Mathematical analysis of a model describing the invasion of bacteria in burn wounds

Mathematical Analysis of a model describing the invasion of bacteria in burn wounds ⋆

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Abstract

We investigate a reaction-diffusion system comprising a parabolic equation coupled with an ordinary differential equation on an unbounded space domain. This system arises as a model for host tissue degradation by bacteria and involves a parameter describing the degradation rate that is typically very large. We prove the existence and uniqueness of solutions to this system and the convergence to a Stefan-like free boundary problem as the degradation rate tends to infinity.

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1 Introduction and main results

The development of alternative treatments for bacterial infections has become a major issue in recent years. The formulation and analysis of suitable mathematical models can serve as a valuable tool in understanding the basic mechanisms underlying such infections and in providing insight into possible means of fighting the spread of bacteria in host tissue. We take our cue here from [KKC+03], where a mathematical model of host tissue degradation by extracellular bacteria was introduced and analysed by formal asymptotic expansion methods. In this paper we continue and complement that investigation by a rigorous mathematical analysis.

The ability of certain extracellular bacterial pathogens, such as *Pseudomonas aeruginosa*, to degrade the tissue of an infected patient (thereby releasing nutrient to the bacteria) can result in the infection becoming lethal. The release by *Pseudomonas aeruginosa* of the virulence determinants that lead to tissue degradation is under the control of a 'quorum-sensing' system that allows the bacterial population to grow to a level at which the host immune defences can be overcome before the virulence factors are expressed (see [KKC+03] and [WKK+04] for further background and references). The rapid development of antibiotic resistance among bacteria has added urgency to the task of enhancing the understanding of such behaviour, with quorum-sensing systems providing possible alternatives targets for treatment. The model with which we are concerned here assumes that the cell signalling between bacteria which underpins quorum sensing has been effective in upregulating (almost) the entire population, leading to the release of virulence determinants wherever the bacteria are present. This obviates the need to keep explicit account of the (growing) bacterial population, the model instead involving just two variables, namely the concentration of virulence determinants and the volume fraction of healthy tissue.

Under such assumptions, the non-dimensionalised model equations take the form (see [KKC+03])

$$\partial_t u = \Delta u - u + w - \gamma k u(1-w), \tag{1.1}$$
$$\partial_t w = k u(1-w), \tag{1.2}$$

where u, w are time and space dependent functions and $\gamma, k > 0$ are fixed parameters. The variable u describes the concentration of degradative enzymes produced by the bacteria, and $(1-w)$ corresponds to the volume fraction of healthy tissue, the population density of bacteria being taken to be proportional to w. The key parameter $k > 0$ is typically very large in practice and governs the degradation rate of the tissue. The upper half space is chosen as the spatial domain and initial and boundary data are prescribed. By a formal asymptotic expansion analysis in [KKC+03], a free boundary problem was obtained in the limit $k \to \infty$. There, for unknowns u, w, the upper half space splits into a region where $u = 0$ and a second region where $u > 0$ and $w = 1$. The common boundary of these two regions moves according to a Stefan-like condition.

We include in our analysis the possibility of a diffusion term in (1.2). This might be of interest in other applications and is convenient for mathematical purposes. To give a precise formulation of the problem, denote the upper half space of $\mathbb{R}^n$ by $\mathbb{R}^n_+ := \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ and by $\vec{e}_n = (0, ..., 0, 1)^T$ the n-th standard unit vector. Moreover, let a time interval $(0, T)$ be given and set

$$\begin{aligned} Q_T &:= (0,T) \times \mathbb{R}^n_+, \\ S_T &:= (0,T) \times (\mathbb{R}^{n-1} \times \{0\}). \end{aligned}$$

We consider initial data $\bar{u}_0, \bar{w}_0$ with

$$\bar{u}_0 \in L^1(\mathbb{R}^n_+), \qquad 0 \le \bar{u}_0 \le 1, \tag{1.3}$$

$$\bar{w}_0 \in L^1(\mathbb{R}^n_+), \qquad 0 \le \bar{w}_0 \le 1. \tag{1.4}$$

In applications a typical choice for w_0 is a characteristic function of a set, thus motivating our assumptions on the regularity of the initial data.
Finally, we define for an open interval $I \subset \mathbb{R}$ and an open set $\Omega \subset \mathbb{R}^n$ the spaces

$$\begin{aligned} W^{1,2}_p(I \times \Omega) \qquad &:= H^{1,p}(I; L^p(\Omega)) \cap L^p(I; H^{2,p}(\Omega)), \\ W^{1,2}_{p,\mathrm{loc}}((0,T) \times \Omega) &:= \bigcap_{\delta, L > 0} W^{1,2}_p\Big((\delta, T) \times (\Omega \cap B_L(0)\Big). \end{aligned}$$

