qquad \text{(lemma 6.3a))} \quad .$$

The completeness of $({}_j\mathcal{B}_0)$ in ${}_j\mathcal{B}_0$ (lemma 5.8) and lemma 6.5e) imply that B_0 may be selected such that ${}_jB_0 \in ({}_j\mathcal{B}_0)$.

b) If $B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\dots)$ and $\lim_{\ell\to\infty} {}_jB_\ell = {}_jB_0$, then (lemma 6.4) $\lim_{\ell\to\infty} \overline{V}^{\mathcal{B}_0}(j,B_\ell) = \overline{V}^{\mathcal{B}_0}(j,B_0)$; hence the same argument as in part a) implies the assertion.

c) The continuity of the mapping $\underline{V}_{\mathcal{B}_0}(j,\cdot)$ from $\mathcal{P}_\sigma$ into R (lemma 6.3b)) and the compactness of $\mathcal{P}_\sigma$ (lemma 4.7a)) imply the assertion.

Lemma 6.8: $\mathcal{B}_0 \subset \mathcal{B}$, $\overline{\mathcal{B}_0} = \mathcal{B}_0$, $\mathcal{B}_0$ is closed, then:

a) $\exists_{B_0} \in \left(\mathcal{B}_0\right) \forall_{B \in \mathcal{B}_0} [B_0 \overline{\lesssim} B]$;

b) $\exists_{B_0} \in \left(\mathcal{B}_0\right) \forall_{B \in \mathcal{B}_0} [B_0 \overline{\lesssim}^{\mathcal{B}_0} B]$.

Proof: Lemma 6.7a)(and b) respectively) implies the existence of ${}_jB_0 \in \left({}_j\mathcal{B}_0\right)$, such that ${}_jB_0 \overline{\lesssim} {}_jB$ (or ${}_jB_0 \overline{\lesssim}^{\mathcal{B}_0} {}_jB$) ($j \in N$), since ${}_j\mathcal{B}_0$ is closed (lemma 2.16). $B_0 \in \left(\mathcal{B}_0\right)$ (lemma 5.3), hence $B_0 \overline{\lesssim} B$ (or $B_0 \overline{\lesssim}^{\mathcal{B}_0} B$ respectively).

** Lemmas 6.7 and 6.8 imply the existence of min-max risk and min-max regret decision rules, for the special subsets of $\mathcal{B}$ which were introduced in earlier sections, since they all have j-restrictions which are closed. They even imply the existence of admissible min-max (j-) decision rules. Combined with results from sections 4 and 5 the assertions of theorem 6.1 are obtained.

Theorem 6.1: a) ($j \in N$)

$$\exists_{{}_jB_0} \in \left({}_j\mathcal{K}_\sigma\right) \forall_{B \in \mathcal{B}} [{}_jB_0 \overline{\lesssim} {}_jB] \quad ;$$

and

$$\exists_{{}_jB_1} \in \left({}_j\mathcal{K}_\sigma\right) \forall_{B \in \mathcal{B}} [{}_jB_1 \overline{\lesssim}^{\mathcal{B}} {}_jB] \quad ;$$

b)
$$\exists_{B_0} \in \left(\overline{\mathcal{K}_\sigma}\right) \forall_{B \in \mathcal{B}} [B_0 \overline{\lesssim} B] \quad ;$$

and

$$\exists_{B_1} \in \left(\overline{\mathcal{K}_\sigma}\right) \forall_{B \in \mathcal{B}} [B_1 \overline{\lesssim}^{\mathcal{B}} B] \quad ;$$

c) when $d_{ki} = d_{11}$ for all $k, i \in N$, then ${}_j\mathcal{K}_\sigma$ in assertion a) may be replaced by ${}_j\mathcal{L}_\sigma$ and $\overline{\mathcal{K}_\sigma}$ in assertion b) may be replaced by $\mathcal{L}_\sigma$.

<u>Proof</u>: a) ${}_j\mathcal{K}_\sigma$ is closed (lemma 4.8), hence (lemma 6.7a)):

$$\exists_{{}_jB_0 \in ({}_j\mathcal{K}_\sigma)} \forall_{B \in \mathcal{K}_\sigma} [{}_jB_0 \,\bar{\lesseqgtr}\, {}_jB] \quad .$$

Then theorem 4.2 implies $\forall_{B \in \mathcal{B}} [{}_jB_0 \,\bar{\lesseqgtr}\, {}_jB]$ (lemma 6.5a)); lemma 6.7b) implies:

$$\exists_{{}_jB_0 \in ({}_j\mathcal{K}_\sigma)} \forall_{B \in \mathcal{K}_\sigma} [{}_jB_0 \,\bar{\lesseqgtr}^{\mathcal{K}_\sigma}\, {}_jB] \quad .$$

Then lemma 6.6a) and theorem 4.2 imply: $\forall_{B \in \mathcal{B}} [{}_jB_0 \,\bar{\lesseqgtr}^{\mathcal{B}}\, {}_jB]$ (lemma 6.5e)).

b) As the proof of assertion a): applying lemma 6.8, corollary 4.2, lemma 6.6b) and lemma 6.5f) instead of lemma 6.7, theorem 4.2 and lemmas 6.6a), 6.5e).

c) The first part proceeds as the proof of assertion a), with theorem 4.3 instead of theorem 4.2.

The second part proceeds as the proof of assertion b), with corollary 4.3 instead of 4.2; the proof is completed by the observation: $V(j_1,B,P) = V(j_2,B,P)$ for all $P \in \mathcal{P}_\sigma$, j_1, $j_2 \in N$ and $B \in \mathcal{L}_\sigma$ (lemma 4.12).

** One easily verifies that - if decision costs are equal - the decision rule which always prescribes action $\cdot \to s_{i_0}$ with $i = i_0$ minimizing $\max_{P \in \mathcal{P}_\sigma} \sum_{\ell=1}^{n} c_{i\ell} p_{i\ell}$ has min-max risk. Some remarks will be made on a criterion for the discrimination between decision rules, which is related to the min-max regret criterion: the "min-max regret somewhere minimum regret" as introduced in [1966, W. Schaafsma, L.J. Smid]. This criterion in fact consists of the selection of a special set $\mathcal{B}_0$ for the criterion $\overline{V}^{\mathcal{B}_0}(j,B)$.

<u>Lemma 6.9</u>: $j \in N$, $\mathcal{B}_o \subset \mathcal{B}$, ${}_j\mathcal{B}_o$ closed,

$\mathcal{B}_j := \{B \in \mathcal{B}_o \mid \exists_{P\in \mathcal{P}_\sigma} [\, V(j,B,P) = \underline{V}_{\mathcal{B}_o}(j,P)]\}$, then:

$$ {}_j\mathcal{B}_j \text{ is closed;} $$

$$ \overline{V}^{\mathcal{B}_o}(j,B) = \overline{V}^{\mathcal{B}_j}(j,B) \quad \text{for each } B \in \mathcal{B} \, ; $$

$$ \left({}_j\mathcal{B}_j\right) \subset \left({}_j\mathcal{B}_o\right) . $$

<u>Proof</u>: Let $B_\ell \in \mathcal{B}_j$, $P_\ell \in \mathcal{P}_\sigma$ $(\ell = 1,2,\ldots)$ and $V(j, B_\ell, P_\ell) = \underline{V}_{\mathcal{B}_o}(j,P_\ell)$. There exist $B_o \in \mathcal{B}_o$, $P_o \in \mathcal{P}_\sigma$ and subsequences $\{B_{\ell_r}\}_{r=1}^{\infty}$ and $\{P_{\ell_r}\}_{r=1}^{\infty}$, with $\lim_{r\to\infty} {}_jB_{\ell_r} = {}_jB_o$ (lemma 2.16) and $\lim_{r\to\infty} P_{\ell_r} = P_o$ (lemma 4.7a)). Hence

$$ \underline{V}_{\mathcal{B}_o}(j,P_o) = \lim_{r\to\infty} \underline{V}_{\mathcal{B}_o}(j,P_{\ell_r}) = \lim_{r\to\infty} V(j,B_{\ell_r},P_{\ell_r}) = V(j,B_o,P_o) $$

(first equality: lemma 6.3b), third equality is proved as lemma 2.14). This implies: $B_o \in \mathcal{B}_j$ and ${}_j\mathcal{B}_j$ is closed.

$\underline{V}_{\mathcal{B}_j}(j,P) = \underline{V}_{\mathcal{B}_o}(j,P)$ for all $P \in \mathcal{P}_\sigma$ (lemma 6.1b), definition 6.1b)), hence $\overline{V}^{\mathcal{B}_o}(j,B) = \overline{V}^{\mathcal{B}_j}(j,B)$ for any $B \in \mathcal{B}$ (definition 6.2).

Let $B \in \mathcal{B}_j$, $V(j,B,P_o) = \underline{V}_{\mathcal{B}_o}(j,P_o)$ and ${}_jB \in \left({}_j\mathcal{B}_j\right)$;

let $B_o \in \mathcal{B}_o$ with ${}_jB_o \prec {}_jB$,

then $V(j,B_o,P_o) \leq V(j,B,P_o) = \underline{V}_{\mathcal{B}_o}(j,P_o)$. Hence $B_o \in \mathcal{B}_j$, which is contradictory; hence $\left({}_j\mathcal{B}_j\right) \subset \left({}_j\mathcal{B}_o\right)$.

<u>Theorem 6.2:</u> $j \in N$, $\mathcal{B}_0 \subset \mathcal{B}$, ${}_j\mathcal{B}_0$ closed,

$\mathcal{B}_j := \{B \in \mathcal{B}_0 \mid \exists_{P \in \mathcal{P}_\sigma} [V(j,B,P) = \underline{V}_{\mathcal{B}_0}(j,P)]\}$, then

$$\exists_{B_0 \in \mathcal{B}_j} [\, {}_jB_0 \in \left({}_j\mathcal{B}_j\right) \text{ and } \forall_{B \in \mathcal{B}_j} ({}_jB_0 \overline{\lesssim}^{\mathcal{B}_0} {}_jB)] \quad .$$

<u>Proof:</u> Apply lemmas 6.7b) and 6.9.

<u>Lemma 6.10:</u> $\forall_{k,i \in N}(d_{ki} = d_{11})$; for no ℓ, $m \in N$:

$\forall_{P \in \mathcal{P}_\sigma} [\sum_{i=1}^{n} c_{\ell i} p_{\ell i} > \sum_{i=1}^{n} c_{mi} p_{mi}$; j, $\mathcal{B}_0$, $\mathcal{B}_j$ as in theorem 6.2, then

$$\mathcal{B}_j = \mathcal{B}_0 \quad .$$

<u>Proof:</u> There exists a subset $\mathcal{P}_1 \subset \mathcal{P}_\sigma$, such that for $P_1 \in \mathcal{P}_1$ $V(j,B,P_1)$ does not vary with B. Hence for any $B \in \mathcal{B}_0$: $V(j,B,P_1) = \underline{V}_{\mathcal{B}_0}(j,P_1)$ (lemma 4.13).

** Lemma 6.10 states that, in the important case of equal decision costs, the "min-max regret" procedure and the "min-max regret somewhere minimum regret" procedure coincide.

It should be emphasized, that the existence of min-max risk and min-max regret j-decision rules for certain $j \in N$ and $\mathcal{B}_0 \subset \mathcal{B}$ — that means j-decision rules in ${}_j\mathcal{B}_0$, which are best in $\overline{\lesssim}$ - or $\overline{\lesssim}^{\mathcal{B}_0}$ - sense — does not necessarily imply the following properties which generally are well-known for risk functions:

$$(6.1) \qquad \min_{B \in \mathcal{B}_0} \max_{P \in \mathcal{P}_\sigma} V(j,B,P) = \max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}_0} V(j,B,P) \quad ,$$

(6.2) $$\min_{B\in\mathcal{B}_0}\ \max_{P\in\mathcal{P}_\sigma}\Big(V(j,B,P) - \underline{V}_{\mathcal{B}_0}(j,P)\Big) = $$
$$= \max_{P\in\mathcal{P}_\sigma}\ \min_{B\in\mathcal{B}_0}\Big(V(j,B,P) - \underline{V}_{\mathcal{B}_0}(j,P)\Big) \quad .$$

Theorem 6.3: $j \in N$, $\mathcal{B}_0 \subset \mathcal{B}$, ${}_j\mathcal{B}_0$ is closed, then: assertion (6.2) is equivalent to $\exists_{B_0\in\mathcal{B}_0}\ \forall_{B\in\mathcal{B}_0}\ [\,{}_jB_0 \preccurlyeq {}_jB\,]$.

Proof: $$\max_{P\in\mathcal{P}_\sigma}\ \min_{B\in\mathcal{B}_0}\Big(V(j,B,P) - \underline{V}_{\mathcal{B}_0}(j,P)\Big) =$$
$$= \max_{P\in\mathcal{P}_\sigma}\Big(\underline{V}_{\mathcal{B}_0}(j,P) - \underline{V}_{\mathcal{B}_0}(j,P)\Big) = 0 \ .$$

$V(j,B,P) \geq \underline{V}_{\mathcal{B}_0}(j,P)$ when $B \in \mathcal{B}_0$. A decision rule $B_0 \in \mathcal{B}_0$ exists with $$\max_{P\in\mathcal{P}_\sigma}\Big(V(j,B_0,P) - \underline{V}_{\mathcal{B}_0}(j,P)\Big) =$$
$$= \min_{B\in\mathcal{B}_0}\ \max_{P\in\mathcal{P}_\sigma}\Big(V(j,B,P) - \underline{V}_{\mathcal{B}_0}(j,P)\Big)$$
(lemma 6.7b)). Hence $V(j,B_0,P) = \underline{V}_{\mathcal{B}_0}(j,P)$ (all $P \in \mathcal{P}_\sigma$) for certain $B_0 \in \mathcal{B}_0$ if and only if (6.2) holds. On the other hand: for such a decision rule B_0 holds ${}_jB_0 \preccurlyeq {}_jB$ ($B \in \mathcal{B}_0$) and reversely.

** Theorem 6.3 proves that the min-max property (6.2) for the regret function only holds for some non-interesting cases. For the risk function, the property (6.1) does also fail to hold generally. For an example not satisfying (6.1) the reader is referred to example A.3 in the appendix. However, in a substantial class of problems the property does hold. A necessary and sufficient condition is presented in lemma 6.12; a situation satisfying this condition in theorem 6.4.

Lemma 6.11: For any $j \in N$, there exists a finite set of decision rules applying pure strategies - say $\mathcal{B}_L := \{B_\ell\}_{\ell=1}^{L} \subset \mathcal{A}$ - such that ${}^{\ell}b^{t}(h,k) = {}^{\ell}b^{t}(h',k)$ for all ℓ $(1 \leq \ell \leq L)$, t $(0 \leq t < T)$, h and $h' \in N^{2t}$, $k \in N$; and $\forall_{B \in \mathcal{B}} \forall_{P \in \mathcal{P}_\sigma} [V(j,B,P) \geq \underline{V}_{\mathcal{B}_L}(j,P)]$;

specially may be demanded:

a) if $T = \infty$: ${}^{\ell}b^{t}(h,k) = {}^{\ell}b^{\tau}(h',k)$ for all ℓ $(1 \leq \ell \leq L)$, t and τ $(0 \leq t, \tau < T)$, $k \in N$, $h \in N^{2t}$, $h' \in N^{2\tau}$;

b) if $\forall_{k,i \in N}(d_{ki} = d_{11})$: $L = n$ and B_ℓ $(\ell = 1,\ldots,n)$ defined by ${}^{\ell}b_{\ell}^{t}(h) = 1$ for all t $(0 \leq t \leq T)$ and $h \in N^{2t+1}$.

Proof: The assertions are implied directly by some results, which will not be proved explicitly, since they are special cases of more general theorems, which will be proved in section 7.
The general result for the case $T < \infty$ is a consequence of corollary 7.5: a decision rule, which is best for $P_0 \in \mathcal{P}_\sigma$ is found by considering a new set $\mathcal{P}_\sigma := \{P_0\}$; then the corresponding set $\mathcal{K}_\sigma \cap \mathcal{A}$ is finite and independent of P_0; in fact it is the set $\{B \in \mathcal{A} \mid b^{t}(h,k) = b^{t}(h',k)$ for all t $(0 \leq t < T)$, $h, h' \in N^{2t}$, $k \in N\}$.
For $T = \infty$, the result is implied by corollary 7.7: the subset of stationnary decision rules of the just mentioned set is finite.
For equal decision costs, the result is implied by theorem 7.8.

** In fact, lemma 6.11 is a consequence of some well-known results on Markovian decision processes: if P is known, then a decision rule applying pure strategies with decision vectors only depending on time and state of decision, is optimal. If $T = \infty$ even the dependence on time may be skipped. See e.g. [1962, 1965, D. Blackwell]. In section 7 these results will be generalized to the case of a known prior distribution for the Markov transition matrix of the basic Markov chain.

Lemma 6.12: $(j \in N)$. Let $\mathcal{B}_L$ be a set of decision rules as implied by lemma 6.11. $\mathcal{B}_0 \subset \mathcal{B}$, ${}_j\mathcal{B}_0$ is closed, $\mathcal{B}_L \subset \mathcal{B}_0$, then assertion (6.1) is equivalent with:

$$\exists_{B_0 \in \mathcal{B}_0} [\overline{V}(j,B_0) = \max_{P \in \mathcal{P}_\sigma} \underline{V}_{\mathcal{B}_L}(j,P)] \quad .$$

Proof: $\underline{V}_{\mathcal{B}_0}(j,P) = \underline{V}_{\mathcal{B}_L}(j,P)$ for each $P \in \mathcal{P}_\sigma$, hence

$$\max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}_0} V(j,B,P) = \max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}_L} V(j,B,P) \quad .$$

Say (6.1) is true: a decision rule $B_0 \in \mathcal{B}_0$ exists with

$$\max_{P \in \mathcal{P}_\sigma} V(j,B_0,P) = \overline{V}(j,B_0) = \min_{B \in \mathcal{B}_0} \max_{P \in \mathcal{P}_\sigma} V(j,B,P) =$$
$$= \max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}_0} V(j,B,P) = \max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}_L} V(j,B,P) =$$
$$= \max_{P \in \mathcal{P}_\sigma} \underline{V}_{\mathcal{B}_L}(j,P) \quad .$$

Reversely: generally

$$\max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}_0} V(j,B,P) \leq \min_{B \in \mathcal{B}_0} \max_{P \in \mathcal{P}_\sigma} V(j,B,P) ;$$

hence the existence of a decision rule $B_0 \in \mathcal{B}_0$ with

$$\max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}_0} V(j,B,P) = \max_{P \in \mathcal{P}_\sigma} V(j,B_0,P)$$

implies (6.1).