We will concern ourselves with the following problem.
Problem (P_k). *Let $k > 0$, $d \ge 0$. Find functions $u_k, w_k : Q_T \to \mathbb{R}$ with*

$$\begin{aligned} &u_k, w_k \text{ smooth in } Q_T, \\ &u_k, w_k \in C^0([0,T]; L^1(\mathbb{R}^n_+)) \end{aligned}$$

if $d > 0$ and

$$\begin{aligned} u_k &\in W^{1,2}_{p,loc}(Q_T) \cap C^0([0,T]; L^1(\mathbb{R}^n_+)), \\ w_k &\in H^{1,\infty}(0,T; L^\infty(\mathbb{R}^n_+)) \cap C^\alpha([0,T]; L^p(\mathbb{R}^n_+)) \end{aligned}$$

for all $1 < p < \infty$, $0 < \alpha < 1$ in the case $d = 0$, such that the equations

$$\partial_t u_k = \Delta u_k - u_k + w_k - \gamma k u_k(1-w_k), \tag{1.5}$$
$$\partial_t w_k = d\Delta w_k + k u_k(1-w_k) \tag{1.6}$$

hold almost everywhere in Q_T, the boundary conditions

$$\nabla u_k \cdot \vec{e}_n = 0, \tag{1.7}$$
$$\nabla w_k \cdot \vec{e}_n = 0 \quad \text{if } d > 0 \tag{1.8}$$

hold almost everywhere on S_T and the initial conditions

$$u_k(0,.) = \bar{u}_0, \qquad w_k(0,.) = \bar{w}_0 \tag{1.9}$$

are satisfied almost everywhere in $\mathbb{R}^n_+$.

Our main result on the existence and uniqueness of solutions for (P_k) is stated in the following theorem.

Theorem 1. *Let $d \geq 0$ and $\bar{u}_0, \bar{w}_0$ be given as in* (1.3), (1.4). *Then there exists for all $k > 0$ a unique solution (u_k, w_k) of Problem (P_k). Moreover u_k, w_k satisfy the estimates*

$$\begin{aligned}
0 \leq u_k, w_k \leq 1,& \\
\int_{\mathbb{R}^n_+} (u_k + \gamma w_k)(t,.) \leq e^{\frac{1}{\gamma}t} \int_{\mathbb{R}^n_+} (\bar{u}_0 + \gamma \bar{w}_0),& \\
\int_0^T \int_{\mathbb{R}^n_+} k u_k (1-w_k) \leq c(\gamma,T) \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n_+)},& \\
\int_0^T \int_{\mathbb{R}^n_+} |\nabla u_k|^2 \leq c(\gamma,T) \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n_+)},&
\end{aligned}$$

and

$$\int_0^T \int_{\mathbb{R}^n_+} d|\nabla w_k|^2 \leq c(\gamma,T) \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n_+)}$$

if $d > 0$, where $c(\gamma,T)$ only depends on γ, T.

The proof of Theorem 1 is given in section 2.

In section 3 we prove that the solutions (u_k, w_k) converge to the unique solution of a Stefan-like free boundary problem as k tends to infinity. Since we lose regularity in the limit we have to give a weak formulation of this problem.

Problem (P_∞). *Let $d \geq 0$. Find functions $u_\infty, w_\infty : Q_T \to \mathbb{R}$ with*

$$\begin{aligned}
&u_\infty \in L^2(0,T;H^{1,2}(\mathbb{R}^n_+)) \cap L^\infty(0,T;L^1(\mathbb{R}^n_+)), \\
&w_\infty \in L^\infty(0,T;L^1(\mathbb{R}^n_+)), \\
&w_\infty \in L^2(0,T;H^{1,2}(\mathbb{R}^n_+)) \text{ if } d > 0,
\end{aligned}$$

such that

$$u_\infty(1-w_\infty)=0 \tag{1.10}$$

is satisfied almost everywhere in Q_T and

$$\partial_t(u_\infty+\gamma w_\infty) = \Delta(u_\infty+\gamma d w_\infty) - u_\infty + w_\infty \quad \text{in } Q_T, \tag{1.11}$$
$$\nabla(u_\infty+\gamma d w_\infty)\cdot \vec{e}_n = 0 \quad \text{on } S_T, \tag{1.12}$$
$$(u_\infty+\gamma w_\infty)(0,.) = \bar{u}_0+\gamma\bar{w}_0 \quad \text{on } \mathbb{R}^n_+ \tag{1.13}$$

hold in the sense that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n_+}\partial_t\zeta\big(u_\infty+\gamma w_\infty-(\bar{u}_0+\gamma\bar{w}_0)\big)\\ =&\int_0^T\int_{\mathbb{R}^n_+}\Big(\nabla\zeta\cdot\nabla(u_\infty+\gamma d w_\infty)+\zeta(u_\infty-w_\infty)\Big)\end{aligned} \tag{1.14}$$

for all $\zeta\in H^{1,2}(Q_T)$ with $\zeta(T,.)=0$ in $\mathbb{R}^n_+$.

Our result on the convergence to the limit problem is stated in the following theorem.

Theorem 2. *Let $d\geq 0$ and $\bar{u}_0,\bar{w}_0$ be given as in* (1.3), (1.4) *and denote for $k\in\mathbb{N}$ by (u_k,w_k) the solution of Problem (P_k) with initial data $(\bar{u}_0,\bar{w}_0)$. Then there are functions u_∞, w_∞ such that*

$$u_k \to u_\infty, \quad w_k \to w_\infty \quad \text{in } L^1(Q_T)$$

as k tends to infinity. Moreover $0\leq u_\infty, w_\infty\leq 1$ holds and (u_∞,w_∞) is the unique solution of Problem (P_∞).

This theorem is proved in section 3 where also the "classical" formulation of this problem is given. We remark that the limit problem can be reformulated as a scalar parabolic equation with nonlinear (and in the case $d=0$ degenerate) diffusion.

Problem ($\tilde{P}_\infty$). *Assume $d\geq 0$ and define $\varphi, h:\mathbb{R}\to\mathbb{R}$,*

$$\varphi(r) := d(r+\gamma)+(1-d)r_+, \qquad h(r) := \frac{1}{\gamma}r-(1+\frac{1}{\gamma})r_+ + 1, \tag{1.15}$$

where $r_+ := \max(r,0)$ denotes the positive part of $r\in\mathbb{R}$.
Let $z_0\in L^1(\mathbb{R}^n_+)\cap L^\infty(\mathbb{R}^n_+)$ be given. Find a function $z:\mathbb{R}^n_+\to\mathbb{R}$ with

$$z\in L^\infty(Q_T),\ z+\gamma, z_+\in L^\infty(0,T;L^1(\mathbb{R}^n_+)),\ \varphi(z)\in L^2(0,T;H^{1,2}(\mathbb{R}^n_+)),$$

such that z satisfies

$$\begin{aligned} \partial_t z &= \Delta\varphi(z) + h(z) && \text{in } Q_T, && (1.16)\\ \nabla\varphi(z)\cdot\vec{e}_n &= 0 && \text{on } S_T, && (1.17)\\ z(0,.) &= z_0 && \text{in } \mathbb{R}^n_+ && (1.18)\end{aligned}$$

in the weak sense that

$$0 = \int_0^T \int_{\mathbb{R}^n_+} \partial_t\zeta(z - z_0) - \nabla\zeta\cdot\nabla\varphi(z) + \zeta h(z) \qquad (1.19)$$

holds for all $\zeta \in H^{1,2}(Q_T)$ with $\zeta(T,.) = 0$ in $\mathbb{R}^n_+$.

We will prove that the problems (P_∞) and $(\tilde{P}_\infty)$ are equivalent in the following sense.

Theorem 3. *Let $\bar{u}_0, \bar{w}_0$ be given as in (1.3), (1.4) and let u_∞, w_∞ be the unique solution of Problem (P_∞) with initial data $\bar{u}_0, \bar{w}_0$. Then*

$$z := u_\infty - \gamma(1 - w_\infty) \qquad (1.20)$$

is the unique solution of Problem $\tilde{P}_\infty$ with initial datum $\bar{u}_0 + \gamma(\bar{w}_0 - 1)$. Conversely let $\hat{z}$ be a solution of Problem $(\tilde{P}_\infty)$ with initial datum $\hat{z}_0 \in L^1(\mathbb{R}^n_+)$ and $-\gamma \le \hat{z}_0 \le 1$. Define functions $\hat{u}, \hat{w}$ and $\hat{u}_0, \hat{w}_0$ by

$$\hat{u} := \hat{z}_+, \qquad \hat{w} := 1 + \frac{1}{\gamma}(\hat{z} - \hat{z}_+) \qquad (1.21)$$

$$\hat{u}_0 := (\hat{z}_0)_+, \qquad \hat{w}_0 := 1 + \frac{1}{\gamma}\Big(\hat{z}_0 - (\hat{z}_0)_+\Big). \qquad (1.22)$$

Then $(\hat{u}, \hat{w})$ is the solution of (P_∞) with $(\hat{u} + \gamma\hat{w})(0,.) = \hat{u}_0 + \gamma\hat{w}_0$.

Let us point out some key ingredients of our analysis. First, due to the monotonic structure of the lower order terms in the equations, an invariant region principle holds for (1.5), (1.6). In particular, if the initial data satisfy (1.3), (1.4) the functions u_k, w_k attain for all times their values in $[0,1]$, which is the range of meaningful values in view of the original model.
If we multiply (1.6) by γ and add (1.5) we deduce that equation (1.11) holds for u_k, w_k. Formally, an integration of (1.11) over $(0,t)\times\mathbb{R}^n_+$ and application of the Gronwall Lemma yields a priori estimates in $L^1(Q_T)$, uniformly in $k > 0$. To make the arguments rigorous, we first assume a positive diffusion coefficient in (1.6) and a bounded space region. The existence of a solution is immediate due to general existence result in [Lun95]. The passage to an unbounded space domain is justified with the help of a comparison principle and a priori estimates for the solutions. To prove the existence of solutions in the case $d = 0$ and to justify the limit as k tends to infinity, we derive uniform estimates

for time and space differences and prove the compactness of the approximate solutions.