Theorem 6.4: $\forall_{k,i \in N}(d_{ki} = d_{11})$, $\mathcal{B}_L$ is the special set implied by lemma 6.11b), $\mathcal{B}_L \subset \mathcal{B}_0 \subset \mathcal{B}$, then for any $j \in N$, such that ${}_j\mathcal{B}_0$ is closed (6.1) is true.

<u>Proof</u>: $V(j,B_\ell,P) = \left(\sum_{t=0}^{T-1} \beta^t\right) \sum_{i=1}^{n} c_{\ell i} p_{\ell i} \quad (1 \leqslant \ell \leqslant n = L,\ P \in \mathcal{P}_\sigma)$.

Apply lemma 6.12 with B_0 equal to the B_ℓ with minimal

$$\max_{P \in \mathcal{P}_\sigma} \sum_{i=1}^{n} c_{\ell i} p_{\ell i} \quad .$$

OPTIMAL DECISION RULES: THE BAYESIAN APPROACH

** The other possibility to induce an ordering in ${}_j\mathcal{B}$, which has been noticed at the beginning of the preceding section, proceeds by the introduction of a weight function on $\mathcal{P}_\sigma$ and averaging V(j,B,P) with respect to this weight function. Such a weight function may be interpreted as a probability distribution on $\mathcal{P}_\sigma$. According to that interpretation one assumes, that before the initiation of the decision process an element P is drawn at random from the set $\mathcal{P}_\sigma$ applying the given probability distribution (prior distribution). One assumes, this specific P - the surveyor of the process does not know which - to be the transition matrix of the underlying Markov chain. The introduction of the concept in question, however, proceeds quite formally by the postulation of a weight function. The properties joined with the probability interpretation will appear gradually as a part of the investigation. The introduction of the weight functions resembles the introduction of mixed decision rules in section 3. This resemblance is also found in the game interpretation of the problem: a prior distribution is a mixed strategy for Nature, which is acting as the opponent of the surveyor.

In order to define a weight function (a normed measure) on $\mathcal{P}$, one needs a σ-algebra of subsets of $\mathcal{P}$. One gets a σ - algebra, which provides an abundance of possibilities for the definition of measures, by interpreting P as a n^2 - vector and introducing the common σ - algebras of Borel sets in R^{n^2}.

Definition 7.1: $\overline{\Phi}_n$ is the σ-algebra of subsets of $\mathcal{P}$, defined by

$$\forall \mathcal{P}_1 \subset \mathcal{P}\left[\mathcal{P}_1 \in \overline{\Phi}_n :\Longleftrightarrow \left\{x \in \mathcal{V}^n \,\middle|\, \exists_{P \in \mathcal{P}_1} \, \forall_{k,i \in N}\left(x_{(k-1)n+i} = p_{ki}\right)\right\} \in \Phi_n\right].$$

Definition 7.2: $\mathcal{F}$ is the set of all normed measures on $(\mathcal{P}, \overline{\Phi}_n)$; elements from $\mathcal{F}$ will be denoted by F, possibly indexed: F_0, F_r; $\mathcal{F}_\sigma$ is the subset of $\mathcal{F}$, containing those $F \in \mathcal{F}$ which satisfy

$$F(\mathcal{P}_\sigma) = 1 \quad .$$

Remark: one easily verifies: $\overline{\Phi}_n$ is a σ-algebra and $\mathcal{P}_\sigma \in \overline{\Phi}_n$.

Lemma 7.1: For any $j \in N$, $B \in \mathcal{B}$, $1 \leq m < 2T + 2$, $h \in N^m$ the following assertion holds:

$\mu_{(j,B,\cdot)}\left(\{h\} \in M^{2T+1-m}\right)$ maps $(\mathcal{P}, \overline{\Phi}_n)$ into $(R, \mathcal{L})$ measurably,

moreover the mapping is integrable with respect to any normed measure on $(\mathcal{P}, \overline{\Phi}_n)$.

Proof: Formulae (2.4) and (2.5) present explicit expressions for this mapping. These expressions immediately imply the measurability and - because of their boundedness - the integrability.

Theorem 7.1: To any $j \in N$, $B \in \mathcal{B}$, $F \in \mathcal{F}$ there corresponds exactly one probability measure

$$\mu^F_{(j,B)} \quad \text{on} \quad (\mathcal{P} \times N^{2T+1}, \overline{\Phi}_n * \Sigma_{2T+1}), \quad \text{such that}$$

$$\forall_{m(1 \leq m < 2T+1)} \, \forall_{h \in N^m} \, \forall_{\mathcal{P}_1 \in \overline{\Phi}_n}:$$

$$\left[\mu^F_{(j,B)}\left(\mathcal{P}_1 \times \{h\} \times N^{2T+1-m}\right) = \int_{\mathcal{P}_1} \mu_{(j,B,P)}\left(\{h\} \times N^{2T+1-m}\right) dF\right] \quad .$$

Proof: Since the proofs of theorem 3.1 and theorem 7.1 are fairly similar, the latter will be omitted.

** A number of definitions and lemmas will introduce the risk concept and some additional results.

Definition 7.3: Mappings u_t $(0 \leq t < T)$ and u from $\mathcal{P} \times N^{2T+1}$ into R are defined by:

$$\forall_{P \in \mathcal{P}} \forall_{h \in N^{2T+1}} [u_t(P,h) := v_t(h),\ u(P,h) := v(h)] \quad .$$

Lemma 7.2: u_t $(0 \leq t < T)$ and u map $(\mathcal{P} \times N^{2t+1}, \overline{\Phi}_n * \Sigma_{2T+1})$ into $(R, \mathcal{B})$ measurably; moreover they are integrable with respect to each probability measure on the first mentioned measurable space.

Proof: The propositions are implied easily by lemma 2.1 (the measurability and integrability of v_t and v), since inverse images of Borel sets under u_t and u are the Cartesian products of $\mathcal{P}$ and the inverse images under v_t and v.

Definition 7.4: $\forall_{j \in N} \forall_{B \in \mathcal{B}} \forall_{F \in \mathcal{F}} \left[U(j,B,F) := \int_{\mathcal{P} \times N^{2T+1}} u \, d\mu^F_{(j,B)} \right]$.

** U(j,B,F) denotes the expected total discounted costs of the decision process including the P - lottery.

Lemma 7.3: $j \in N,\ B \in \mathcal{B},\ F \in \mathcal{F}$, then:

a) $$U(j,B,F) = \sum_{t=0}^{T-1} \int_{\mathcal{P} \times N^{2T+1}} u_t d\mu^F_{(j,B)} \quad ;$$

b) $$\int_{\mathcal{P} \times N^{2T+1}} u_t d\mu^F_{(j,B)} = \int_{\mathcal{P}} \left[\int_{N^{2T+1}} v_t d\mu_{(j,B,P)} \right] dF \qquad (0 \leq t < T) \quad ;$$

c) $U(j,B,F) = \int_{\mathcal{P}} V(j,B,P)dF$.

Proof: a) Lebesgue's theorem for dominated convergence.

b) $$\int_{\mathcal{P}\times N^{2T+1}} u_t d\mu^F_{(j,B)} =$$

$$= \sum_{h\in N^{2t}} \sum_{k,i,\ell\in N} \beta^t(d_{ki}+c_{i\ell})\mu^F_{(j,B)}\left(\mathcal{P}\times\{h,k,i,\ell\}\times N^{2(T-t-1)}\right)$$

$$= \sum_{h\in N^{2t}} \sum_{k,i,\ell\in N} \beta^t(d_{ki}+c_{i\ell})\int_{\mathcal{P}}\mu_{(j,B,P)}\left(\{h,k,i,\ell\}\times N^{2(T-t-1)}\right)dF$$

(theorem 7.1)

$$= \int_{\mathcal{P}}\left[\int_{N^{2T+1}} v_t d\mu_{(j,B,P)}\right]dF$$ (transposing finite summation and integration).

c) $$U(j,B,F) = \sum_{t=0}^{T-1}\int_{\mathcal{P}\times N^{2T+1}} u_t d\mu^F_{(j,B)}$$ (assertion a))

$$= \sum_{t=0}^{T-1}\int_{\mathcal{P}}\left[\int_{N^{2T+1}} v_t d\mu_{(j,B,P)}\right]dF$$ (assertion b))

$$= \int_{\mathcal{P}}\left[\sum_{t=0}^{T-1}\int_{N^{2T+1}} v_t d\mu_{(j,B,P)}\right]dF$$ (Lebesgue's theorem, applying lemma 2.6)

$$= \int_{\mathcal{P}} V(j,B,P)dF$$.

Lemma 7.4: $j \in N$, $B \in \mathcal{B}$, $F \in \mathcal{F}$ and $\mathcal{P}_1 \in \overline{\Phi}_n$, $H \in \Sigma_{2T+1}$, then:

$$\mu^F_{(j,B)}(\mathcal{P}_1\times H) = \int_{\mathcal{P}_1}\mu_{(j,B,P)}(H)dF$$.

Proof: When it is known that for any $H \in \Sigma_{2T+1}$ $\mu_{(j,B,\cdot)}(H)$ maps $(\mathcal{P}, \overline{\Phi}_n)$ into $(R, \mathcal{B})$ measurably, the lemma follows easily. Namely, measurability implies integrability (because of the boundedness) and the integrals of the assertion define a nonnegative set function on $\overline{\Phi}_n \times \Sigma_{2T+1}$ coinciding with $\mu^F_{(j,B)}$ for sets of the type $\mathcal{P}_1 \times \{h\} \times N^{2T+1-m}$ $(1 \leq m < 2T+2,\ h \in N^m)$ and furthermore satisfying the relevant properties of a probability measure.
Hence because of the uniqueness of its extension to a probability measure (theorem 7.1):

$$\forall_{\mathcal{P}_1 \in \overline{\Phi}_n}\ \forall_{H \in \Sigma_{2T+1}} \left[\mu^F_{(j,B)}(\mathcal{P}_1 \times H) = \int_{\mathcal{P}_1} \mu_{(j,B,P)}(H)dF \right] .$$

a) $T < \infty$. The measurability of $\mu_{(j,B,\cdot)}(H)$ follows from the fact that H is the union of a finite number of allowed histories until time T; hence $\mu_{(j,B,\cdot)}(H)$ is the sum of a finite number of measurable functions (lemma 7.1) and therefore measurable.

b) $T = \infty$. It suffices to prove for any $H \in \Sigma_\infty$ and any $p \in U$:

$$\mathcal{P}_1 := \{P \in \mathcal{P} \mid \mu_{(j,B,P)}(H) \geq p\} \in \overline{\Phi}_n .$$

Define $H_m \in \Sigma^\infty_m$ for m natural by: $H_m := \{(h_1,h_2) \in N^\infty \mid h_1 \in N^m,$ $\exists_{h \in N^\infty}((h_1, h) \in H)\}$. Then the sequence $\{H_m\}^\infty_{m=1}$ converges (monotonically decreasing) to H. Hence for any $P \in \mathcal{P}$:

$\mu_{(j,B,P)}(H_m) \searrow \mu_{(j,B,P)}(H)$ when $m \to \infty$. This implies

$$\mathcal{P}_1 = \bigcap_{m=1}^{\infty} \{P \in \mathcal{P} \mid \mu_{(j,B,P)}(H_m) \geq p\} .$$

Hence $\mathcal{P}_1$ has been written as an intersection of a countable number of measurable sets (lemma 7.1), which proves the proposition.

Lemma 7.5: If $F_1 \in \mathscr{F}$, $P_1 \in \mathscr{P}$ with $F_1(\{P_1\}) = 1$, then for any $j \in N$, $B \in \mathscr{B}$:

a) $\mu^{F_1}_{(j,B)}(\mathscr{P} \times H) = \mu_{(j,B,P_1)}(H) \qquad (H \in \Sigma_{2T+1})$;

b) $U(j,B,F_1) = V(j,B,P_1)$.

Proof: a) lemma 7.4.

b) lemma 7.3c).

** According to lemma 7.5, the theory for the situation with known weight function for the matrix of Markov transition probabilities is a direct generalization of the situation with known Markov transition probabilities.

Lemma 7.6: $j \in N$, $B \in \mathscr{B}$, $F \in \mathscr{F}$, $0 \leq t < T$, $h \in N^{2t+1}$, $\mu^{F}_{(j,B)}(\mathscr{P} \times \{h\} \times N^{2(T-1)}) = 0$, then $U(j,B,F)$ does not depend on $b^t(h)$.

Proof: Lemma 7.4 implies: $\mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)}\right) = 0$, except on a set $\mathscr{P}_1 \in \overline{\Phi}$ with $F(\mathscr{P}_1) = 0$.

Lemma 4.3b) implies for $P \notin \mathscr{P}_1$: $V(j,B,P)$ does not depend on $b^t(h)$.

Hence (lemma 7.3c)) $U(j,B,F)$ does not depend on the choice of $b^t(h)$.

Lemma 7.7: $j \in N$, B_1, $B_2 \in \mathscr{B}$, then

$$j^{B_1} = j^{B_2} \Longrightarrow \forall_{F \in \mathscr{F}}[U(j,B_1,F) = U(j,B_2,F)] \quad .$$

Proof: The assertion follows easily by combination of lemmas 2.7 and 7.3c), or as a consequence of lemma 7.6.

** A weight function $F \in \mathcal{F}$ provides the possibility to discriminate between the decision rules:

Definition 7.5: $F \in \mathcal{F}$; $B_1, B_2 \in \mathcal{B}$, then:

a) $(j \in N)$ $\quad {}_jB_1 \overset{F}{\leqslant} {}_jB_2 \;:\Longleftrightarrow U(j,B_1,F) \leqslant U(j,B_2,F)$;

b) $\quad B_1 \overset{F}{\leqslant} B_2 \;:\Longleftrightarrow \forall_{j \in N}\left({}_jB_1 \overset{F}{\leqslant} {}_jB_2\right)$.

** The $\overset{F}{\leqslant}$ -concept defines relations in the sets ${}_j\mathcal{B}$, since all $B \in \mathcal{B}$ with ${}_jB = {}_jB_1$ possess the same $U(j,B,F)$ for any $F \in \mathcal{F}$ (lemma 7.7) and every ${}_jB$ is the j-restriction of at least one decision rule (lemma 2.1b)).

Lemma 7.8: $F \in \mathcal{F}$, then:

a) $(j \in N)$ $\quad \overset{F}{\leqslant}$ defines a weak ordering in ${}_j\mathcal{B}$;

b) $\quad \overset{F}{\leqslant}$ defines a partial weak ordering in $\mathcal{B}$.

Lemma 7.9:

a) $\forall_{j \in N}\, \forall_{B_1, B_2 \in \mathcal{B}}\left[{}_jB_1 \leqslant {}_jB_2 \Longleftrightarrow \forall_{F \in \mathcal{F}_\sigma}\left({}_jB_1 \overset{F}{\leqslant} {}_jB_2\right)\right]$;

b) $\quad \forall_{B_1, B_2 \in \mathcal{B}}\left[B_1 \leqslant B_2 \Longleftrightarrow \forall_{F \in \mathcal{F}_\sigma}\left(B_1 \overset{F}{\leqslant} B_2\right)\right]$.

Proof: a) "$\Longrightarrow$" on application of lemma 7.3c);

"$\Longleftarrow$" to any $P_0 \in \mathcal{P}_\sigma$, there corresponds a $F_0 \in \mathcal{F}_\sigma$, such that $F_0(\{P_0\}) = 1$. Lemma 7.5b) implies:

$$ {}_jB_1 \overset{F_0}{\leqslant} {}_jB_2 \Rightarrow V(j,B_1,P_0) \leqslant V(j,B_2,P_0) \quad . $$

b) by combination of assertion a) with the definitions of the $\leqslant$- and $\overset{F}{\leqslant}$- concepts in $\mathcal{B}$.

Lemma 7.10: For any $j \in N$, $F \in \mathcal{F}$ the mapping $U(j,\cdot,F)$ from $\mathcal{B}$ into R is continuous;

even: if $B_\ell \in \mathcal{B}$ $(\ell = 0,1,2,\ldots)$ and $\lim\limits_{\ell\to\infty} {}_jB_\ell = {}_jB_o$, then

$$\lim_{\ell\to\infty} U(j,B_\ell,F) = U(j,B_o,F) \quad \text{(uniformly in F)} .$$

Proof: $$|U(j,B_\ell,F) - U(j,B_o,F)| = \left|\int_{\mathcal{P}}\Big(V(j,B_\ell,P) - V(j,B_o,P)\Big)dF\right|$$
$$\leqslant \int_{\mathcal{P}} |V(j,B_\ell,P) - V(j,B_\ell,P)|\, dF$$

(lemma 7.3c)).

This implies the assertion, since $\lim\limits_{\ell\to\infty} V(j,B_\ell,P) = V(j,B_o,P)$ uniformly in P (according to lemma 2.17).

** For any $F \in \mathcal{F}$, there exists a decision rule, which is the best in the sense of the risk U:

Theorem 7.2: $\forall_{F\in\mathcal{F}}\ \exists_{B_o\in\mathcal{B}}\ \forall_{B\in\mathcal{B}}\left(B_o \overset{F}{\leqslant} B\right)$.

Proof: $F \in \mathcal{F}$; the mapping $U(j,\cdot,F)$ from the compact set $\mathcal{B}$ (lemma 2.12) into R is continuous (lemma 7.10) for every $j \in N$. Hence, there exist decision rules $B_j \in \mathcal{B}$ $(j \in N)$, such that

$$\forall_{B\in\mathcal{B}}[U(j,B_j,F) \leqslant U(j,B,F)] \quad .$$

Define B_o by: ${}_jB_o := {}_jB_j$ $(j \in N)$, hence $\forall_{B\in\mathcal{B}}\left(B_o \overset{F}{\leqslant} B\right)$ (lemma 7.7).

** Theorem 7.2 asserts, that the ordering of $\mathcal{B}$, which is induced
 by a weight function for the Markov transition matrix of the ba-

sic process, guarantees the existence of a decision rule which is at least as good as all others with respect to the Bayes risk. Reviewing the results of section 4, one may expect, that the search for a F-best decision rule may be restricted to a class of decision rules, which is substantially smaller than $\mathcal{B}$.