It is striking that the expressions of the limit free boundary problem (1.10)-(1.13) or (1.16)-(1.18) hold in both cases, i.e. $d > 0$ and $d = 0$. In the case that the free boundary is smooth, we show in Proposition 6 that a Rankine-Hugoniot type condition is satisfied on the free boundary. This condition reduces to a transmission condition across the free boundary in the case that $d > 0$ (see Remark 2) and to a Stefan condition in the case that $d = 0$ with the initial data being that from the biological model. Related results were proven before; we refer to [DHMP99], [CDH$^+$04] and [HIMN01] for the case that $d > 0$ and to [HvdHP00], [HvdHP97], [HvdHP96] and [EHvdHP01] for the case that $d = 0$, but this is the first time that a single expression has been given for the limit problem which permits in particular the recovery both of a two-phase Stefan problem with 'zero latent heat' in the case that $d > 0$ and of a one-phase Stefan problem in the case that $d = 0$ with the special initial condition holding.

In a forthcoming paper, [HKR], we will prove the existence of one-dimensional travelling wave solutions for (1.5), (1.6) and investigate a nonlinear selection principle for the minimal speed of travelling waves.

2 Existence of solutions for the RD system

In this section we prove Theorem 1. Instead of working in the upper half-space, it is convenient to consider the problem on the whole space. We extend the initial data to the whole of $\mathbb{R}^n$ by

$$\bar{u}_0(x_1, ..., x_n) := \bar{u}_0(x_1, ..., -x_n), \quad \bar{w}_0(x_1, ..., x_n) := \bar{w}_0(x_1, ..., -x_n), \tag{2.1}$$

for $x = (x_1, ..., x_n)$ with $x_n < 0$. Then, Theorem 1 will follow from the corresponding result for the whole space and from the uniqueness of solutions.
Moreover, in addition to (1.3), (1.4) we first assume that both initial data are continuous and have compact support, so that the extended initial data satisfy

$$\bar{u}_0 \in C_c^0(\mathbb{R}^n), \qquad 0 \le \bar{u}_0 \le 1, \tag{2.2}$$
$$\bar{w}_0 \in C_c^0(\mathbb{R}^n), \qquad 0 \le \bar{w}_0 \le 1. \tag{2.3}$$

2.1 Uniformly parabolic system, bounded domain

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary, such that the supports of $\bar{u}_0, \bar{w}_0$ are included in Ω, and set $\Omega_T := (0,T) \times \Omega$. For an arbitrary positive constant $d > 0$ we consider the equations (1.5), (1.6) in Ω_T and obtain the following existence result.

Proposition 1. *Let $k, d > 0$ and assume that $\bar{u}_0, \bar{w}_0$ satisfy* (2.2), (2.3)*. Then there exists a solution u, w of* (1.5), (1.6) *with*

$$u, w \in C^{1,2}([\delta, T] \times \overline{\Omega}) \cap C^0(\overline{\Omega_T}) \tag{2.4}$$

for all $\delta > 0$, such that

$$\begin{aligned} u &= 0, & w &= 0 & &\text{on } (0,T) \times \partial\Omega, & (2.5)\\ u(0,.) &= \bar{u}_0, & w(0,.) &= \bar{w}_0 & &\text{in } \Omega. & (2.6) \end{aligned}$$

Moreover, u, w satisfy the estimates

$$\begin{aligned} 0 \;\le\; u, w \;&\le\; 1, & (2.7)\\ \int_\Omega (u + \gamma w)(t,.) \;&\le\; e^{\frac{1}{\gamma}t} \int_\Omega (\bar{u}_0 + \gamma \bar{w}_0), & (2.8)\\ \int_{\Omega_T} ku(1-w) \;&\le\; c(\gamma, T)\|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\Omega)}. & (2.9) \end{aligned}$$

We give the proof of Proposition 1 after the next lemma, in which we establish a comparison principle. To apply this to functions with values possibly not in $[0,1]$ we modify the nonlinear terms in (1.5), (1.6) by terms with a linear growth rate. Define a cut-off function

$$g(r) := \begin{cases} r & \text{if } -1 < r < 2, \\ -1 & \text{if } r \le -1, \\ 2 & \text{if } r \ge 2, \end{cases} \tag{2.10}$$

and consider the equations

$$\begin{aligned} \partial_t u &= \Delta u - u + w - \gamma k u(1 - g(w)), & (2.11)\\ \partial_t w &= d\Delta w + ku(1 - g(w)). & (2.12) \end{aligned}$$

Notice that for $0 \le u, w \le 1$ these equations are identical to (1.5), (1.6).

Lemma 1. *Let u, w and $\tilde{u}, \tilde{w}$ be two solutions of* (2.11), (2.12) *as in* (2.4) *with* $0 \leq u, w \leq 1$ *or* $0 \leq \tilde{u}, \tilde{w} \leq 1$ *such that*

$$\begin{aligned} u(0,.) \geq \tilde{u}(0,.), \quad w(0,.) &\geq \tilde{w}(0,.), \qquad &&\text{in } \Omega, &(2.13)\\ u \geq \tilde{u}, \qquad w &\geq \tilde{w} &&\text{on } (0,T)\times\partial\Omega &(2.14)\end{aligned}$$

are satisfied. Then the inequalities

$$u \geq \tilde{u}, \qquad w \geq \tilde{w} \tag{2.15}$$

hold in Ω_T.

Proof. We prove the claim for the case $0 \leq \tilde{u}, \tilde{w} \leq 1$. Define

$$U := u - \tilde{u}, \qquad W := w - \tilde{w}$$

and observe that U, W satisfy

$$0 = U_t - \Delta U + U - W + \gamma k(1 - g(w))U - \gamma k\tilde{u}(g(w) - g(\tilde{w})), \tag{2.16}$$

$$0 = W_t - d\Delta W - k(1 - g(w))U + k\tilde{u}(g(w) - g(\tilde{w})). \tag{2.17}$$

Let $U_- := \min(0, U), W_- := \min(0, W)$ denote the negative parts of U, W, and note that $U_-(t,x) = W_-(t,x) = 0$ holds if $x \in \partial\Omega$ or $t = 0$. Multiplying (2.16) by U_- and integrating over $(t_0, t)\times\Omega$ we obtain after some partial integrations

$$\begin{aligned} 0 &= \int_\Omega \frac{1}{2}U_-^2(t,.) - \int_\Omega \frac{1}{2}U_-^2(t_0,.) + \int_{t_0}^t\int_\Omega \Big(|\nabla U_-|^2 + U_-^2\Big)\\ &\quad - \int_{t_0}^t\int_\Omega \Big(U_- W - \gamma k(1-g(w))U_-^2 + \gamma k\tilde{u}(g(w) - g(\tilde{w}))U_-\Big)\\ &\geq \int_\Omega \frac{1}{2}U_-^2(t,.) - \int_\Omega \frac{1}{2}U_-^2(t_0,.)\\ &\quad - \int_{t_0}^t\int_\Omega \Big(U_- W_- + \gamma k U_-^2 + \gamma k\tilde{u}(g(w) - g(\tilde{w}))U_-\Big) \end{aligned} \tag{2.18}$$

for all $t \in (0,T)$. For the last term on the right-hand side we observe that

$$\begin{aligned} \int_{t_0}^t\int_\Omega -\gamma k\tilde{u}(g(w) - g(\tilde{w}))U_- &= \int_{t_0}^t\int_{\{W\geq 0\}} -\gamma k\tilde{u}(g(w) - g(\tilde{w}))U_-\\ &\quad + \int_{t_0}^t\int_{\{W<0\}} -\gamma k\tilde{u}(g(w) - g(\tilde{w}))U_-\\ &\geq \int_{t_0}^t\int_{\{W<0\}} -\gamma k\tilde{u}(g(w) - g(\tilde{w}))U_-, \end{aligned}$$

since g is monotonically increasing and $\tilde{u} \geq 0$. Using the Lipschitz continuity of g we find

$$\int_{t_0}^t \int_{\{W<0\}} -k\tilde{u}(g(w)-g(\tilde{w}))U_- \geq -\int_{t_0}^t \int_\Omega \gamma k W_- U_-.$$

Recalling (2.18), using the Cauchy inequality and letting $t_0 \searrow 0$ we obtain

$$\int_\Omega \frac{1}{2}U_-^2(t,.) \leq \int_0^t \int_\Omega \frac{1}{2}U_-^2 + \frac{1}{2}W_-^2 + \gamma k U_-^2 + \frac{\gamma k}{2}W_-^2 + \frac{\gamma k}{2}U_-^2. \tag{2.19}$$

Analogously, multiplying (2.17) by W_- and integrating over $(t_0,t)\times\Omega$ yields

$$\begin{aligned}
\int_\Omega \frac{1}{2}W_-^2(t,.) &= \lim_{t_0\to 0}\Big(\int_\Omega \frac{1}{2}W_-^2(t,.) - \int_\Omega \frac{1}{2}W_-^2(t_0,.)\Big)\\
&\leq \lim_{t_0\to 0}\int_{t_0}^t \int_\Omega \Big(k(1-g(w))UW_- - k\tilde{u}(g(w)-g(\tilde{w}))W_-\Big)\\
&\leq \lim_{t_0\to 0}\int_{t_0}^t \int_\Omega k(1-w)U_-W_-\\
&\leq \int_0^t \int_\Omega \frac{k}{2}(U_-^2 + W_-^2)
\end{aligned}$$

for all $t \in (0,T)$, where in the second inequality we have used that g is monotonically increasing and that $g(w) = w \leq 1$ holds in $\{W<0\}$. Adding this inequality to (2.19) we obtain for all $t>0$

$$\int_\Omega \Big(U_-^2 + W_-^2\Big)(t,.) \leq C(\gamma,k)\int_0^t \int_\Omega \Big(U_-^2 + W_-^2\Big).$$

Thus the Gronwall Lemma yields $U_-, W_- = 0$ in Ω_T, which proves (2.15). $\square$

Proof of Proposition 1. According to [Lun95] Proposition 7.3.2 there exist $0 < T_* \leq T$ and $u,w \in C^{1,2}([\delta,T_*]\times\overline{\Omega})$, $u,w \in C^0(\overline{\Omega_{T_*}})$ for all $\delta>0$ such that (u,w) is a solution of (2.11), (2.12) with homogeneous Dirichlet boundary values on $\partial\Omega$ and initial data $\bar{u}_0, \bar{w}_0$. We observe that the constant functions $(1,1)$ and $(0,0)$ are also solutions of (2.11), (2.12). Since $0 \leq \bar{u}_0, \bar{w}_0 \leq 1$, we obtain from the comparison principle Lemma 1 that u,w are uniformly bounded by zero and unity, which gives (2.7). This implies that solutions exist on the whole time interval, thus $T_* = T$ holds. Recalling that $g(r) = r$ on $[0,1]$ we get that (u,w) solve (1.5), (1.6) in Ω_T with boundary and initial conditions (2.5), (2.6).