However, before establishing some assertions on this point, attention will be directed on a general structural property of any decision rule which is best in F-sense. This property is related to the interpretation as a prior probability distribution of the weight function F. Given T_0 $(0 \leq T_0 < T)$ and $h \in N^{2T_0}$, the j-decision rule ${}_{(h,j)}B$ for the problem with $T - T_0$ steps is best for the weight function which is the posterior distribution derived from F for (h,j) realized, when $B \in \mathcal{B}$ is F-best.

Definition 7.6: For any T_0 $(0 \leq T_0 < T + 1)$, any $h = (k_0, i_0, k_1, \ldots) \in N^{2T_0+1}$ and any $F \in \mathcal{F}$ with

$\int_{\mathcal{P}} \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF \neq 0$, the realvalued function F^h on $\bar{\Phi}_n$ is defined

by: $$F^h(\mathcal{P}_1) := \frac{\int_{\mathcal{P}_1} \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF}{\int_{\mathcal{P}} \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF} \qquad \text{for } \mathcal{P}_1 \in \bar{\Phi}_n$$

(empty products equal 1).

Lemma 7.11: If $0 \leq T_0 < T + 1$, $h = (k_0, i_0, \ldots) \in N^{2T_0+1}$, $F \in \mathcal{F}$ and $\int_{\mathcal{P}} \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF \neq 0$, then $F^h \in \mathcal{F}$;

if moreover $F \in \mathcal{F}_\sigma$, then: $F^h \in \mathcal{F}_\sigma$.

Proof: One easily verifies, that F^h defines a normal measure on the measurable space $(\mathcal{P}, \overline{\Phi})$.

If $F \in \mathcal{F}_\sigma : F^h(\mathcal{P}_\sigma) = 1$.

Lemma 7.12: If $F \in \mathcal{F}$, then all allowed histories until any time (say T_0) $h = (k_0, i_0, \ldots) \in N^{2T_0+1}$ with equal information matrices and $\int_{\mathcal{P}} \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF \neq 0$ define the same normed measure F^h; if $F \in \mathcal{F}_\sigma$, the equality of the information matrices in the primal assertion may be replaced by sub-equality.

Proof: Remark, that only histories until the same time can have equal information matrices. However, sub-equality is possible for allowed histories until different times. The primal assertion is directly implied by definition 7.6, since the integrands in numerator and denominator are the same for allowed histories until T_0 with equal information matrices.

If $F \in \mathcal{F}_\sigma$, the integral ranges in the definition of F^h, may be restricted to $\mathcal{P}_1 \cap \mathcal{P}_\sigma$ and $\mathcal{P}_\sigma$. In these ranges, the integrands in numerator and denominator are equal and both may be rewritten as product of a constant and a factor depending on P. The constants in numerator and denominator cancel each other, whereas the other factors are the same for allowed histories until any time with sub-equal information matrices.

** The following lemma states: if the weight function is such, that - in probability interpretation - the rows of the matrix P are chosen independently, then F^h (when defined) again has this property. Furthermore, the marginal distribution of a row is the prior one, only corrected for the realized Markov transitions in h starting in the corresponding state.

Lemma 7.13: Let $F \in \mathcal{F}$ have the property that the corresponding measure on $(\mathcal{V}^n, \Phi_n)$ - see definition 7.1 - is the product measure, generated by n normed measures $\{\varphi_i\}_{i=1}^n$ on $(\mathcal{V}, \Phi)$.

Then F^h $\left(\text{with } h = (k_0, i_0, \ldots) \in N^{2T_0+1}, \ 0 \leq T_0 < T+1 \text{ and } \int_{\mathcal{P}} \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF \neq 0\right)$ corresponds to a measure on $(\mathcal{V}^n, \Phi_n)$, which is the product measure generated by n normed measures $\{\varphi_i^h\}_{i=1}^n$ on $(\mathcal{V}, \Phi)$ with for every $i \in N$ and $Y \in \Phi$:

$$\varphi_i^h(Y) := \frac{\int_Y \prod_{t=0,\ i_t=i}^{T_0-1} p_{i_t k_{t+1}} \, d\varphi_i}{\int_{\mathcal{V}} \prod_{t=0,\ i_t=i}^{T_0-1} p_{i_t k_{t+1}} \, d\varphi_i} ,$$

where p_{ik} $(k \in N)$ denotes the k-th component of the integration point in $\mathcal{V}$.

Proof: The normed measures φ_i^h are well-defined.

Applying definition 7.6 one may verify that the measure on $(\mathcal{V}^n, \Phi_n)$ corresponding to F^h of a product set $Y = Y_1 \times \ldots \times Y_n \in \Phi^n$ with $Y_i \in \Phi$ $(i \in N)$ equals

$$\prod_{i=1}^n \varphi_i^h(Y_i) \quad .$$

Since the extension of such a set function on the class of product sets Φ^n to a measure on Φ_n is unique, this measure just equals the stated product measure.

Lemma 7.14: $j \in N$, $0 \leq T_0 < T+1$, $B \in \mathcal{B}$, $F \in \mathcal{F}$, $h \in N^{2T_0+1}$ and $\mu^F_{(j,B)}\left(\mathcal{P} \times \{h\} \times N^{2(T-T_0)}\right) \neq 0$, then for any $\mathcal{P}_1 \in \overline{\Phi}_n$:

$$F^h(\mathcal{P}_1) = \mu^F_{(j,B)}\left(\mathcal{P}_1 \times N^{2T+1} \,\middle|\, \mathcal{P} \times \{h\} \times N^{2(T-T_0)}\right) \quad .$$

Proof: The right hand part of the equation equals (definition and lemma 7.4):

$$\frac{\int_{\mathcal{P}_1} \mu_{(j,B,P)}\left(\{h\} \times N^{2(T-T_0)}\right) dF}{\int_{\mathcal{P}} \mu_{(j,B,P)}\left(\{h\} \times N^{2(T-T_0)}\right) dF} \quad .$$

The assertion is derived by applying formula (2.5) on the integrands in this quotient.

** Lemma 7.14 shows that actually F^h is a posterior distribution for the Markov transition matrix, based on the prior distribution F and the realized state history h.

Lemma 7.15: $0 \leq T_0 < T + 1$, $h = (k_0, i_0, \ldots) \in N^{2T_0+1}$, $F \in \mathcal{F}$ and $\int_{\mathcal{P}} \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF \neq 0$; let G be an integrable function on the probability space $(\mathcal{P}, \overline{\Phi}_n, F^h)$, then for any $\mathcal{P}_1 \in \overline{\Phi}_n$:

$$\int_{\mathcal{P}_1} G(P) dF^h = \frac{\int_{\mathcal{P}_1} G(P) \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF}{\int_{\mathcal{P}} \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF} \quad .$$

Proof: If G is a step function, then the assertion is implied directly by definition 7.6. Otherwise, there exists a sequence of step functions $\{G_\ell\}_{\ell=1}^{\infty}$ with

$$\lim_{\ell\to\infty} \int_{\mathcal{P}_1} G_\ell(P)dF^h = \int_{\mathcal{P}_1} G(P)dF^h$$

and

$$\lim_{\ell\to\infty} \int_{\mathcal{P}_1} G_\ell(P) \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF = \int_{\mathcal{P}_1} G(P) \prod_{t=0}^{T_0-1} p_{i_t k_{t+1}} \, dF \quad .$$

Lemma 7.16: $B \in \mathcal{B}$, $j \in N$, $F \in \mathcal{F}$, then:

$$U(j,B,F) = \sum_{\tau=0}^{t-1} \int_{\mathcal{P}} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} dF +$$

$$+ \beta^t \sum_{h\in N^{2t}} \sum_{k\in N} \mu^F_{(j,B)}\left(\mathcal{P}\times \{h,k\} \times N^{2(T-t)}\right) U\left(k, {}_hB, F^{(h,k)}\right)$$

for $0 \leq t < T$, where any U-factor in the latter sum is defined, when the corresponding $\mu^F_{(j,B)}$ - factor does not equal zero; otherwise their product is defined to be zero.

Proof: lemmas 5.16 and 7.3b), c) imply:

$$(7.1) \quad U(j,B,F) = \sum_{\tau=0}^{t-1} \int_{\mathcal{P}} \int_{N^{2T+1}} v_\tau d\mu_{(j,B,P)} +$$

$$+ \beta^t \sum_{h\in N^{2t}} \sum_{k\in N} \int_{\mathcal{P}} V(k, {}_hB, P)\mu_{(j,B,P)}\left(\{h,k\} \times N^{2(T-t)}\right)dF \quad .$$

$$(7.2) \quad \int_{\mathcal{P}} V(k, {}_hB, P)\mu_{(j,B,P)}\left(\{h,k\} \times N^{2(T-t)}\right)dF =$$

(say $h = (k_0, i_0, \ldots, i_{t-1})$, $k = k_t$)

$$= \delta_{jk_0} \left(\prod_{\tau=0}^{t-1} b^\tau_{i_\tau}(k_0, i_0, \ldots, k_\tau)\right) \int_{\mathcal{P}} V(k_t, {}_hB, P) \prod_{\tau=0}^{t-1} p_{i_\tau k_{\tau+1}} \, dF$$

(formula (2.5)).

Applying lemma 7.15 one obtains, if the first integral in (7.3) does not equal zero:

$$(7.3) \quad = \delta_{jk_0} \left(\prod_{\tau=0}^{t-1} b_{i_\tau}^{\tau}(k_0, i_0, \ldots, k_\tau) \right) \int_{\mathcal{P}} \prod_{\tau=0}^{t-1} p_{i_\tau k_{\tau+1}} \, dF \cdot$$

$$\cdot \int_{\mathcal{P}} V(k_t, {}_hB, P) dF^{(h,k_t)} =$$

(formula (2.5) and lemma 7.4)

$$(7.4) \quad = \mu^F_{(j,B)} \left(\mathcal{P} \times \{h, k_t\} \times N^{2(T-t)} \right) \int_{\mathcal{P}} V(k_t, {}_hB, P) dF^{(h,k_t)} \quad .$$

Theorem 7.3: $B_0 \in \mathcal{B}$, $F \in \mathcal{F}$, $\forall_{B \in \mathcal{B}} \; (B_0 \overset{F}{\leqslant} B)$;

$0 \leqslant t < T$; $h_0 \in N^{2t}$; $j, k_t \in N$ and

$\mu^F_{(j,B_0)} \left(\mathcal{P} \times \{h_0, k_t\} \times N^{2(T-t)} \right) \neq 0$, then:

a) for the problem with $T-t$ steps:

$$(h_0, k_t)^{B_0} \overset{F^{(h_0,k_t)}}{\leqslant} (h_0, k_t)^{B} \qquad (\text{for any } B \in \mathcal{B}) \quad ;$$

b) modification of $(h_0, k_t)^{B_0}$ does not destroy the F-bestness of B_0, as long as property a) is maintained.

Proof: a) When $U(j, B_0, F)$ is written in the form (7.1), it appears that $(h_0, k_t)^{B}$ only influences its term (7.2) with $h = h_0$ (lemma 4.3a) and the proof of lemma 5.16 with lemma 4.2b)).

Hence $(h_0, k_t)^{B_0}$ should minimize (7.2) with $h = h_0$ or (7.4) with $h = h_0$. In (7.4) only the integral is influenced by $(h_0, k_t)^{B_0}$ (formula (2.5)), hence lemma 7.3c) proves the assertion.

b) $(h_o,k_t)^{B_o}$ only influences (7.4) with $h = h_o$ as a term in the representation for $U(j,B_o,F)$ of lemma 7.16.

Hence any modification of $(h_o,k_t)^{B_o}$, which maintains $U\left(k_t,{}_{h_o}B_o,F^{(h_o,k_t)}\right)$ minimal, does not destroy the F-optimality of B_o.

** In addition to their usefullness in the sequel of this section, lemma 7.16 and theorem 7.3 are of value for computational purposes.

<u>Lemma 7.17</u>: $0 \leq t < T$, $B \in \mathcal{B}$, $j \in N$, $h \in N^{2t+1}$, then $V_t(j,B,\cdot|h)$ is a measurable mapping from $(\mathcal{P},\overline{\Phi}_n)$ into $(R,\mathcal{L})$; furthermore the mapping is integrable with respect to any normed measure on the first mentioned measurable space.

<u>Proof</u>: The mapping E_τ from $(\mathcal{P},\overline{\Phi}_n)$ into $(R,\mathcal{L})$, defined by:

$$E_\tau(P) = \begin{cases} \int\limits_{N^{2T+1}} v_\tau d\mu^{(h)}_{(j,B,P)} & \text{, when } \mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)-1}\right) \neq 0 \\ 0 & \text{, otherwise} \end{cases}$$

is measurable for any τ with $(t \leq \tau < T)$. Namely $\mathcal{P}_1 := \left\{P \in \mathcal{P} \,\middle|\, \mu_{(j,B,P)}\left(\{h\} \times N^{2(T-t)-1}\right) = 0\right\} \in \overline{\Phi}_n$ and the integral is measurable on $\mathcal{P}\setminus\mathcal{P}_1$, since it is a finite linear combination of functions like $\mu^{(h)}_{(j,B,P)}\left(\{h_1\} \times N^{2(T-\rho)-1}\right)$, which are measurable (compare lemma 7.1).

$V_t(j,B,P \mid h) = \sum_{\tau=t}^{T-1} E_\tau(P)$ (definition 4.1b)) is measurable for that reason.

The integrability is implied by the boundedness.

** Now it will be proved that the search for a F-best decision rule may be restricted to $\mathcal{K}_\sigma$:

<u>Theorem 7.4</u>: $\forall_{F \in \mathcal{F}_\sigma} \exists_{B_0 \in \mathcal{K}_\sigma} \forall_{B \in \mathcal{B}} [B_0 \overset{F}{\lesssim} B]$.

<u>Proof</u>: Say $F \in \mathcal{F}_\sigma$.

a) There exists a decision rule B_*, which is best in F-sense (theorem 7.2). There exists a decision rule $B_T \in \overline{\mathcal{K}_\sigma}$, which is sub-equivalent to B_* (corollary 4.2). Hence (lemma 7.9b)) B_T is best in F-sense. In the sequel of this proof, B_T will be modified into a decision rule $B_0 \in \mathcal{K}_\sigma$ with $U(j,B_0,F) = U(j,B_T,F)$ for every $j \in N$.

b) The modification of $B_T \in \overline{\mathcal{K}_\sigma}$ into $B_0 \in \mathcal{K}_\sigma$ will be accomplished by an induction process. An induction step will be proved in this part. The process, which is of the same type as the induction process in the proof of theorem 4.1, is treated in part c).

Say $0 \leq r < T$, $B_{r+1} \in \mathcal{B}$, with $U(j,B_{r+1},F) = U(j,B_T,F)$ for every $j \in N$, with ${}^{r+1}b^t = {}^{T}b^t$ for $0 \leq t < r+1$ and B_{r+1} satisfies the condition of definition 4.6 for $r+1 \leq t < T$.

The existence will be proved of a decision rule B_r, with ${}^{r}b^t = {}^{r+1}b^t$ for $t \neq r$, the same U-values as B_{r+1} and satisfying the condition of definition 4.6 for $r \leq t < T$. Lemmas 4.1, 7.3b), 7.1, 7.17 imply for every $j \in N$:

 (7.5) $U(j,B_{r+1},F) = \sum_{\tau=0}^{r-1} \int_{\mathcal{P}} \left[\int_{N^{2T+1}} v_\tau \, d\mu(j,B_{r+1},P) \right] dF +$

$$+ \sum_{h \in N^{2r+1}} \int_{\mathcal{P}} \mu(j, B_{r+1}, P)\left(\{h\} \times N^{2(T-r)-1}\right) V_r(j, B_{r+1}, P \,|\, h) dF \,.$$

The value of the first sum in the right hand part of equation (7.5) will not be changed by altering ${}^{r+1}b^r$ (lemma 4.3). The second sum in the right hand part may be rewritten as a finite sum of (finite) subsums, such that each subsum collects all terms corresponding with allowed histories until time r, with information matrices, which are sub-equal to a certain given matrix and all observe the same state at time r. Regard the subsum for matrix K and state $s_{k_r} \in S$:

$$(7.6) \quad \sum_{\substack{h \in N^{2r} \\ K(h,k_r) \overset{\sigma}{=} K}} \quad \sum_{i_r \in N} \int_{\mathcal{P}} \mu(j, B_{r+1}, P)\left(\{h, k_r, i_r\} \times N^{2(T-r)-1}\right) V_r\left(j, B_{r+1}, P \,\middle|\, h, k_r i_r\right) dF \quad .$$

The factors V_r in this expression neither depend on ${}^{r+1}b^r$ (lemma (4.2), nor on j,h (lemma 4.9). These factors will be denoted by $V_r(K, k_r, i_r, P)$. The factors $\mu_{(j, B_{r+1}, P)}$ are determined by formula (2.4) and hence all contain the same number of subfactors p_{ik} for those indices $(i,k) \notin I_\sigma$. The product of these subfactors will be denoted by $\Pi(P,K)$. The product of the remaining π-factors is denoted by $\Pi(\{\pi\}; h, k_r)$. The mapping ${}^{r+1}b^r$ only depends on j,K (reduced), k_r and images will therefore be denoted by ${}^{r+1}b^r(j, K, k_r)$ in this proof. Expression (7.6) may be rewritten as:

$$(7.7) \quad \left[\sum_{i_r \in N} {}^{r+1}b^r_{i_r}(j, K, k_r) \int_{\mathcal{P}} \Pi(P,K) V_r(K, k_r, i_r, P) dF \right] \times$$

$$\times \left[\sum_{\substack{h = (k_0, \dots, i_{r-1}) \in N^{2r} \\ K(h,k_r) \overset{\sigma}{=} K}} \delta_{jk_0} \Pi(\{\pi\}; h, k_r) \sum_{\tau=0}^{r-1} {}^{r+1}b^\tau_{i_\tau}(k_0, i_0, \dots, k_\tau) \right].$$

Expression (7.7) is the product of two rather complicated factors. The second factor is nonnegative. In the first factor the ${}^{r+1}b^r_{i_r}(j,K,k_r)$ ($i_r \in N$) have been selected, such that (7.7) is minimal (these b's do not occur in any other subsum). However, the integrals in the first factor do not depend on $j \in N$. Hence one can select the same ${}^{r+1}b^r(j,K,k_r)$ - values for every $j \in N$ in order to minimize (7.7).

c) For $T < \infty$, the assertion follows directly on application of the induction step: the induction process may be started with $r = T-1$. For the case $T = \infty$, the establishment of the assertion in somewhat more complicated:

$$(7.8) \quad U(j,B,F) = \sum_{\tau=0}^{t-1} \int_{\mathcal{P}\times N^\infty} u_\tau d\mu^F_{(j,B)} + \sum_{\tau=t}^{\infty} \int_{\mathcal{P}\times N^\infty} u_\tau d\mu^F_{(j,B)}$$

$$(0 \leq t < \infty, B \in \mathcal{B}, j \in N) \text{ (lemma 7.3a)).}$$

The first sum on the right hand side of equation (7.8) is equal to the U-value for j and F of the decision process with t steps and the decision rule coinciding with B for those steps. Hence, according to the assertion for $T < \infty$, one can find $B_{to} \in \mathcal{K}_\sigma$, such that

$$\sum_{\tau=0}^{t-1} \int_{\mathcal{P}\times N^\infty} u_\tau d\mu^F_{(j,B)}$$

is minimal with respect to B, for each $j \in N$. The sequence $\{B_{to}\}_{t=o}^{\infty}$ possesses a convergent subsequence with limit $B_o \in \mathcal{K}_\sigma$ (lemma 4.8): $\{B_{t_\ell o}\}_{\ell=1}^{\infty}$. Then for all $j \in N$:

$$\lim_{\ell\to\infty} U(j,B_{t_\ell o},F) = U(j,B_o,F) \qquad \text{(lemma 7.10)} .$$

Hence for all $j \in N$ (a finite number) and any $\varepsilon > 0$:

$$U(j,B_0,F) \leq U(j,B_{t_\ell 0},F) + \varepsilon \qquad (\ell \text{ sufficiently large})$$

$$\leq \min_{B\in\mathcal{B}} \sum_{\tau=0}^{t_\ell-1} \int_{\mathcal{P}\times N^\infty} u_\tau d\mu^F_{(j,B)} + 2\varepsilon \ (\ell \text{ sufficiently large}).$$

Namely the second sum in (7.8) possesses an absolute value smaller than or equal to $\frac{\beta^t}{1-\beta} \max_{k,i,\ell \in N} |d_{ki} + c_{i\ell}|$ (independent of B and j; lemma 2.6a) and 7.3b)).