To prove (2.8) we multiply equation (1.6) by γ and add equation (1.5) to obtain

$$\partial_t(u+\gamma w) = \Delta(u+\gamma d w) - u + w. \tag{2.20}$$

If we integrate this equation over Ω we deduce

$$\partial_t \int_\Omega (u+\gamma w)(t,.) = \int_\Omega \Delta(u+\gamma dw)(t,.) - (u-w)(t,.)$$
$$\leq \frac{1}{\gamma}\int_\Omega (u+\gamma w)(t,.) \tag{2.21}$$

for almost all $t \in (0,T)$, where in the second inequality we have used that the outer normal derivative of $(u+\gamma w)(t,.)$ on $\partial\Omega$ is nonpositive since $u(t,.), w(t,.)$ equal zero on $\partial\Omega$ and are nonnegative in Ω. By the Gronwall Lemma (2.21) yields (2.8). If we now integrate equation (1.5) over $(t_0,t)\times\Omega$ we obtain

$$\begin{aligned}
&\int_{t_0}^t \int_\Omega \gamma k u(1-w) \\
&\leq \int_\Omega u(t_0,.) - \int_\Omega u(t,.) - \int_{t_0}^t \int_\Omega (u-w) \\
&\leq \int_\Omega u(t_0,.) + \int_{t_0}^t \int_\Omega w \\
&\leq \int_\Omega u(t_0,.) + e^{\frac{T}{\gamma}} \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\Omega)}
\end{aligned}$$

for all $t_0, t \in (0,T)$, where we have used (2.8). Letting $t_0 \to 0$ this gives (2.9). □

For use in the following sections we now derive some additional estimates. We start by proving that u, w are uniformly bounded in $L^2(0,T;\mathrm{H}^{1,2}(\Omega))$. If we multiply (1.5) and (1.6) by u and w respectively and integrate over $(t_0,t)\times\Omega$, where $0 < t_0 \leq t < T$, we obtain

$$\frac{1}{2}\int_\Omega u^2(t,.) + \int_{t_0}^t \int_\Omega \Big(|\nabla u|^2 + u^2\Big) \leq \frac{1}{2}\int_\Omega u(t_0,.)^2 + \int_{t_0}^t \int_\Omega u$$

and

$$\frac{1}{2}\int_\Omega w^2(t,.) + \int_{t_0}^t \int_\Omega d|\nabla w|^2 \leq \frac{1}{2}\int_\Omega w(t_0,.)^2 + \int_{t_0}^t \int_\Omega ku(1-w).$$

Letting t_0 tend to zero and using (2.8) we obtain the *energy estimates*

$$\frac{1}{2}\int_\Omega u^2(t,.) + \int_0^t \int_\Omega |\nabla u|^2 \leq \frac{1}{2}\int_\Omega \bar{u}_0^2 + tC(\gamma,T)\|\bar{u}_0 + \gamma\bar{w}_0\|_{L^1(\Omega)}, \tag{2.22}$$
$$\frac{1}{2}\int_\Omega w^2(t,.) + \int_0^t \int_\Omega d|\nabla w|^2 \leq \frac{1}{2}\int_\Omega \bar{w}_0^2 + \int_0^t \int_\Omega ku(1-w). \tag{2.23}$$

Next we derive estimates for u, w in $W^{1,2}_{p,loc}(\Omega_T)$. Let $\delta > 0$ and $L > 0$ such that $B_{2L}(0) \subset \Omega$. Then it follows from [Lie96], Theorem 7.13, that

$$\|u\|_{W_p^{1,2}((\delta,T)\times B_L(0))} \leq C(\delta,L)\|u-w+\gamma ku(1-w)\|_{L^p(\Omega_T)}, \tag{2.24}$$

$$\|w\|_{W_p^{1,2}((\delta,T)\times B_L(0))} \leq C(\delta,L,d)\|ku(1-w)\|_{L^p(\Omega_T)} \tag{2.25}$$

hold for all $1 < p < \infty$. We can now estimate the norm on the right hand side of (2.24) via

$$\begin{aligned}
\int_{\Omega_T} |u-w+\gamma ku(1-w)|^p &\leq C_p \int_{\Omega_T} \Big(u^p + w^p + \gamma^p k^p u^p (1-w)^p\Big) \\
&\leq C_p \Big(\int_{\Omega_T} u + \int_{\Omega_T} w + \gamma^p k^{p-1} \int_{\Omega_T} ku(1-w)\Big) \\
&\leq C(p,\gamma,T,k)\|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\Omega)},
\end{aligned}$$

where we have used (2.8), (2.9) in the last inequality. Thus, using (2.24), we deduce that

$$\|u\|_{W_p^{1,2}((\delta,T)\times B_L(0))} \leq C(\delta,L,p,\gamma,T,k)\|\bar{u}_0 + \gamma\bar{w}_0\|_{L^1(\Omega)}^{1/p} \tag{2.26}$$

holds for all $1 < p < \infty$. In a similar way we can prove from (2.25) that

$$\|w\|_{W_p^{1,2}((\delta,T)\times B_L(0))} \leq C(\delta,L,d,p,\gamma,T,k)\|\bar{u}_0 + \gamma\bar{w}_0\|_{L^1(\Omega)}^{1/p} \tag{2.27}$$

is satisfied for all $1 < p < \infty$.

2.2 Uniformly parabolic system, unbounded domain

Next we prove the existence of solutions for the uniformly parabolic system on $(0,T)\times\mathbb{R}^n$.

Proposition 2. *Let $\bar{u}_0, \bar{w}_0$ satisfy (2.2), (2.3) and let $d,k > 0$. Then there exists a pair of functions (u,w) with*

$$\begin{aligned}
&u, w \text{ smooth in } (0,T)\times\mathbb{R}^n, \\
&u, w \in C^0([0,T]; L^1(\mathbb{R}^n))
\end{aligned}$$

such that (u,w) satisfy equations (1.5), (1.6) on $(0,T)\times\mathbb{R}^n$ and the initial conditions

$$u(0,.) = \bar{u}_0, \qquad\qquad w(0,.) = \bar{w}_0 \tag{2.28}$$

in $\mathbb{R}^n$*, and such that the following estimates hold:*

$$0 \le u, w \le 1, \tag{2.29}$$

$$\int_{\mathbb{R}^n} (u+\gamma w)(t,.) \le e^{\frac{1}{\gamma}t} \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)}, \tag{2.30}$$

$$\int_0^T \int_{\mathbb{R}^n} ku(1-w) \le c(\gamma, T)\|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)} \tag{2.31}$$

and, moreover,

$$\frac{1}{2}\int_{\mathbb{R}^n} u^2(t,.) + \int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 \le \frac{1}{2}\int_{\mathbb{R}^n} \bar{u}_0^2 + tc(\gamma, T)\|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)}, \tag{2.32}$$

$$\frac{1}{2}\int_{\mathbb{R}^n} w^2(t,.) + \int_0^t \int_{\mathbb{R}^n} d|\nabla w|^2 \le \frac{1}{2}\int_{\mathbb{R}^n} \bar{w}_0^2 + \int_0^t \int_{\mathbb{R}^n} ku(1-w) \tag{2.33}$$

for almost all $t \in (0,T)$ *and*

$$\|u\|_{W_p^{1,2}((\delta,T)\times B_L(0))} \le C(\delta, L, p, \gamma, T, k)\|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)}^{1/p}, \tag{2.34}$$

$$\|w\|_{W_p^{1,2}((\delta,T)\times B_L(0))} \le C(\delta, L, d, p, \gamma, T, k)\|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)}^{1/p} \tag{2.35}$$

for all $\delta, L > 0$*,* $1 < p < \infty$*.*

Proof. Choose $R_0 > 0$ such that the supports of $\bar{u}_0, \bar{w}_0$ are included in $B_{R_0}(0)$. For $R > R_0$ denote by (u^R, w^R) the solution of (1.5), (1.6) on $(0,T) \times B_R(0)$ with initial data $(\bar{u}_0, \bar{w}_0)$ and zero Dirichlet boundary data on $(0,T) \times \partial B_R(0)$. The existence of such a solution is ensured by Proposition 1. We show that u^R, w^R converge to solutions of (1.5), (1.6) on $(0,T) \times \mathbb{R}^n$.
First we observe that u^R, w^R, extended by zero to functions on the whole of $(0,T) \times \mathbb{R}^n$, are monotone in $R > 0$; for all $R > R_0$

$$0 \le u^R \le u^{R+1} \le 1,$$
$$0 \le w^R \le w^{R+1} \le 1$$

hold in $(0,T) \times \mathbb{R}^n$. In fact, both u^R, w^R and u^{R+1}, w^{R+1} solve (1.5), (1.6) in $(0,T) \times B_R(0)$ with $u^R(0,.) = u^{R+1}(0,.)$ and $u^R \le u^{R+1}$ on $(0,T) \times \partial B_R(0)$. The comparison principle Lemma 1 yields $u^R \le u^{R+1}$ in $B_R(0)$, whereas on $B_{R+1}(0) \setminus B_R(0)$ we immediately find $u^R = 0 \le u^{R+1}$.
The Monotone Convergence Theorem and (2.7), (2.8) yield that the limits

$$u := \lim_{R\to\infty} u^R, \tag{2.36}$$

$$w := \lim_{R\to\infty} w^R, \tag{2.37}$$

exist in $L^1((0,T) \times \mathbb{R}^n)$ and that (2.29), (2.30), (2.31) hold. Since we have extended the functions u^R, w^R by zero to the whole of R^n we see from (2.22), (2.23) that u^R, w^R are uniformly bounded in $L^2(0,T;H^{1,2}(\mathbb{R}^n))$. Therefore we we obtain