Hence for all $j \in N$:

$$U(j,B_0,F) \leq \min_{B\in\mathcal{B}} U(j,B,F) + 3\varepsilon \qquad \text{(same reasoning)} ,$$

hence the same assertion has been proved.

** Refinement of the proof of theorem 7.5 even leads to a much stronger result than the assertion of that theorem. In order to minimize (7.7) one may always select a ${}^r b^r\ (j,K,k_r)$, which only consists of zeros and a one. Hence it becomes apparent that a sub-information decision rule applying pure strategies is optimal in F-sense.

<u>Theorem 7.5</u>: $\forall_{F\in\mathcal{F}_\sigma} \forall_{B_0 \in \mathcal{K}_\sigma \cap \mathcal{A}} \forall_{B\in\mathcal{B}} [B_0 \overset{F}{\preccurlyeq} B]$.

<u>Proof</u>: In order to prove the assertion, one may pursue the proof of theorem 7.4. Only the induction step needs a slight modification.

Let B_{r+1} be best in F-sense. Let B_{r+1} satisfy the condition of definition 4.6 for all $t \geq r + 1$ with ${}^{r+1}b^t = {}^T b^t$ for $t < r + 1$, while ${}^{r+1}b^t(h)$ consists of zeros and ones for all $t \geq r + 1$ and $h \in N^{2t+1}$. By altering ${}^{r+1}b^r$ the decision rule B_r may be constructed, which is best in F-sense and satisfies the condition of definition 4.6 for all $t \geq r$ with ${}^{r}b^{t} = {}^{T}b^{t}$ for $t < r$, while ${}^{r}b^{t}(h)$ consists of zeros and ones for all $t \geq r$ and $h \in N^{2t+1}$.

For the case $T = \infty$ the same reasoning as in the proof of theorem 7.4 (part c)) can be presented, since B_{to} may be selected from $\mathcal{R}_\sigma \cap \mathcal{A}$, which is compact (lemmas 2.12 and 4.8).

Corollary 7.5: $P_0 \in \mathcal{P}$; there exists a decision rule $B_0 \in \mathcal{A}$, such that

$$\forall_{t(o \leq t < T)} \forall_{h_1, h_2 \in N^{2t}} \forall_{k \in N} [{}^{o}b^{t}(h_1, k) = {}^{o}b^{t}(h_2, k)]$$

and

$$\forall_{j \in N} \forall_{B \in \mathcal{B}} [V(j, B_0, P_0) \leq V(j, B, P_0)] \quad .$$

Proof: Suppose: $\mathcal{P}_\sigma = \{P_0\}$. Then $I_\sigma = N^2$.

The assertion is implied by theorem 7.5 on application of lemma 7.5b).

** Theorem 7.5 substantially restricts the search for an F-best decision rule.

In fact, the set $\mathcal{R}_\sigma \cap \mathcal{A}$ is finite when T is finite.

In view of theorem 4.3 it seems obvious to investigate the possibility of restricting the search for a F-best decision rule in the case of equal decision costs to the set of state-free sub-information decision rules applying pure strategies $\mathcal{L}_\sigma \cap \mathcal{A}$.

Theorem 7.6: If $\forall_{k,i \in N}(d_{ki} = d_{11})$, then

$$\forall_{F \in \mathcal{F}_\sigma} \exists_{B_0 \in \mathcal{L}_\sigma \cap \mathcal{A}} \forall_{B \in \mathcal{B}} \left[B_0 \overset{F}{\leq} B\right] \quad .$$

 Proof: The proof will not be exhibited completely, since it fairly resembles the proof of theorem 7.4 combined with the proof of theorem 7.5:

Theorem 7.2 implies the existence of a F-best decision rule in $\mathcal{B}$.

Corollary 4.3 implies the existence of a decision rule $B_T \in \overline{\mathcal{L}}_\sigma$ which is sub-equivalent to the F-best decision rule and hence F-best itself (in fact lemmas 7.3c) and 4.12 imply directly that a $B_0 \in \mathcal{L}_\sigma$ is F-best; however to find a $B_0 \in \mathcal{L}_\sigma \cap \mathcal{A}$ the step-by-step procedure is still wanted).

A step-by-step procedure as in the proofs of theorems 7.4 and 7.5 provides the possibility to modify B_T into a decision rule $B_0 \in \mathcal{L}_\sigma \cap \mathcal{A}$, which is F-best.

Corollary 7.6.1: $\forall_{k,i \in N}(d_{ki} = d_{11})$, then:

$$\forall_{F \in \mathcal{F}_\sigma} \forall_{B \in \mathcal{B}}\Big[(B \text{ is best in F-sense}) \Rightarrow \forall_{j,\ell \in N}\big(U(j,B,F) = U(\ell,B,F)\big)\Big].$$

Proof: $F \in \mathcal{F}_\sigma$, $B \in \mathcal{B}$

a) Combination of lemmas 7.3c) and 4.12 delivers:

$$B \in \mathcal{L}_\sigma \Rightarrow \forall_{j,\ell \in N}\big(U(j,B,F) = U(\ell,B,F)\big) \quad ;$$

b) The assertion is proved by combining result a) with theorem 7.6.

Corollary 7.6.2: $\forall_{k,i \in N}(d_{ki} = d_{11})$; $I \subset N$ with $I \times N \subset I_\sigma$; $\ell \in N \setminus I$;

$$\forall_{P \in \mathcal{P}_\sigma}\Big(\min_{i \in I} \sum_{k \in N} c_{ik}\,\pi_{ik} \geq \sum_{r \in N} c_{\ell r}\,p_{\ell r}\Big) \quad ;$$

$\mathcal{L}_1$ defined as in corollary 5.3, then:

$$\forall_{F \in \mathcal{F}_\sigma} \exists_{B_0 \in \mathcal{L}_1 \cap \mathcal{A}} \forall_{B \in \mathcal{B}}\Big[B_0 \overset{F}{\leqslant} B\Big] .$$

Proof: $F \in \mathcal{F}_\sigma$. The proof of the assertion proceeds as the proof of theorem 7.6 with corollary 4.3 replaced by corollary 5.3:

A decision rule $B_T \in \overline{\mathcal{L}_1}$ exists, which is F-best.

The same step-by-step procedure as in the proof of theorem 7.6 provides a modification of B_T into a decision rule B_0 which is still F-best and which is an element of $\mathcal{L}_1 \cap \mathcal{A}$.

Lemma 7.18: $\forall_{k,i \in N}(d_{ki} = d_{11})$; $B \in \mathcal{L}_\sigma$, $F \in \mathcal{F}_\sigma$, $0 \leq t < T$, K is a $n \times n$-matrix of nonnegative integers, then:

For the problem with $T-t$ steps, the value $U\left(k_t, {}_hB, F^{(h,k_t)}\right)$ is the same for all $h = (k_0, i_0, \dots, i_{t-1}) \in N^{2t}$, $k_t \in N$ with

$$K(h,k_t) \overset{\sigma}{=} K \quad \text{and}$$

$$\int_{\mathcal{P}} \prod_{\tau=0}^{t-1} p_{i_\tau k_{\tau+1}} \, dF \neq 0 \quad .$$

Proof: $F^{(h,k_t)}$ is the same for all h, k_t considered (lemma 7.12): call this one $F^{(K)}$.

Let (h,k_t) and (h',k_t') be two histories until time t which satisfy the conditions of the assertion.

For those allowed histories until any time (for the problem with $T-t$ steps) starting with k_t the decision rule ${}_hB$ prescribes the same decision vectors as ${}_{h'}B$ for those starting with k_t'. Those decision vectors are determining for the values of $U\left(k_t, {}_hB, F^{(K)}\right)$ and $U\left(k_t', {}_{h'}B, F^{(K)}\right)$ respectively (lemma 7.7). Furthermore, since ${}_hB$ and ${}_{h'}B \in \mathcal{L}_\sigma$ (lemma 5.14) part a) in the proof of corollary 7.6.1 implies:

$$U\left(k_t, {}_hB, F^{(K)}\right) = U\left(k_t', {}_{h'}B, F^{(K)}\right) \quad .$$

** So far it has been proved that, when a weight function for the transition matrix of the basic Markov chain has been given, decisions may be based on pure strategies only using a part of the information provided by the realized allowed history until the time of decision. Namely, the information matrix (up to sub-equality), the time of decision and the state observed at that time (this last item may be skipped in the case of equal decision costs). It is interesting to know, whether the item "time if decision" may be skipped in some situations. It is obvious, that this may not be asked generally (compare example A.1b) in the appendix with $\frac{1}{2} < \pi < \frac{5}{9}$). However, it will be proved, that this is allowed in the case $T = \infty$ (theorem 7.7) and in the case of equal decision costs with only complete rows of the Markov transition matrix known (theorem 7.8).

Definition 7.7: A sub-information decision rule - say B - is said to be stationary, if and only if:

$$\forall_{t(0 \leq t < T)} \forall_{\tau(0 \leq \tau < T)} \forall_{h_1 \in N^{2t}} \forall_{h_2 \in N^{2\tau}} \forall_{k \in N}$$

$$[K(h_1,k) \overset{\sigma}{=} K(h_2,k) \Longrightarrow b^t(h_1,k) = b^\tau(h_2,k)] \quad .$$

Theorem 7.7: When $T = \infty$, then:

$$\forall_{F \in \mathcal{F}_\sigma} \exists_{B_0 \in \mathcal{K}_\sigma \cap \mathcal{A}} \left[B_0 \text{ is stationary and } \forall_{B \in \mathcal{B}} \left(B_0 \overset{F}{\leqslant} B \right) \right].$$

Proof: The assertion is a direct result of the theorems 7.3 and 7.5. In the case $T = \infty$, the h-restriction of a decision rule B (for any $h \in N^{2t}$, $0 \leq t < \infty$) is again a decision rule for the problem with T steps.

Suppose: $B_0 \in \mathcal{K}_\sigma \cap \mathcal{A}$, $F \in \mathcal{F}_\sigma$ and $\forall_{B \in \mathcal{B}} \left(B_0 \overset{F}{\leqslant} B \right)$ (theorem 7.5).

Applying theorem 7.3, B_0 may be modified into a decision rule in $\mathcal{K}_\sigma \cap \mathcal{A}$, which is stationary, without loosing its F-bestness property.

Such a modification may be defined in a doubly inductive way:

1. Induction with respect to the possible values of $I(h) := \sum_{(i,k) \notin I_\sigma} K_{ik}(h)$, with $K_{ik}(h)$ denoting the element labelled (i,k) of the information matrix of h (allowed history until certain time);

2. For any possible value of I the modification is performed by induction with respect to the time of decision t.

The result of every modification is again denoted by B_0:

a) For t subsequently equal to 1,2,3,... replace ${}_{(h,k)}B_0$ by ${}_kB_0$ for all $h \in N^{2t}$, $k \in N$ with $I(h,k) = 0$. B_0 stays F-best (theorem 7.3).

b) Say $0 < m < \infty$, B_0 is F-best and for every t $(0 \leq t < \infty)$, $h_1 \in N^{2t}$, $k \in N$ with $I(h_1,k) < m$ holds:

$$\exists_{i \in N}\left({}^0b_i^t(h_1,k) = 1\right) \quad \text{and}$$

$$\forall_{\tau(0 \leq \tau < \infty)} \forall_{h_2 \in N^{2\tau}} [K(h_1,k) \leqq K(h_2,k) \Rightarrow {}^0b^t(h_1,k) = {}^0b^\tau(h_2,k)] .$$

Then consider $H_m := \{(h,k) \in \bigcup_{t=m}^{\infty} N^{2t+1} \mid k \in N,\ I(h,k) = m\}$.

Divide H_m in subsets $H_m(K,k)$, which contain all elements of H_m with last component k and information matrix sub-equal to K.

Suppose $(h_0,k) \in H_m(K,k)$ for certain matrix K and $k \in N$, such that $F^{(h_0,k)}$ is defined. A decision rule $B_{(h_0,k)} \in \mathcal{K}_\sigma \cap \mathcal{A}$ exists, which is $F^{(h_0,k)}$ - best (theorem 7.5).

For t subsequently equal to m,m+1,...

replace ${}_{(h,k)}B_0$ by ${}_kB_{(h_0,k)}$ for all $h \in N^{2t}$ with $K(h,k) \overset{\sigma}{=} K(h_0,k)$.

When for certain K,k no $(h_0,k) \in H_m(K,k)$ with defined $F^{(h_0,k)}$ exists, $B_{(h_0,k)}$ may be chosen arbitrarily from $\mathcal{K}_\sigma \cap \mathcal{A}$, since those allowed histories do not contribute substantially to U.

B_0 stays F-best (theorem 7.3).

** The well-known result (e.g. [1962, 1965, D. Blackwell]) that, if the Markov transition matrix is known and $T = \infty$, there exists an optimal decision rule, which applies pure strategies and only depends on the state observed, is contained in theorem 7.7:

Corollary 7.7: $T = \infty$, $P_0 \in \mathcal{P}$; then there exists a decision rule $B_0 \in \mathcal{A}$, such that:

$$\forall_{j \in N} \forall_{B \in \mathcal{B}} [V(j,B_0,P_0) \leq V(j,B,P_0)] \quad ;$$

$$\forall_{t(0 \leq t < \infty)} \forall_{h_1 \in N^{2t}} \forall_{k \in N} \forall_{\tau(0 \leq \tau < \infty)} \forall_{h_2 \in N^{2\tau}} \left({}^0b^t(h_1,k) = {}^0b^\tau(h_2,k)\right) \quad .$$

Proof: Suppose: $\mathcal{P}_\sigma = \{P_0\}$; hence $I_\sigma = N^2$. The assertion is directly implied by theorem 7.7 on application of lemma 7.5b).

Lemma 7.19: Any sub-information decision rule B, which is state-free and stationary satisfies:

$$\forall_{t(0 \leq t < T)} \forall_{\tau(0 \leq \tau < T)} \forall_{h_1 \in N^{2t+1}} \forall_{h_2 \in N^{2\tau+1}}$$

$$\left[K(h_1) \overset{\sigma}{=} K(h_2) \Rightarrow b^t(h_1) = b^\tau(h_2)\right] \quad .$$

** The following lemma presents a key assertion on the case of equal decision costs with the known Markov transition probabilities filling some rows of the Markov transition matrix. Say that at time t, for certain "reduced" information matrix, it is necessary to decide on transforming to s_ℓ (the elements labelled (ℓ,k) of the Markov transition matrix are known to be equal to $\pi_{\ell k}$ for all $k \in N$) in order to obtain a F-best decision rule. Then the same is necessary at time $t+1$. Actually lemma 7.20 is a generalization of lemma 4.1 in [1956, R.N. Bradt e.a], which only treats the case: $\beta = 1$, $T < \infty$, $n = 2$, $c_{11} = c_{22} = -1$, $c_{12} = c_{21} = 0$ and $I = \emptyset$.

<u>Lemma 7.20</u>: $\forall_{k,i \in N}(d_{ki} = d_{11})$; $I \subset N$, $I \times N \subset I_\sigma$; $\ell \in N \setminus I$, $\{\ell\} \times N \subset I_\sigma$;

$\mathcal{L}_1$ defined as in corollary 5.3;

$B_0 \in \mathcal{L}_1 \cap \mathcal{A}$, $F \in \mathcal{F}_\sigma$, $\forall_{B \in \mathcal{B}}\left(B_0 \overset{F}{\leqslant} B\right)$;

$0 \leqslant t < T - 1$, K is a $n \times n$-matrix of nonnegative integers;

B_0 is such, that: $\forall_{h \in N^{2t+1}}[K(h) \overset{\sigma}{=} K \Rightarrow {}^0b_\ell^t(h) = 1]$, furthermore, there does not exist a $B_1 \in \mathcal{L}_1 \cap \mathcal{A}$, which is best in F-sense, with ${}^1b^\tau = {}^0b^\tau$ for all τ $(0 \leqslant \tau < t)$ and ${}^1b_\ell^t(h) \neq 1$ for certain $h \in N^{2t+1}$ with $K(h) \overset{\sigma}{=} K$.

Then: ${}^0b_\ell^{t+1}(h) = 1$ for all $h \in N^{2t+3}$ with $K(h) \overset{\sigma}{=} K$.

<u>Proof</u>: The necessity of ${}^0b_\ell^t(h) = 1$ for those $h \in N^{2t+1}$ with $K(h) \overset{\sigma}{=} K$ implies:

$$\sum_{\substack{h \in N^{2t+1} \\ K(h) \overset{\sigma}{=} K}} \mu^F_{(j,B_0)}\left(\mathcal{P} \times \{h\} \times N^{2(T-t)}\right) > 0 \text{ for all } j \in N \text{ (lemma 7.6)}$$

Hence:

$$\sum_{\substack{h \in N^{2t+3} \\ K(h) \overset{\sigma}{=} K}} \mu^F_{(j,B_0)}\left(\mathcal{P} \times \{h\} \times N^{2(T-t-1)}\right) > 0 \quad \text{for all } j \in N$$

(lemma 7.4, formula (2.5), ${}^{o}b_{\ell}^{t}(h) = 1$ for $h \in N^{2t+1}$ with $K(h) \stackrel{\sigma}{=} K$).
For all $h \in N^{2t+1} \cup N^{2t+3}$ with $K(h) \stackrel{\sigma}{=} K$, the weight function F^{h} (when defined) is the same (lemma 7.12), say $F^{(K)}$.
Theorem 7.3 shows, that ${}_{h}B_{o}$ should be $F^{(K)}$-best for the problem with $T-t$ steps when $h \in N^{2t+1}$, $K(h) \stackrel{\sigma}{=} K$ and for the problem with $T-t-1$ steps when $h \in N^{2t+3}$, $K(h) \stackrel{\sigma}{=} K$.
Hence it suffices to consider $t = 0$ and K consisting of zeros:
$F^{(K)} := F$.

Suppose ${}^{o}b_{m}^{1}(h) = 1$ $(m \in N \setminus I,\ m \neq \ell)$ for $h \in N^{3}$ with $K(h) \stackrel{\sigma}{=} K$.
$U(j,B_{o},F)$ does not differ with $j \in N$ (part a) in the proof of corollary 7.6.1) and equals (lemma 7.3):

$$(7.9)\quad \left(d_{11} + \sum_{k=1}^{n} c_{\ell k}\,\pi_{\ell k}\right) + \sum_{\tau=1}^{T-1} \int_{\mathcal{P}\times N^{2T+1}} u_{\tau}\, d\mu^{F}_{(j,B_{o})} \quad .$$

In short-hand notation (7.9) may be written as: $\bar{\pi}_{\ell} + \beta U_{T-1}$. A more detailed expression for $V(j,B_{o},F)$ is (lemma 7.3):

$$(7.10)\quad \left(d_{11} + \sum_{k=1}^{n} c_{\ell k}\,\pi_{\ell k}\right) + \beta\left(d_{11} + \int_{\mathcal{P}} \sum_{k=1}^{n} c_{mk}\, p_{mk}\, dF\right) + $$
$$+ \sum_{\tau=2}^{T-1} \int_{\mathcal{P}\times N^{2T+1}} u_{\tau}\, d\mu^{F}_{(j,B_{o})} \quad .$$

In short-hand notation (7.10) may be written as: $\bar{\pi}_{\ell} + \beta\bar{p}_{m} + \beta^{2} U^{1}_{T-2}$.
The suppositions lead to the following inequalities:

$$(7.11)\quad \bar{\pi}_{\ell} + \beta U_{T-1} < \bar{p}_{m} + \beta U_{T-1} \quad ,$$

$$\text{hence}\quad \bar{\pi}_{\ell} < \bar{p}_{m} \quad .$$

(7.12) $\bar{\pi}_\ell + \beta\bar{p}_m + \beta^2 U^1_{T-2} < \bar{p}_m + \beta\bar{\pi}_\ell + \beta^2 U^1_{T-2}$,

$$\text{hence} \quad (1-\beta)\bar{\pi}_\ell < (1-\beta)\bar{p}_m \quad .$$

(7.13) $\bar{\pi}_\ell + \beta\bar{p}_m + \beta^2 U^1_{T-2} < \bar{p}_m + \beta U^1_{T-2} + \beta^{T-1}\bar{\pi}_\ell$,

$$\text{hence} \quad \beta(1-\beta)U^1_{T-2} + (1-\beta)\bar{p}_m > \bar{\pi}_\ell(1-\beta^{T-1}) \quad .$$

(7.14) $\bar{\pi}_\ell + \beta\bar{p}_m + \beta^2 U^1_{T-2} \leq \bar{\pi}_\ell \sum_{\tau=0}^{T-1} \beta^\tau$,

$$\text{hence} \quad \beta U^1_{T-2} + \bar{p}_m \leq \bar{\pi}_\ell \sum_{\tau=0}^{T-2} \beta^\tau \quad .$$

($\beta^{T-1} := 0$, when $T = \infty$)

(7.12) is contradictory for $\beta = 1$.

(7.11) contradicts (7.12) in case $\beta > 1$.

(7.13) contradicts (7.14) in case $0 < \beta < 1$.

Hence ${}^0b^1_m(h) = 0$ for all $m \in N \setminus I$, $m \neq \ell$, $h \in \mathbf{N}^3$ with $K(h) \overset{\sigma}{=} K$.

<u>Theorem 7.8</u>: $\forall_{k,i\in N}(d_{ki} = d_{11})$; $N_\sigma \subset N$ with $I_\sigma = N_\sigma \times N$, then

$$\forall_{F\in\mathcal{F}_\sigma} \exists_{B_T \in \mathcal{L}_\sigma \cap \mathcal{A}} \left[B_T \text{ is stationary and } \forall_{B\in\mathcal{B}}\left(B_T \overset{F}{\preccurlyeq} B\right)\right].$$

<u>Proof</u>: In case $N_\sigma = \emptyset$ (hence $I_\sigma = \emptyset$), the assertion is the same as theorem 7.6. Suppose subsequently $N_\sigma \neq \emptyset$.

Be $\ell \in N_\sigma$, such that $\sum_{k=1}^{n} c_{\ell k}\pi_{\ell k} = \min_{i\in N_\sigma} \sum_{k=1}^{n} c_{ik}\pi_{ik}$; $I := N_\sigma \setminus \{\ell\}$.

$\mathcal{L}_1$ is defined as in corollary 5.3.

 Be $F \in \mathcal{F}_\sigma$.

Corollary 7.6.2 guarantees that:

$$\mathcal{A}_0 := \left\{ B_0 \in \mathcal{L}_1 \cap \mathcal{A} \,\middle|\, \forall_{B \in \mathcal{B}} \left(B_0 \overset{F}{\leqslant} B \right) \right\}$$

is nonempty.

Lemma 7.20 provides the possibility to construct a decision rule in $\mathcal{A}_0$ which is stationary. The construction proceeds by induction with respect to time.

Choose a decision rule from $\mathcal{A}_0$, say B_1, if possible with ${}^1b^0_\ell(j) = 0$ for all $j \in N$.

Let a decision rule B_r $(1 \leqslant r < T)$ be chosen from $\mathcal{A}_0$, such that the condition of definition 7.7 is satisfied for all t and τ with $0 \leqslant t,\ \tau < r$. Then B_{r+1} is chosen, such that $B_{r+1} \in \mathcal{A}_0$, ${}^{r+1}b^t = {}^rb^t$ for $0 \leqslant t < r$. The equivalence classes of N^{2r+1} with respect to sub-equality of the information matrices are numbered and ${}^{r+1}b^r$-values for these classes are fixed subsequently - if possible with ${}^{r+1}b^r_\ell(\cdot) = 0$. B_{r+1} satisfies the conditions of definition 7.7 for all t and τ with $0 \leqslant t,\ \tau < r+1$.

When $T < \infty$, B_T emerges in a finite number of steps. If $T = \infty$, B_T may be selected as $\lim_{r \to \infty} B_r$.

** In the appendix (A.5) it is demonstrated that in the general case of equal decision costs it may occur that there does not exist a stationary decision rule which is best in F-sense.

Theorem 7.9 combines a number of results of this section.

<u>Theorem 7.9</u>: $\forall_{F \in \mathcal{F}_\sigma} \exists_{B_0 \in (\mathcal{B}_0)} \forall_{B \in \mathcal{B}} \left(B_0 \overset{F}{\leqslant} B \right)$, with $\mathcal{B}_0$ equal to

a) $\mathcal{K}_\sigma \cap \mathcal{A}$ (all cases);

b) the subset of stationary decision rules in $\mathcal{K}_\sigma \cap \mathcal{A}$, if $T = \infty$;

c) $\mathcal{L}_\sigma \cap \mathcal{A}$, if $\forall_{k,i\in N}(d_{ki} = d_{11})$;

d) the subset of stationary decision rules in $\mathcal{L}_\sigma \cap \mathcal{A}$, if $\forall_{k,i\in N}(d_{ki} = d_{11})$ and $I_\sigma = N_\sigma \times N$.

Proof: Since all $\mathcal{B}_0$ involved are closed sets, the sets $(\mathcal{B}_0)$ are complete in $\mathcal{B}_0$ (lemma 5.11).

Lemma 7.9 combined with theorems 7.5, 7.7, 7.6 and 7.8 respectively implies the assertions.

ADMISSIBILITY AND THE BAYESIAN APPROACH

** As noticed in section 5, the admissibility of ${}_jB_0$ in ${}_j\mathcal{B}$ implies the existence of a weight function $F \in \mathcal{F}_\sigma$, such that for all $B \in \mathcal{B}$

$$ {}_jB_0 \overset{F}{\leqslant} {}_jB \quad . $$

The assertion, which provides a characterization of admissibility will be proved in this section (theorem 8.2).

Theorem 7.9a) states (for any $F \in \mathcal{F}_\sigma$) the existence of a decision rule $B_0 \in \left(\mathcal{K}_\sigma \cap \mathcal{A}\right)$ with for all $B \in \mathcal{B}$

$$ B_0 \overset{F}{\leqslant} B \quad , \quad \text{hence} \quad \forall_{j \in N} \left({}_jB_0 \overset{F}{\leqslant} {}_jB \right) \quad . $$

Reviewing theorems 7.9a) and 8.2, it is tempting to conjecture for any $B_0 \in \left(\mathcal{K}_\sigma \cap \mathcal{A}\right)$ the existence of a weight function $F \in \mathcal{F}_\sigma$ with for all $B \in \mathcal{B}$

$$ B_0 \overset{F}{\leqslant} B \quad . $$

This conjecture, however, appears to be false. A.6 in the appendix provides a counter example.

In section 6, it has been mentioned for the expected total costs $V(j,B,P)$, that the property

$$ \min_{B \in \mathcal{B}} \max_{P \in \mathcal{P}_\sigma} V(j,B,P) = \max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}} V(j,B,P) $$

is not satisfied generally.

Consider $\mathcal{F}_\sigma$, the set of weight functions on $\mathcal{P}_\sigma$, as the set of strategies for Nature (the opponent of the surveyor of the process in the game interpretation of the problem). Then the similar property for the expected total costs U(j,B,F) is satisfied generally. This is proved in theorem 8.1, since in the proof of theorem 8.2 this assertion will be applied.

There exist two generally applied methods in order to proof an assertion like theorem 8.1. The first is based on the application of a fixed point theorem for point-to-set mappings. The second one, which leads to success in this case, basically applies a separation property of convex sets in finitely dimensional Euclidean space.

<u>Theorem 8.1</u>: $\inf_{B \in \mathcal{B}} \sup_{F \in \mathcal{F}_\sigma} U(j,B,F) = \sup_{F \in \mathcal{F}_\sigma} \inf_{B \in \mathcal{B}} U(j,B,F)$. $(j \in N)$.

<u>Proof</u>: $j \in N$.

Lemmas 2.6b) and 7.3c) imply the boundedness of U(j,B,F); hence both parts of the asserted equation are finite real numbers.

Trivially holds:

$$\inf_{B \in \mathcal{B}} \sup_{F \in \mathcal{F}_\sigma} U(j,B,F) \geq \sup_{F \in \mathcal{F}_\sigma} \inf_{B \in \mathcal{B}} U(j,B,F) \quad .$$

Hence the proof will consist of justifying:

$$(8.1) \qquad \inf_{B \in \mathcal{B}} \sup_{F \in \mathcal{F}_\sigma} U(j,B,F) \leq \sup_{F \in \mathcal{F}_\sigma} \inf_{B \in \mathcal{B}} U(j,B,F) \quad .$$

Select $a \in R$, with $\inf_{B \in \mathcal{B}} \sup_{F \in \mathcal{F}_\sigma} U(j,B,F) > a$.

Define for any $P \in \mathcal{P}_\sigma$: $\mathcal{B}_P := \{B \in \mathcal{B} \mid V(j,B,P) > a\}$.

All sets $\mathcal{B}_P$ are open in $\mathcal{B}$ (lemma (2.13).

$\{\mathcal{B}_P\}_{P \in \mathcal{P}_\sigma}$ constitutes an open covering of $\mathcal{B}$.

The compactness of $\mathcal{B}$ (lemma 2.12) implies the existence of a finite subcovering: $\{\mathcal{B}_{P_r}\}_{r=1}^{\ell}$.

Hence:

$$(8.2) \qquad \inf_{B\in\mathcal{B}} \max_{1\leq r\leq \ell} V(j,B,P_r) \geq a \quad .$$

Define: $V_a := \sup_{\{p_r\}} \inf_{B\in\mathcal{B}} \sum_{r=1}^{\ell} p_r V(j,B,P_r)$,

with $\sup_{\{p_r\}}$ denoting the supremum over the set

$$\{(p_1,\dots,p_\ell) \in R^\ell \mid p_r \geq 0 \text{ for } r=1,\dots,\ell \text{ and } \sum_{r=1}^{\ell} p_r = 1\} \quad .$$

Since $\sup_{F\in\mathcal{F}_\sigma} \inf_{B\in\mathcal{B}} U(j,B,F) \geq V_a$ (lemma 7.3c), it suffices to prove: $V_a \geq a$.

Namely then: $\sup_{F\in\mathcal{F}_\sigma} \inf_{B\in\mathcal{B}} U(j,B,F) \geq a$,

For any $a \in R$, with

$$\inf_{B\in\mathcal{B}} \sup_{F\in\mathcal{F}_\sigma} U(j,B,F) > a \quad ,$$

which implies (8.1).

Because of (8.2), it suffices to prove:

$$(8.3) \quad \sup_{\{p_r\}} \inf_{B\in\mathcal{B}} \sum_{r=1}^{\ell} p_r V(j,B,P_r) \geq \inf_{B\in\mathcal{B}} \sup_{\{p_r\}} \sum_{r=1}^{\ell} p_r V(j,B,P_r) \quad .$$

Define: $V := \Big\{v = (v_1,\dots,v_\ell) \in R^\ell \Big| \exists_{B\in\mathcal{B}} \forall_{r(1\leq r\leq \ell)}$

$$\Big(v_r = V(j,B,P_r)\Big)\Big\}$$

for $v_0 \in R$: $W(v_0) := \{w = (w_1,\dots,w_\ell) \in R^\ell \mid \forall_{r(1 \leq r \leq \ell)}(w_r < v_0)\}$

$b := \sup\{v_0 \in R \mid V \cap W(v_0) = \emptyset\}$

hence $b \in R$.

This implies: $V \cap W(b + \frac{1}{m}) \neq \emptyset$ for natural m.

Suppose $v^m \in V \cap W(b + \frac{1}{m})$,

and $B_m \in \mathcal{B}$ satisfies: $v^m_r = V(j, B_m, P_r)$ for $r = 1, \dots, \ell$.

Then:

$$V(j, B_m, P_r) \leq b + \frac{1}{m} \qquad (r = 1, \dots, \ell, \text{ natural } m)$$

or

$$\sum_{r=1}^{\ell} p_r V(j, B_m, P_r) \leq b + \frac{1}{m} \quad (\text{natural } m) \text{ for } p_r \geq 0, \sum_{r=1}^{\ell} p_r = 1.$$

(8.4) This implies: $\inf_{B \in \mathcal{B}} \sup_{\{p_r\}} \sum_{r=1}^{\ell} p_r V(j, B, P_r) \leq b$.

$W(b)$ is an open, convex subset of R^ℓ.

V is an convex subset of R^ℓ, namely:

$$\forall_{\lambda(0 \leq \lambda \leq 1)} \forall_{B_1, B_2 \in \mathcal{B}} \exists_{B_\lambda \in \mathcal{B}} \forall_{r(1 \leq r \leq \ell)}$$

$$[V(j, B_\lambda, P_r) = \lambda V(j, B_1, P_r) + (1 - \lambda) V(j, B_2, P_r)] \quad .$$

This is a consequence of theorem 3.4.

$V \cap W(b) = \emptyset$.

A separation theorem for disjoint convex sets - compare [1959, C. Berge, page 171] - implies the existence of nonnegative real numbers $q_1, \dots, q_\ell$ summing to 1, such that

$$\forall_{v \in V} \forall_{w \in W(b)} \left[\sum_{r=1}^{\ell} q_r v_r > \sum_{r=1}^{\ell} q_r w_r\right] \quad .$$

Hence

$$\forall_{v \in V}\left[\sum_{r=1}^{\ell} q_r v_r \geq b\right] .$$

This implies:

$$\inf_{B \in \mathcal{B}} \sum_{r=1}^{\ell} q_r V(j,B,P_r) \geq b \quad ,$$

or:

$$(8.5) \qquad \sup_{\{p_r\}} \inf_{B \in \mathcal{B}} \sum_{r=1}^{\ell} p_r V(j,B,P_r) \geq b \quad .$$

(8.4) and (8.5) imply (8.3).

** In fact, however, the proof of theorem 8.2 needs a somewhat different result. The proof of that assertion proceeds exactly as the proof of theorem 8.1. Whence the assertion is presented as a corollary to theorem 8.1:

<u>Corollary 8.1.1</u>: $B_1 \in \mathcal{B}$, $j \in N$, then

$$(8.6) \quad \inf_{B \in \mathcal{B}} \sup_{F \in \mathcal{F}_\sigma}\Big(U(j,B,F) - U(j,B_1,F)\Big) = \sup_{F \in \mathcal{F}_\sigma} \inf_{B \in \mathcal{B}}\Big(U(j,B,F) - U(j,B_1,F)\Big) .$$

** Actually "inf" and "sup" in theorem 8.1 may be replaced by "min" and "max". This will be asserted in a second corollary to this theorem. However, the proof needs some auxiliary measure and integration theoretic results. These will be presented in two lemmas. The developments are based on [1956, H. Richter].

<u>Lemma 8.1</u>: ℓ is a natural number, Q is a closed and bounded subset of R^ℓ (in the natural topology).

Let $\{\omega_r\}_{r=1}^{\infty}$ be a sequence of normed measures on $(R^\ell, \mathcal{L}_\ell)$, with corresponding distribution functions $\{G_r\}_{r=1}^{\infty}$; such that

$$\omega_r(Q) = 1 \qquad (r = 1,2,3,\dots) \quad .$$

Then: there exist a normed measure ω_0 on $(R^\ell, \mathcal{L}_\ell)$ with corresponding distribution function G_0 and a subsequence of the sequence of natural numbers $\{r_\rho\}_{\rho=1}^{\infty}$, such that

$$\omega_0(Q) = 1$$

and for any $x \in R^\ell$, which is a continuity point of G_0:

$$\lim_{\rho\to\infty} G_{r_\rho}(x) = G_0(x) \quad .$$

Proof: Theorem V.7.3 in [1956, H. Richter] implies the existence of a measure defining function G_0 and hence a measure ω_0, which satisfy nearly all the conditions.
It only remains to be proved that the G_0 and ω_0 found are such that G_0 is a distribution function and $\omega_0(Q) = 1$.
That G_0 is a distribution function follows from the boundedness of Q: there exist points $x^{(1)}, x^{(2)} \in R^\ell$ with $G_0\left(x^{(1)}\right) = 1$, $G_0\left(x^{(2)}\right) = 0$ and for no $x \in R^\ell$ holds $G_0(x) < 0$ or $G_0(x) > 1$.
Select for any natural k finitely many real numbers

$$\alpha_{k0} < \alpha_{k1} < \dots < \alpha_{km_k} \quad ,$$

such that no discontinuity coordinate of G_0 occurs and

$$\alpha_{k\nu} - \alpha_{k\nu-1} \leq \frac{1}{k} \qquad (\nu = 1,2,\dots,m_k) \quad ,$$

$$Q \subset (\alpha_{k0}, \alpha_{km_k}]^\ell$$ (compare theorem I.5.20 in [1956, H. Richter]).

Define $\mathcal{J}^{(k)}$ as the set of all "semi open - semi closed" intervals in R of the type:

$$I(\nu_1,\dots,\nu_\ell) := \{x \in R^\ell \mid \alpha_{k\nu_\lambda} < x_\lambda \leq \alpha_{k\nu_\lambda+1} \quad \text{for } \lambda = 1,2,\dots,\ell\}.$$

$A^{(k)}$ is the union of those elements of $\mathfrak{J}^{(k)}$, which have a non-vacuous intersection with Q. Hence $A^{(k)} \supset Q$.
For any $I \in \mathfrak{J}^{(k)}$ holds:

$$\lim_{\rho \to \infty} \omega_{r_\rho}(I) = \omega_0(I) \quad .$$

Hence for $A^{(k)}$ applies:

$$1 = \lim_{\rho \to \infty} \omega_{r_\rho}\left(A^{(k)}\right) = \omega_0\left(A^{(k)}\right) \quad .$$

For any natural $\varkappa$ holds: $\omega_0\left(\bigcap_{k=1}^{\varkappa} A^{(k)}\right) = 1$.

Thus $\bigcap_{k=1}^{\infty} A^{(k)} = Q$ implies

$$\omega_0(Q) = 1 \quad .$$

Lemma 8.2: ℓ is a natural number, Q is a closed and bounded subset of R^ℓ, Y is a set.

Let $\{f_y\}_{y \in Y}$ be a family of equicontinuous mappings of Q into R. Furthermore this family is supposed to be uniformly bounded.

Let $\{\omega_r\}_{r=0}^{\infty}$ be a sequence of normed measures on $(R^\ell, \mathcal{L}_\ell)$, with corresponding distribution functions $\{G_r\}_{r=0}^{\infty}$, such that

$$\omega_r(Q) = 1 \quad \text{for} \quad r = 0,1,\ldots$$

and for any $x \in R^\ell$, which is a continuity point of G_0:

$$\lim_{r \to \infty} G_r(x) = G_0(x) \quad .$$

Then:

$$\lim_{r \to \infty} \int_Q f_y \, d\omega_r = \int_Q f_y \, d\omega_0 \qquad \text{(uniformly on Y)} .$$

Remark: The equicontinuity of the family means, that to any $\varepsilon > 0$ there corresponds a number $\delta > 0$, such that

$$|f_y(x^{(1)}) - f_y(x^{(2)}) < \varepsilon \text{ whenever } y \in Y, x^{(1)}, x^{(2)} \in Q \text{ with } |x^{(1)} - x^{(2)}| < \delta \quad .$$

Proof: Suppose: $\varepsilon > 0$;

select $\delta(\varepsilon) > 0$, such that

$$\left| f_y(x^{(1)}) - f_y(x^{(2)}) \right| < \frac{\varepsilon}{4}$$

whenever $y \in Y$, $x^{(1)}, x^{(2)} \in Q$ with

$$|x_\lambda^{(1)} - x_\lambda^{(2)}| < \delta(\varepsilon) \quad \text{for} \quad \lambda = 1, \ldots, \ell \quad ;$$

select finitely many real numbers $\alpha_0 < \alpha_1 < \ldots < \alpha_m$, such that no α_ν $(\nu = 0, \ldots, m)$ is a discontinuity coordinate of G_0 and

$$\alpha_\nu - \alpha_{\nu-1} < \delta(\varepsilon) \qquad (\nu = 1, \ldots, m)$$

and $Q \subset (\alpha_0, \alpha_m]^\ell = A$ (compare theorem I.5.20 in [1956, H. Richter]).

The f_y are extended to functions on R^ℓ, by defining

$$f_y(x) := 0 \quad \text{for} \quad y \in Y,\ x \in R^\ell \setminus Q \quad .$$

Then:

$$\int_A f_y \, d\omega_r = \int_Q f_y \, d\omega_r \qquad (y \in Y,\ r = 0, 1, \ldots) \quad .$$

$\mathfrak{I}$ is the set of all "semi open - semi closed" intervals in R^ℓ of the type

$$I(\nu_1, \ldots, \nu_\ell) := \left\{ x \in R^\ell \,\middle|\, \alpha_{\nu_\lambda} < x_\lambda \leq \alpha_{\nu_\lambda + 1} \quad \text{for} \quad \lambda = 1, \ldots, \ell \right\} \quad ,$$

 having a nonvacuous intersection with Q.

Define for $I \in \mathfrak{I}$ and $y \in Y$: $f_y^I := \inf_{x \in I \cap Q} f_y(x)$;

for $x \in I$: $\varepsilon(y,x) := \begin{cases} f_y(x) - f_y^I & \text{when } x \in Q \,, \\ 0 & \text{when } x \notin Q \,. \end{cases}$

Hence $0 \leq \varepsilon(y,x) \leq \frac{\varepsilon}{4}$ for those x with $\varepsilon(y,x)$ defined.

Now

$$\left| \int_A f_y \, d\omega_r - \int_A f_y \, d\omega_o \right| \leq \sum_{I \in \mathfrak{I}} \left| \int_I f_y \, d\omega_r - \int_I f_y \, d\omega_o \right| =$$

$$= \sum_{I \in \mathfrak{I}} \left| f_y^I \left(\int_I d\omega_r - \int_I d\omega_o \right) + \int_I \varepsilon(y,x) d\omega_r - \int_I \varepsilon(y,x) d\omega_o \right|$$

$$\leq \sum_{I \in \mathfrak{I}} \left| f_y^I \right| \cdot \left| \int_I d\omega_r - \int_I d\omega_o \right| + \frac{\varepsilon}{4} \sum_{I \in \mathfrak{I}} \left(\omega_r(I) + \omega_o(I) \right)$$

$$\leq |f|_{max} \sum_{I \in \mathfrak{I}} \left| \int_I dG_r - \int_I dG_o \right| + \frac{\varepsilon}{2} \quad (\text{choose } |f|_{max} > 0) \,.$$

G_o is continuous for the vertices of any $I \in \mathfrak{I}$.

Hence: $\lim_{r \to \infty} \int_I dG_r = \int_I dG_o$ for $I \in \mathfrak{I}$.

Then, when r sufficiently large

$$\left| \int_I dG_r - \int_I dG_o \right| < \frac{1}{|f|_{max}} \cdot \frac{1}{m^\ell} \cdot \frac{\varepsilon}{2} \quad \text{for all } I \in \mathfrak{I} \,.$$

Hence for such r:

$$\left| \int_Q f_y \, d\omega_r - \int_Q f_y \, d\omega_o \right| < \varepsilon \qquad \text{for any } y \in Y \,.$$

Corollary 8.1.2:

$$\min_{B \in \mathcal{B}} \max_{F \in \mathcal{F}_\sigma} U(j,B,F) = \max_{F \in \mathcal{F}_\sigma} \min_{B \in \mathcal{B}} U(j,B,F) \qquad (j \in N).$$

Proof: $j \in N$.

It suffices to prove that "inf" and "sup" in theorem 8.1 may be replaced by "min" and "max".

a) Say $B \in \mathcal{B}$. There exists a sequence $\{F_r\}_{r=1}^{\infty} \subset \mathcal{F}_\sigma$, such that

$$\lim_{r \to \infty} U(j,B,F_r) = \sup_{F \in \mathcal{F}_\sigma} U(j,B,F) \quad .$$

$(\mathcal{P}, \overline{\Phi}_n)$ and $(\mathfrak{V}^n, \Phi_n)$ are fairly similar (definition 7.1). Hence $F \in \mathcal{F}_\sigma$ (definition 7.2) induces a normed measure on $(\mathfrak{V}^n, \Phi_n)$ in a natural way. Therefore any $F \in \mathcal{F}_\sigma$ induces by extension a normed measure ω on $\left(R^{n^2}, \mathcal{L}_{n^2}\right)$ with a distribution function G and the property that the ω-measure of the $\mathcal{L}_{n^2}$-measurable set Q which corresponds to $\mathcal{P}_\sigma \in \overline{\Phi}_n$ is equal to 1.

Then lemma 8.1 implies the existence of a subsequence $\{F_{r_\rho}\}_{r=1}^{\infty}$ and a weight function $F_0 \in \mathcal{F}_\sigma$, which satisfy (according to lemma 7.3c) and lemma 8.2):

$$\lim_{\rho \to \infty} U(j,B,F_{r_\rho}) = U(j,B,F_0)$$

(the y of lemma 8.2 is a dummy variable in this case; the continuity of f_y - in this case $V(j,B,\cdot)$ - is asserted by lemma 2.9, while the compactness of $\mathcal{P}_\sigma$ implies its uniformity).

Hence:

$$\sup_{F \in \mathcal{F}_\sigma} U(j,B,F) = U(j,B,F_0) = \max_{F \in \mathcal{F}_\sigma} U(j,B,F) \quad .$$

b) $\max_{F \in \mathcal{F}_\sigma} U(j,\cdot,F)$ is continuous on $\mathcal{B}$ (compact according to lemma 2.12):

$$\left| \max_{F \in \mathcal{F}_\sigma} U(j,B,F) - \max_{F \in \mathcal{F}_\sigma} U(j,B_\varepsilon,F) \right| \leq \max_{F \in \mathcal{F}_\sigma} \left| U(j,B,F) - U(j,B_\varepsilon,F) \right|$$

and application of lemma 7.10.

Hence the first "inf" may be replaced by "min".

c) Theorem 7.2 provides the justification of the "min" in the right hand side.

d) There exists a sequence $\{F_r\}_{r=1}^{\infty} \subset \mathcal{F}_\sigma$, such that

$$\lim_{r \to \infty} \min_{B \in \mathcal{B}} U(j,B,F_r) = \sup_{F \in \mathcal{F}_\sigma} \min_{B \in \mathcal{B}} U(j,B,F) \quad .$$

As in part a) of this proof, lemma 8.1 implies the existence of a subsequence $\{F_{r_\rho}\}_{\rho=1}^{\infty}$ and a weight function $F_0 \in \mathcal{F}_\sigma$, such that the corresponding distribution functions satisfy: $\lim_{\rho \to \infty} G_{r_\rho}(x) = G_0(x)$ for each continuity point $x \in R^{n^2}$ for G_0.

Then:

$$\left| \min_{B \in \mathcal{B}} U(j,B,F_{r_\rho}) - \min_{B \in \mathcal{B}} U(j,B,F_0) \right| \leq \max_{B \in \mathcal{B}} \left| U(j,B,F_{r_\rho}) - U(j,B,F_0) \right| .$$

This maximum is less than any preassigned $\varepsilon > 0$ for ρ sufficiently large according to lemma 7.3c) and lemma 8.2 (the role of y in lemma 8.2 is played by B in this case; the equicontinuity of the family $\{f_y\}_{y \in Y}$ - in this case $\{V(j,B,\cdot)\}_{B \in \mathcal{B}}$ - is asserted by lemma 2.10 and the compactness of $\mathcal{P}_\sigma$).

Hence:

$$\sup_{F \in \mathcal{F}_\sigma} \min_{B \in \mathcal{B}} U(j,B,F) = \min_{B \in \mathcal{B}} U(j,B,F_0) = \max_{F \in \mathcal{F}_\sigma} \min_{B \in \mathcal{B}} U(j,B,F) \quad .$$

Theorem 8.2: $B_o \in \mathcal{B}$, $j \in N$, ${}_jB_o \in ({}_j\mathcal{B})$, then

$$\exists_{F \in \mathcal{F}_\sigma} \forall_{B \in \mathcal{B}} \left({}_jB_o \overset{F}{\lesssim} {}_jB\right) \quad .$$

Proof: Another formulation of the theorem is:

$${}_j\mathcal{B}_b := \left\{B_b \in \mathcal{B} \;\middle|\; \exists_{F \in \mathcal{F}_\sigma} \forall_{B \in \mathcal{B}} \left({}_jB_b \overset{F}{\lesssim} {}_jB\right)\right\} \supset \left({}_j\mathcal{B}\right) \quad .$$

Therefore it suffices to prove:

$${}_j\mathcal{B}_b \text{ is complete in } {}_j\mathcal{B} \text{ (lemma 5.5f)).}$$

Hence attention may be restricted to essential completeness. For completeness is implied by essential completeness in this case.

Thus the problem is:

Be $B_1 \in \mathcal{B}$. Then find $B_* \in \mathcal{B}$, such that

$${}_jB_* \in {}_j\mathcal{B}_b \quad \text{and} \quad {}_jB_* \in {}_jB_1 \quad .$$

Or: find $B_* \in \mathcal{B}$, such that for certain $F \in \mathcal{F}_\sigma$:

$$U(j,B_*,F) \leq U(j,B,F) \qquad \text{(for any } B \in \mathcal{B}\text{)} \quad ,$$

and $$V(j,B_*,P) \leq V(j,B_1,P) \qquad \text{(for any } P \in \mathcal{P}_\sigma\text{)} \quad .$$

Hence it suffices to find $B_* \in \mathcal{B}$, such that:

(8.7) $$\forall_{r(\text{natural})} \exists_{F_r \in \mathcal{F}_\sigma} \forall_{B \in \mathcal{B}} \left[U(j,B_*,F_r) \leq U(j,B,F_r) + \frac{1}{r}\right] ,$$

(8.8) and $$\forall_{P \in \mathcal{P}_\sigma} \left[V(j,B_*,P) \leq V(j,B_1,P)\right] \quad .$$

 Namely, according to the same reasoning as in part a) of the proof of corollary 8.1.2 there exist a subsequence $\{F_{r_\rho}\}_{\rho=1}$ and a weight function $F_0 \in \mathcal{F}_\sigma$, such that for all $B \in \mathcal{B}$:

$$\lim_{\rho\to\infty} U(j,B,F_{r_\rho}) = U(j,B,F_0) \quad .$$

Define: $U_1(j,B,F) := U(j,B,F) - U(j,B_1,F) \quad (B \in \mathcal{B},\ F \in \mathcal{F})$.

It suffices to prove the existence of a decision rule B_* satisfying

(8.9) $\forall_{r(\text{natural})} \exists_{F_r \in \mathcal{F}_\sigma} \forall_{B\in\mathcal{B}} [U_1(j,B_*,F_r) \leq U_1(j,B,F_r) + \frac{1}{r}]$,

(8.10) and $\quad \sup_{F\in\mathcal{F}_\sigma} U_1(j,B_*,F) = \inf_{B\in\mathcal{B}} \sup_{F\in\mathcal{F}_\sigma} U_1(j,B,F) \quad .$

Namely: (8.9)$\Longleftrightarrow$(8.7) and (8.10) implies:

$$\sup_{F\in\mathcal{F}_\sigma} U_1(j,B_*,F) \leq \sup_{F\in\mathcal{F}_\sigma} U_1(j,B_1,F) = 0 \quad ;$$

hence: $\quad U_1(j,B_*,F) \leq 0$ for any $F \in \mathcal{F}_\sigma$, which implies (8.8) (via lemma 7.5b)).

Then it suffices to prove:

(8.11) $\quad \inf_{B\in\mathcal{B}} \sup_{F\in\mathcal{F}_\sigma} U_1(j,B,F) = \sup_{F\in\mathcal{F}_\sigma} \inf_{B\in\mathcal{B}} U_1(j,B,F)$

and the existence of $B_* \in \mathcal{B}$ with

(8.12) $\quad \sup_{F\in\mathcal{F}_\sigma} U_1(j,B_*,F) = \inf_{B\in\mathcal{B}} \sup_{F\in\mathcal{F}_\sigma} U_1(j,B,F) \quad .$

Namely: (8.11) and (8.12) imply:

$$\sup_{F\in\mathcal{F}_\sigma} U_1(j,B_*,F) = \sup_{F\in\mathcal{F}_\sigma} \inf_{B\in\mathcal{B}} U_1(j,B,F) \quad ,$$

which implies the existence of a sequence $\{F_r\}_{r=1}^{\infty} \subset \mathcal{F}_\sigma$ with

$$U_1(j,B_*,F_r) \leq \inf_{B \in \mathcal{B}} U_1(j,B,F_r) + \frac{1}{r} \quad .$$

(8.11) is true according to corollary 8.1.1.

The existence of B_* satisfying (8.12) is implied by the compactness of $\mathcal{B}$ and the continuity of $\sup_{F \in \mathcal{F}_\sigma} U_1(j, \cdot ,F)$ on $\mathcal{B}$ (applying the same argument as in part b) of the proof of corollary 8.1.2).

Corollary 8.2: $\forall_{k,i \in N}(d_{ki} = d_{11})$, $B_0 \in \left(\mathcal{L}_\sigma\right)$, then

$$\exists_{F \in \mathcal{F}_\sigma} \forall_{B \in \mathcal{B}}\left(B_0 \overset{F}{\lesssim} B\right) \quad .$$

Proof: $B_0 \in \left(\mathcal{L}_\sigma\right) \Rightarrow \forall_{j \in N}\left({}_jB_0 \in \left({}_j\mathcal{B}\right)\right)$ (lemma 5.4e), c), a)).

${}_1B_0$ is F-best for certain $F \in \mathcal{F}_\sigma$ (theorem 8.2).

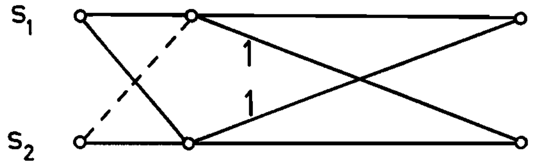
Then B_0 is best in F-sense (corollary 7.6.1), since $B_0 \in \mathcal{L}_\sigma$ implies $U(j,B_0,F) = U(\ell,B_0,F)$ (part a) of the proof of corollary 7.6.1).

APPENDIX

** In this appendix some examples have been collected. No example has been included for its value in applications. The only purpose of this appendix is to illustrate some features mentioned in the main text of this study.

The first example already illustrates the fact that the $<$ - and $\leqslant$- relations in $\mathcal{B}$ do not provide the means for the selection of a "best" decision rule. The two investigated varieties of this example (T = 1 and T = 2) furthermore illustrate the value of information collecting on behalf of decision-making.

A.1



$n = 2;\ d_{11} = d_{12} = d_{22} = 0,\ d_{21}$ large;

$\beta = 1;\ c_{11} = c_{22} = 0,\ c_{12} = c_{21} = 1;$

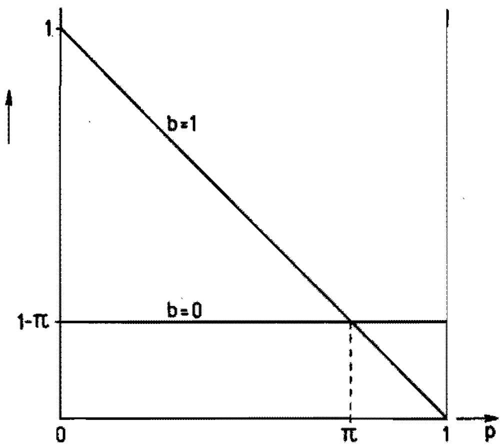
$I_\sigma = \{(2,1),(2,2)\},\ \pi_{22} =: \pi,\ p_{11} =: p.$

a) $T = 1.$

$V(1,B,P) = b_1^0(1) \cdot (1-p) + b_2^0(1) \cdot (1-\pi) =$

$$= b_1^0(1) \cdot (\pi - p) + 1 - \pi$$

call: $b_1^0(1) =: b$.



$$\left({}_1\mathcal{B}\right) = {}_1\mathcal{B}$$

$$U(1,B,F) = b\left(\pi - \int_0^1 p\,dF\right) + 1 - \pi \qquad \left(F \in \mathcal{F}_\sigma\right)$$

Hence $b = 0$ is F-optimal, if $\pi \geqslant \int_0^1 p\,dF$; otherwise $b = 1$.

For min-max risk: choose $b = 0$.

For min-max regret: choose $b = 1 - \pi$.

b) $T = 2$.

Only consider those ${}_1B$ with: $b_2^1(1,2,2) = b_2^1(1,1,2) = 1$.

$$V(1,B,P) = b_1^0(1) \cdot [(1-p) + p\{b_1^1(1,1,1) \cdot (1-p) +$$

$$+ b_2^1(1,1,1) \cdot (1-\pi)\} + (1-p)(1-\pi)]+$$

$$+ b_2^0(1)\cdot[(1-\pi)+\pi(1-\pi)+(1-\pi)\{b_1^1(1,2,1)\cdot(1-p)+$$
$$+ b_2^1(1,2,1)\cdot(1-\pi)\}].$$

call: $b_1^0(1) =: b,\ b_1^1(1,1,1) =: b_1,\ b_1^1(1,2,1) =: b_2.$

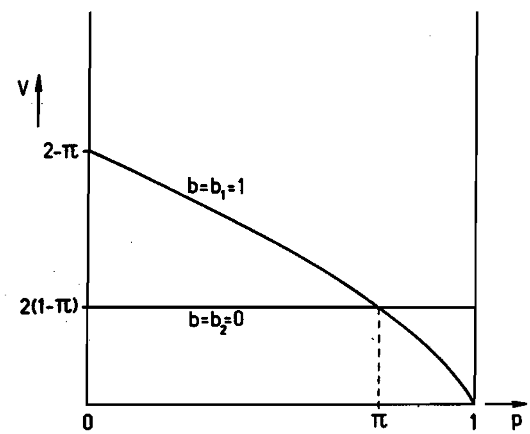
Then:

$$V(1,B,P) = -bb_1p^2 + [b(b_1\pi-1)-(1-b)(1-\pi)b_2]p +$$
$$+ (b+b_2-2-b_2\pi-bb_2+bb_2\pi)\pi+2.$$

Hence $V(1,B,P)$ is for any B a part of a parabole (or a straight line).

$$({}_1\mathcal{B}) = \{{}_1B \mid b_1=1,\ b_2=0\} \cup \{{}_1B \mid b=0,\ b_2=0\} \cup \{{}_1B \mid b=1,\ b_2=1\}$$

(the first set is the essential one)



$$U(1,B,F) = 2(1-\pi) - b[\nu_2 + (1-\pi)\nu_1 - \pi]\ ,\ \text{with}\ \nu_i = \int_0^1 p^i\, dF\ ,$$

when $b_1 = 1,\ b_2 = 0$ and $F \in \mathcal{F}_\sigma$.

Hence, when F is uniform for p : $\nu_1 = \frac{1}{2},\ \nu_2 = \frac{1}{3}$ and $b=0$ is F-optimal if $\pi \geq \frac{5}{9}$, otherwise $b=1$.

For min-max risk: choose $b=0$ (and $b_1=1,\ b_2=0$).

For min-max regret: choose $b = \frac{2(1-\pi)}{2-\pi}$ (and $b_1 = 1$, $b_2 = 0$).

Let ${}_1B_1$ and ${}_2B_2$ both have $b_1 = 1$ and $b_2 = 0$, and b-values smaller than $\frac{2(1-\pi)}{2-\pi}$ (${}_1B_1$ has the smallest one), then:

$$ {}_1B_1 \,\bar{\lesseqgtr}\, {}_1B_2 \,, \quad \text{but not} \quad {}_1B_1 \,\bar{\lesseqgtr}^{\mathcal{B}}\, {}_1B_2 \quad \text{(section 6)} \;. $$

** The second example serves to demonstrate, that in the case of equal decision costs an admissible decision rule may prescribe the application (in an essential way) of an action with expected costs, which are dominated (uniformly in P) by the expected costs of another action (section 5).

A.2

$n = 3$; D consists of zeros; $\beta = 1$; $T = 2$; $I_\sigma = \{(1,3),(2,3),(3,3)\}$, $\pi_{13} = \pi_{23} = \pi_{33} = 0$.

$$ C = \begin{pmatrix} 16 & 0 & \cdot \\ 32 & 12 & \cdot \\ 19 & 17 & \cdot \end{pmatrix} \qquad \text{(the third column is indifferent).} $$

Hence action 3 is dominated by action 1.

It will be demonstrated, that the decision rule B_0 with ${}^0b_3^0(\cdot) = {}^0b_1^1(\cdot,3,1) = {}^0b_1^1(\cdot,3,2) = 1$ is not improved by any decision rule B with $b_3^t = 0$ for $t = 0,1$.

$$ V(\cdot,B_0,P) = 19p_{31} + 17p_{32} + 16p_{31}p_{11} + p_{32}(20p_{21} + 12) \;. $$

Suppose: $V(\cdot,B_0,P) \geq V(\cdot,B,P)$ for certain B and all $P \in \mathcal{P}_\sigma$.

Then certainly for those P with $p_{11} = 1$, $p_{21} = 0$ and

$p_{11} = 1$, $p_{21} = 1$.

Or: for all $p_{31} \in [0,1]$:

$$19p_{31} + 17p_{32} + 16p_{31} + 12p_{32} \geq 16b_1^0(\bullet) + 12\left(1 - b_1^0(\bullet)\right) +$$
$$+ b_1^0(\bullet)\left[16b_1^1(\bullet,1,1) + 12\left(1 - b_1^1(\bullet,1,1)\right)\right] +$$
$$+ \left(1 - b_1^0(\bullet)\right)\left[16b_1^1(\bullet,2,2) + 12\left(1 - b_1^1(\bullet,2,2)\right)\right]$$

and:

$$19p_{31} + 17p_{32} + 16p_{31} + 32p_{32} \geq 16b_1^0(\bullet) + 32\left(1 - b_1^0(\bullet)\right) +$$
$$+ b_1^0(\bullet)\left[16b_1^1(\bullet,1,1) + 32\left(1 - b_1^1(\bullet,1,1)\right)\right] +$$
$$+ \left(1 - b_1^0(\bullet)\right)\left[16b_1^1(\bullet,2,1) + 32\left(1 - b_1^1(\bullet,2,1)\right)\right] .$$

Or:
$$29 \geq 12 + 4b_1^0(\bullet) + b_1^0(\bullet)\left(4b_1^1(\bullet,1,1) + 12\right) +$$
$$+ \left(1 - b_1^0(\bullet)\right)\left(4b_1^1(\bullet,2,2) + 12\right)$$

and:
$$35 \geq 16\left[2 - b_1^0(\bullet) + 2b_1^0(\bullet) - b_1^0(\bullet)b_1^0(\bullet,1,1) +\right.$$
$$\left. + \left(1 - b_1^0(\bullet)\right)\left(2 - b_1^1(\bullet,2,1)\right)\right] .$$

Whenever this is true, then for $b_1^1(\bullet,2,1) = 1$, $b_1^1(\bullet,2,2) = 0$:

$$\frac{5}{4} \geq b_1^0(\bullet) + b_1^0(\bullet)b_1^1(\bullet,1,1) ,$$

$$\frac{13}{16} \leq b_1^0(\bullet)b_1^1(\bullet,1,1) .$$

However the second inequality implies:

$$b_1^0(\bullet) + b_1^0(\bullet)b_1^1(\bullet,1,1) \geq 2b_1^0(\bullet)b_1^1(\bullet,1,1) \geq \frac{13}{8} .$$

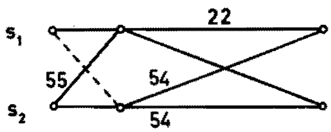
This contradicts the first inequality.

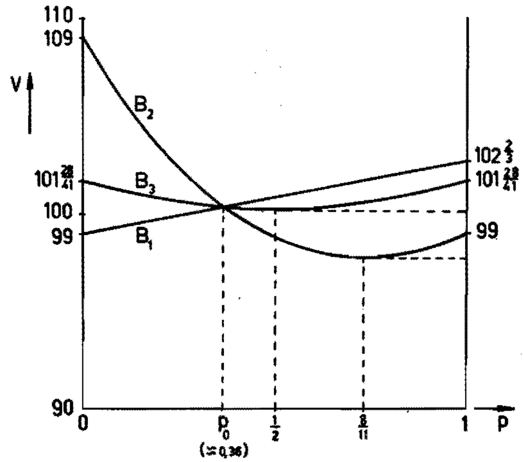
** The next example does not satisfy (6.1) (the min-max property for $V(j,B,P)$).

A.3

$n = 2$; $T = 2$; $\beta = 1$, $d_{11} = d_{22} = 0$, $d_{21} = 55$, d_{12} large; $C = \begin{pmatrix} 22 & 0 \\ 54 & 54 \end{pmatrix}$;

$I_\sigma = \{(2,1),(2,2)\}$, $\pi_{22} = \frac{5}{6}$, $p_{11} =: p$.





B_1 and B_2 are such that: ${}^1b_2^0(2) = {}^1b_2^1(2,2,2) = {}^1b_1^1(2,2,1) = 1$,

${}^2b_1^0(2) = {}^2b_2^1(2,1,2) = {}^2b_1^1(2,1,1) = 1$.

For any p one of B_1, B_2 is best (starting in s_2).

$V(2,B_1,P) = \frac{11}{3}p + 99$, $V(2,B_2,P) = 22\left(p - \frac{8}{11}\right)^2 + \frac{1071}{11}$.

$$\max_{P \in \mathcal{P}_\sigma} \min_{B \in \mathcal{B}} V(2,B,P) = V(2,B_1,p_0) \approx 100{,}32 .$$

B_3 satisfies: ${}^3b_1^0(2) = \frac{11}{41}$, ${}^3b_1^1(2,2,2) = {}^3b_1^1(2,1,2) = 0$.

B_3 minimizes the maximum risk:

$$\max_{P \in \mathcal{P}_\sigma} V(2,B_3,P) = V(2,B_3,p=0) = 101\frac{28}{41} .$$

** An example of an admissible decision rule, which does not minimize $V(j,B,P)$ for a single $P \in \mathcal{P}_\sigma$ is presented next (section 5).

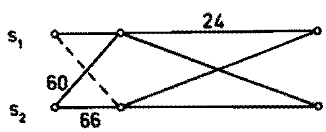
A.4

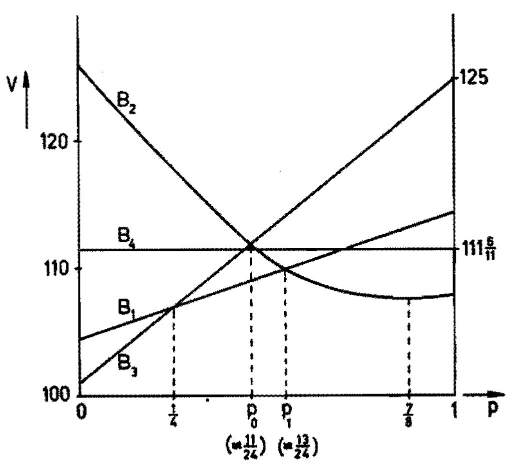
$n = 2$; $T = 2$; $\beta = 1$; $d_{11} = 0$, $d_{22} = 66$, $d_{21} = 60$, d_{12} large; $c_{11} = 24$, $c_{12} = c_{22} = c_{21} = 0$; $I_\sigma = \{(2,1),(2,2)\}$; $\pi_{22} = \frac{7}{12}$, $p_{11} =: p$.

B_1, B_2, B_3 are such that: ${}^1b_2^0(2) = {}^1b_2^1(2,2,2) = {}^1b_1^1(2,2,1) = 1$,

$${}^2b_1^0(2) = {}^2b_2^1(2,1,2) = {}^2b_1^1(2,1,1) = 1 ,$$

$${}^3b_2^0(2) = {}^3b_1^1(2,2,2) = {}^3b_1^1(2,2,1) = 1 .$$





For any p one of B_1, B_2, B_3 is best (starting in s_2).

$V(2,B_1,P) = 10p + 104\frac{1}{2}$, $V(2,B_2,P) = 24\left(p - \frac{7}{8}\right)^2 + 107\frac{5}{8}$,

$V(2,B_3,P) = 24p + 101$.

B_4 satisfies: ${}^4b_1^0(2) = \frac{5}{11}$, ${}^4b_1^1(2,1,2) = {}^4b_2^1(2,2,2) = 1$.

$V(2,B_4,P) = 111\frac{6}{11}$.

${}_2B_4$ is admissible in ${}_2\mathcal{B}$, but is best for no $P \in \mathcal{P}_\sigma$.

** Example A.5 presents a situation with equal decision costs, but no stationary sub-information decision rule which is best in F-sense for certain $F \in \mathcal{F}_\sigma$ (section 7).

A.5

$n = 3$; $T = 2$; $\beta = 1$; $d_{ki} = 0$ $(k,i \in N)$; $c_{31} = c_{32} = c_{33} =$ large, $c_{21} = c_{22} = c_{23} = -1$, $c_{11} = -2$, $c_{12} = 0$, $c_{13} = -\frac{9}{10}$;
$I_\sigma = \{(1,3),(2,1),(2,2),(2,3)\}$, $\pi_{13} = \frac{1}{4}$; F is uniform for p_{11} on $[0,\frac{3}{4}]$.

$B \in \mathcal{B}$ is optimal with respect to F, only if

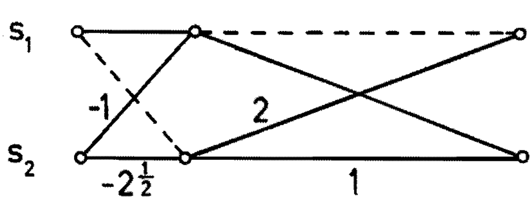
$$b_1^0(\cdot) = b_1^1(\cdot,1,1) = b_2^1(\cdot,1,2) = b_2^1(\cdot,1,3) = 1 \quad .$$

Whereas stationarity requires:

$$b_1^0(\cdot) = b_1^1(\cdot,1,3) \quad .$$

** The last example shows, that $B_0 \in \left(\mathcal{K}_\sigma \cap \mathcal{A}\right)$ is not necessarily F-optimal for certain $F \in \mathcal{F}_\sigma$ (section 8).

A.6



$n = 2$; $\beta = \frac{1}{2}$; $T = 2$; $d_{11} = 0$, d_{12} large, $d_{21} = -1$, $d_{22} = -2\frac{1}{2}$;

 $C = \begin{pmatrix} \cdot & 0 \\ 2 & 1 \end{pmatrix}$; $I_\sigma = \{(1,1),(1,2)\}$, $\pi_{11} = 0$, call $p_{22} =: p$.

$B_a, \dots, B_f$ such that:

$$^{a}b_1^1(1,1,2) = 1 \quad , \qquad \text{then} \quad V(1,B_a,P) = -\tfrac{1}{2} \quad ;$$

$$^{b}b_1^1(1,1,2) = 1 \quad , \qquad V(1,B_b,P) = -\tfrac{1}{2}p - \tfrac{1}{4} \quad ;$$

$$^{c}b_1^0(2) = {}^{c}b_1^1(2,1,2) = 1 \quad , \qquad V(2,B_c,P) = -1\tfrac{1}{2} \quad ;$$

$$^{d}b_1^0(2) = {}^{d}b_2^1(2,1,2) = 1 \quad , \qquad V(2,B_d,P) = -\tfrac{1}{2}p - 1\tfrac{1}{4} \quad ;$$

$$^{e}b_2^2(2) = {}^{e}b_1^1(2,2,2) = 1 \quad , \qquad V(2,B_e,P) = -1\tfrac{1}{2}p - \tfrac{1}{2} \quad ;$$

$$^{f}b_2^0(2) = {}^{f}b_2^1(2,2,2) = 1 \quad . \qquad V(2,B_f,P) = -\tfrac{1}{2}p^2 - 1\tfrac{1}{4}p - \tfrac{1}{2}.$$

$\mathcal{K}_\sigma \cap \mathcal{A}$ contains the following combinations:

B_1 with ${}_1B_1 = {}_1B_a$ and ${}_2B_1 = {}_2B_c$;

B_2 with ${}_1B_2 = {}_1B_a$ and ${}_2B_2 = {}_2B_e$;

B_3 with ${}_1B_3 = {}_1B_a$ and ${}_2B_3 = {}_2B_f$;

B_4 with ${}_1B_4 = {}_1B_b$ and ${}_2B_4 = {}_2B_d$;

B_5 with ${}_1B_5 = {}_1B_b$ and ${}_2B_5 = {}_2B_e$;

B_6 with ${}_1B_6 = {}_1B_b$ and ${}_2B_6 = {}_2B_f$.

All are admissible.

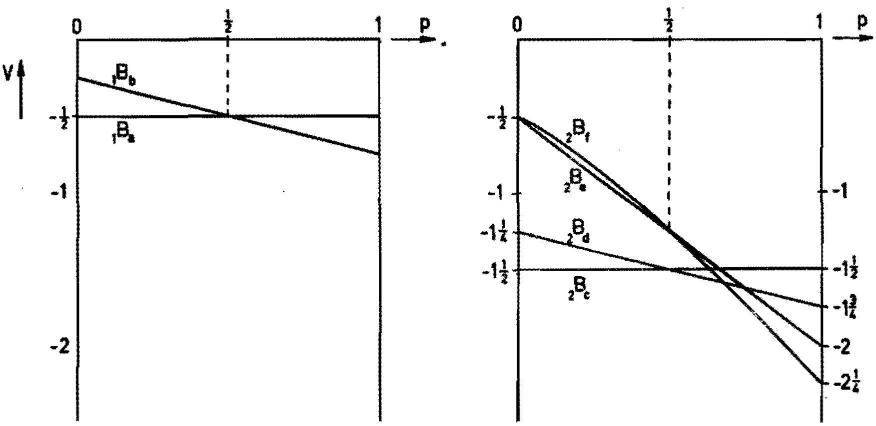
Suppose that B_3 is best for certain $F \in \mathcal{F}_\sigma$, then a contradiction is obtained by considering the requirements:

1. ${}_1B_a \overset{F}{\lesssim} {}_1B_b$;

2. ${}_2B_f \overset{F}{\lesssim} {}_2B_c$,

$$ {}_2B_f \overset{F}{\preccurlyeq} {}_2B_d \quad , $$

$$ {}_2B_f \overset{F}{\preccurlyeq} {}_2B_e \quad . $$



REFERENCES

1959, C. Berge, Espaces topologiques.
Dunod, Paris.

1962, D. Blackwell, Discrete dynamic programming.
Ann. Math. Statist. 33, 719 - 726.

1965, D. Blackwell, Discounted dynamic programming.
Ann. Math. Statist. 36, 226 - 235.

1954, D. Blackwell, M.A. Girshick, Theory of games and statistical decisions.
J. Wiley, New York.

1956, R.N. Bradt, S.M. Johnson, S. Karlin, On sequential designs for maximizing the sum of n observations.
Ann. Math. Statist. 27, 1061 - 1074.

1960, R. Howard, Dynamic programming and Markov processes.
M.I.T.-Technology Press, J. Wiley, New York.

1955, J.L. Kelley, General topology.
D. Van Nostrand Comp., New York.

1965, G. de Leve, Generalized Markovian decision processes I, II.
Mathematisch Centrum, Amsterdam.

1955, M. Loève, Probability theory.
D. Van Nostrand, New York.

1956, H. Richter, Wahrscheinlichkeitstheorie.
Springer-Verlag, Berlin.

1966, W. Schaafsma, L.J. Smid, Most stringent somewhere most powerful tests against alternatives restricted by a number of linear inequalities.
Ann. Math. Statist. 37, 1161 - 1172.

1966, D. Sworder, Optimal adaptive control systems.
Academic Press, New York.

SAMENVATTING

Het onderwerp van dit proefschrift behoort tot de theorie van de stochastische beslissingsprocessen. Een stochastisch beslissingsproces is een stochastisch proces dat van buiten af beïnvloed kan worden. In deze studie is het autonome proces een Markov keten met eindig veel toestanden, terwijl de overgangswaarschijnlijkheden niet van de tijd afhangen. Om bij een veel voorkomende praktische situatie aan te sluiten, wordt niet aangenomen dat de overgangswaarschijnlijkheden precies bekend zijn. Het proces werkt gedurende eindig veel of (aftelbaar) oneindig veel perioden. Bij de start van elke periode mag ingegrepen worden; vervolgens is er gelegenheid voor een Markov stap.

Een beslissingsregel is een voorschrift, dat in elke mogelijk voorkomende situatie (een situatie wordt niet uitsluitend bepaald door de toestand op een bepaald moment, doch mede door de hele voorgeschiedenis, want die beïnvloedt onze kennis van de onbekende overgangswaarschijnlijkheden) aangeeft hoe ingegrepen dient te worden.

Een beslissingsregel, een starttoestand en een matrix van overgangswaarschijnlijkheden bepalen tezamen een stochastisch proces. Dit wordt aangetoond in de paragrafen 2 en 3: in § 2 voor beslissingsregels die op elk tijdstip van ingrijpen loting toestaan tussen de diverse ingrijpmogelijkheden; in § 3 voor beslissingsregels die vóór de start loting toestaan tussen beslissingsregels van het type uit § 2. In § 3 wordt voorts aangetoond, dat de klassen van beslissingsregels met verschillende soorten loting

(loting alleen op de beslissingsmomenten, loting alleen bij de start, loting zowel bij de start als op de beslissingsmomenten) in wezen dezelfde verzamelingen stochastische processen opleveren. Op grond van dit resultaat kunnen de beschouwingen verder beperkt worden tot beslissingsregels met uitsluitend loting op de beslissingstijdstippen. Deze klasse wordt gekozen omdat de beslissingsregels uit deze klasse de gunstigste mogelijkheden bieden bij de bestudering van de eigenschappen van de resulterende stochastische processen, bovendien zijn ze het gemakkelijkste toe te passen. De resultaten van § 3 kunnen bovendien in volgende paragrafen nog enige keren nuttig worden gebruikt.

Verondersteld wordt voorts, dat aan de ingrepen (ingrijpen wil zeggen: het systeem in een andere toestand brengen) en aan de toestandsveranderingen ten gevolge van het autonome proces kosten zijn toegekend: respektievelijk beslissingskosten en proceskosten. Als kriterium voor de kwaliteit van een beslissingsregel wordt ingevoerd: de verwachting van de totale (verdiskonteerde) kosten van het stochastische proces. Deze verwachting wordt, behalve door de gekozen beslissingsregel, bepaald door de geldende matrix van overgangswaarschijnlijkheden en door de begintoestand van het proces.
In deze verkennende studie worden eigenschappen onderzocht van de verzameling beslissingsregels met betrekking tot de verwachte totale kosten: de z.g. risikofunktie. In het bijzonder wordt een aantal eigenschappen behandeld die van belang zijn voor de keuze van een beslissingsregel.

In § 4 wordt aangetoond, dat men zich wat betreft de ingevoerde risikofunktie, kan beperken tot het beschouwen van een deelklasse (genaamd $\overline{\mathcal{R}}_\sigma$) van $\mathcal{B}$ (de in § 2 geïntroduceerde klasse van beslissingsregels). Bij beslissingsregels uit $\overline{\mathcal{R}}_\sigma$ worden de afzonderlijke ingrepen niet bepaald door de volledige geschiedenissen tot en met het beslissingstijdstip, doch slechts door de starttoestand, het tijdstip, de momentane toestand en de aantallen geregistreerde autonome toestandsovergangen waarvan de bijbehorende overgangswaarschijnlijkheden onbekend zijn. In het geval de beslissingskosten gelijk zijn, behoeft ook niet naar de momentane toestand gekeken te worden (de betreffende verzameling beslissingsregels heet $\overline{\mathcal{L}}_\sigma$).

In § 5 wordt de partiële ordening van de beslissingsregels bestudeerd, die geïntroduceerd wordt door dominantie van het risiko als funktie van de matrix van overgangswaarschijnlijkheden. Deze partiële ordening bepaalt in het algemeen geen "beste" beslissingsregel. Daarvoor zijn andere kriteria nodig. Enige uit andere problemen welbekende kriteria, die gebaseerd zijn op de risikofunktie, worden behandeld. Namelijk: maximum risiko en maximum spijt (regret) in § 6; gewogen risiko in § 7.
Voor maximum risiko en maximum spijt bestaan optimale beslissingsregels uit $\overline{\mathcal{R}}_\sigma$. Als de beslissingskosten gelijk zijn, is voor beide kriteria een beslissingsregel uit $\mathcal{L}_\sigma$ optimaal ($\mathcal{L}_\sigma$ bevat de beslissingsregels uit $\overline{\mathcal{L}}_\sigma$ waarbij niet naar de starttoestand gekeken wordt voor de momentane ingrepen).
Bij het kriterium "gewogen risiko" bestaat een optimale beslissingsregel uit $\mathcal{R}_\sigma$ waarbij nergens geloot wordt. Als $T = \infty$ is er een optimale beslissingsregel met een zekere mate van tijdsonafhankelijkheid. Dit is ook het geval als de beslissingskosten gelijk zijn en bovendien de bekende overgangswaarschijnlijkheden volledige rijen in de Markov matrix vullen. Tevens wordt de struktuur van optimale beslissingsregels aangegeven.
Iedere beslissingsregel waarvan bij een gegeven begintoestand het risiko als funktie van de Markov matrix niet door de risikofunktie van enige andere beslissingsregel gedomineerd wordt, is optimaal voor zekere weging van de mogelijke Markov matrices (§ 8).

CURRICULUM VITAE

De schrijver van dit proefschrift werd geboren te Amsterdam op 19 januari 1939.

In 1956 behaalde hij het diploma H.B.S.-b. Vervolgens studeerde hij tot 1963 aan de Universiteit van Amsterdam. In dat jaar deed hij doktoraal examen wiskunde met als specialisatie kansrekening en statistiek en als bijvak mechanika.

Van 1960 tot 1963 was hij werkzaam als assistent van Prof.dr. J. Hemelrijk.

Sinds 1963 is hij als wetenschappelijk medewerker verbonden aan de Onderafdeling der Wiskunde van de Technische Hogeschool te Eindhoven.

Eindhoven, 30 januari 1968.

STELLINGEN

behorende bij het proefschrift van J. Wessels.

1. Door het invoeren van een belastingfunktie voor schendingen van de beperkingen, kunnen verschillende typen van stochastische programmeringsproblemen vanuit één gezichtspunt worden beschouwd.

J. Wessels, Stochastic programming.
Statistica Neerlandica 21 (1967) 39 - 53.

2. Ten onrechte beweert D.J. Wilde, dat de methode van de contourraakvlakken om het maximum te vinden van een differentieerbare, sterk eentoppige funktie op een begrensd gebied een rij punten oplevert die naar het maximumpunt konvergeert, als bij iedere stap het door hem gedefinieerde middenpunt van het overgebleven gebied wordt gekozen en dit middenpunt in het inwendige van het overgebleven gebied ligt.

D.J. Wilde, Optimization by the Method of Contour Tangents.
A.I.Ch.E.-Journal 9 (1963) 186 - 190.

3. De eerste inbeddingsstelling van Sobolev geldt voor open gebieden in R^n die voldoen aan een eenvoudige kegelvoorwaarde.
Dit volgt reeds op betrekkelijk elementaire wijze uit de oorspronkelijke uitspraak van S.L. Sobolev voor begrensde open stervormige gebieden.

S.L. Sobolev, Einige Anwendungen der Funktional-Analysis auf Gleichungen der Mathematischen Physik.
Akademie-Verlag, Berlin 1964.

4. Zij K een deelverzameling van R^n.
Voor $x \in R^n$ wordt de verzameling $A(x)$ van bereikbare richtingen gedefinieerd als de verzameling van alle $a \in R^n$, waarvoor - als a niet de nulvektor is - een n-vektorwaardige funktie Ψ van een reële variabele θ bestaat, zodat:

$$\Psi(0) = x \quad \text{en} \quad \Psi(\theta) \neq x \quad \text{als} \quad \theta \neq 0 \quad ,$$

terwijl voor zekere $\theta_1 > 0$:

$$\Psi(\theta) \in K \quad \text{als} \quad 0 < \theta < \theta_1 \quad ;$$

Ψ heeft in $\theta = 0$ een rechterafgeleide en wel:

$$\lim_{\theta \downarrow 0} \frac{\Psi(\theta) - x}{\theta} = a \quad .$$

Als er een konvexe verzameling $K_0 \subset K$ bestaat met $\bar{K}_0 = \bar{K}$, dan is $A(x)$ voor iedere $x \in R^n$ een gesloten en konvexe kegel.

5. Bij het berekenen van de kansverdeling voor de werkingsduur van een systeem uit de kansverdelingen voor de werkingsduur van de samenstellende komponenten, kan men soms op nuttige en elegante wijze gebruik maken van de theorie van de Markov processen.
Het verdient echter geen aanbeveling om, zoals G.H. Sandler doet, op deze wijze te werk te gaan bij problemen die ook eenvoudig op te lossen zijn met behulp van elementaire kansrekening.

G.H. Sandler, System reliability engineering.
Prentice-Hall, Englewood Cliffs 1963.

6. Stel in een systeem worden komponenten van een bepaald type toegepast. Aangenomen wordt dat voor de levensduur van de komponenten vermoeidheid door langdurige belasting geen rol speelt: de "conditional failure rate" λ op tijdstip t hangt slechts af van de momentane belasting $b(t)$, dus $\lambda = \lambda(b(t))$ (voor terminologie zie bv. het bij de vorige stelling genoemde boek van G.H. Sandler).
In het systeem kunnen deze komponenten op twee manieren worden toegepast:

Bij <u>methode A</u> wordt steeds één exemplaar belast en dit wordt vervangen zodra het faalt; totaal mogen n exemplaren gebruikt worden.

Bij <u>methode B</u> worden n exemplaren parallel geschakeld en gelijktijdig belast; stel: als de belasting van het systeem $b(t)$ bedraagt en er nog k exemplaren werken is de "conditional failure rate" van elk van deze exemplaren gelijk aan $a_k \lambda(b(t))$.

De kans dat het systeem op tijdstip t nog werkt bedraagt:

1) bij methode A en bij methode B als alle ka_k gelijk (en dus = 1) zijn:

$$e^{-\Lambda(t)} \sum_{j=0}^{n-1} \frac{\Lambda^j(t)}{j!} \quad ;$$

2) bij methode B als alle $ka_k = \alpha_k$ verschillend zijn:

$$\sum_{k=1}^{n} A_k e^{-\alpha_k \Lambda(t)} \quad , \quad \text{waarin} \quad A_k = \prod_{\substack{\ell=1 \\ \ell \neq k}}^{n} \frac{\alpha_\ell}{\alpha_\ell - \alpha_k} \quad .$$

Hierin is $\Lambda(t) = \int_0^t \lambda(b(\tau))d\tau$; daarbij wordt aangenomen dat deze integraal voor elke $t > 0$ bestaat.

7. Van de beide volgende uitspraken verdient de eerste de voorkeur:
"de kans dat geen verjaardagen samenvallen is reeds kleiner dan een half bij een groep van 23 door loting aangewezen personen";
"de kans dat geen verjaardagen samenvallen is nog groter dan een half bij een groep van 22 door loting aangewezen personen".

8. Bij stochastische beslissingsprocessen waarvan het kansmechanisme onvolledig gespecificeerd is, verdient, ter verkrijging van een optimaliteitskriterium, een Bayesiaanse behandeling van de gekozen risikofunktie overweging.

9. Laat $f_{\mu,\sigma}$ (μ reëel, $\sigma > 0$) de kansdichtheid zijn die een normale verdeling karakteriseert met als verwachting en standaardafwijking respektievelijk μ en σ. Stel $0 < p < 1$. De kansdichtheid h wordt gedefinieerd door:

$$h(x) = pf_{\mu_1,\sigma_1}(x) + (1-p)f_{\mu_2,\sigma_2}(x) \quad .$$

Als p, σ_1, σ_2 vast gekozen zijn en $\mu_1 - \mu_2$ doorloopt het interval $[0, \infty)$, dan is h op den duur tweetoppig. Echter, het kan voorkomen dat h eerst een stuk eentoppig is, dan tweetoppig, vervolgens weer eentoppig en daarna pas definitief tweetoppig.

J. Wessels, Multimodality in a family of probability densities, with application to a linear mixture of two normal densities.
Statistica Neerlandica <u>18</u> (1964) 267 - 282.

10. In § 7 van dit proefschrift wordt bewezen, dat bij een gegeven gewichtsfunktie op de parameterverzameling $\mathcal{P}_\sigma$ een sub-informatie beslissingsregel bestaat die optimaal is. Bovendien wordt aangetoond, dat er een dergelijke beslissingsregel is die werkt met zuivere strategieën. Het laatste is een nevenresultaat bij het bewijs van het eerste. De existentie bij een gegeven gewichtsfunktie op $\mathcal{P}_\sigma$ van een optimale beslissingsregel met zuivere strategieën kan ook worden bewezen met behulp van de resultaten over gemengde beslissingsregels in § 3 van dit proefschrift.

11. Invoering van het onderwerp Statistiek in het wiskunde programma van het Voorbereidend Wetenschappelijk Onderwijs is wenselijk ter demonstratie van de basis van elke empirische wetenschap: de mogelijkheid te oordelen over de waarde van een theorie op grond van waarnemingsmateriaal.