$$u^R \rightharpoonup u \text{ weakly in } L^2(0,T;H^{1,2}(\mathbb{R}^n)), \tag{2.38}$$
$$w^R \rightharpoonup w \text{ weakly in } L^2(0,T;H^{1,2}(\mathbb{R}^n)), \tag{2.39}$$

for $R \to \infty$ since we can identify any weak limit point of u^R with u and any weak limit point of w^R with w, due to (2.36), (2.37). Moreover, by the lower-semicontinuity of the norm under weak convergence we deduce that (2.32), (2.33) hold.
The bounds (2.26), (2.27) show that

$$u, w \in W^{1,2}_{p,loc}((0,T)\times\mathbb{R}^n) \tag{2.40}$$

with (2.34), (2.35) for all $1 < p < \infty$, since we can first find the weak convergence of subsequences of (u^R, w^R) to (u, w) and then, by the lower-semicontinuity of the norm under weak convergence, the estimate (2.34), (2.35). The smoothness of the solutions in $(0,T)\times\mathbb{R}^n$ follows using interior L^2-regularity results for parabolic equations (see for example [Eva98] §7.1, Theorem 6, and the Remark after Theorem 7) and a bootstrapping argument. If we multiply equations (1.5), (1.6) for u^R, w^R by $\zeta \in H^{1,2}((0,T)\times\mathbb{R}^n)$ with $\zeta(T,.) = 0$ and integrate we find

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} \partial_t\zeta(u^R - \bar{u}_0) \\
&= \int_0^T \int_{\mathbb{R}^n} \nabla\zeta\cdot\nabla u^R + \zeta\Big(u^R - w^R + \gamma k u^R(1-w^R)\Big), && (2.41)\\
&\int_0^T \int_{\mathbb{R}^n} \partial_t\zeta(w^R - \bar{w}_0) \\
&= \int_0^T \int_{\mathbb{R}^n} d\nabla\zeta\cdot\nabla w^R - \zeta k u^R(1-w^R). && (2.42)
\end{aligned}$$

Due to (2.36), (2.37) and (2.38), (2.39) we can pass to the limit $R \to \infty$ in these equations and obtain

$$0 = \int_0^T \int_{\mathbb{R}^n} \Big(\partial_t\zeta(u - \bar{u}_0) - \nabla\zeta\cdot\nabla u - \zeta\big(u - w + \gamma k u(1-w)\big)\Big), \tag{2.43}$$
$$0 = \int_0^T \int_{\mathbb{R}^n} \Big(\partial_t\zeta(w - \bar{w}_0) - d\nabla\zeta\cdot\nabla w + \zeta k u(1-w)\Big) \tag{2.44}$$

for all $\zeta \in H^{1,2}((0,T)\times\mathbb{R}^n)$ with $\zeta(T,.) = 0$. Substituting into (2.43), (2.44) a test function $\zeta \in C_c^\infty((0,T)\times\mathbb{R}^n)$ we can, due to (2.40), perform a partial integration and find that

$$0 = \int_0^T \int_{\mathbb{R}^n} \zeta\Big(\partial_t u - \Delta u + u - w + \gamma k u(1-w)\Big),$$
$$0 = \int_0^T \int_{\mathbb{R}^n} \zeta\Big(\partial_t w - d\Delta w - k u(1-w)\Big)$$

which proves that (1.5), (1.6) hold almost everywhere in $(0,T) \times \mathbb{R}^n$.
Let us now prove that $u(t,.), w(t,.)$ are continuous in $L^1(\mathbb{R}^n)$ at $t = 0$ and attain the initial data $\bar{u}_0, \bar{w}_0$. We first show the convergence in $L^2(\mathbb{R}^n)$. Letting $t \to 0$ in (2.32) we obtain that

$$\limsup_{t\to 0} \int_{\mathbb{R}^n} u^2(t,x)\,dx \le \int_{\mathbb{R}^n} u_0^2(x)\,dx. \tag{2.45}$$

On the other hand, if we define, for $t \in (0,T)$ and $h > 0$, the function $\zeta^h(\tau,x) := \eta^h(\tau)\xi(x)$, where $\xi \in H^{1,2}(\mathbb{R}^n)$ and

$$\eta^h(\tau) := \begin{cases} 1 & \text{if } \tau \in [0,t), \\ 1 - \frac{1}{h}(\tau - t) & \text{if } \tau \in [t, t+h), \\ 0 & \text{if } \tau \in [t+h, T], \end{cases}$$

and substitute ζ^h in (2.43) we can deduce

$$\begin{aligned} &-\frac{1}{h}\int_t^{t+h} \Big(\int_{\mathbb{R}^n} \xi(u(\tau,.) - \bar{u}_0)\Big)\,d\tau \\ &= \int_0^{t+h} \eta^h(\tau)\Big(\int_{\mathbb{R}^n} \nabla\xi\cdot\nabla u + \xi\Big(u - w + \gamma k u(1-w)\Big)\Big)\,d\tau, \end{aligned}$$

and by letting $h \to 0$ we obtain

$$-\int_{\mathbb{R}^n} \xi(u(t,.) - \bar{u}_0) = \int_0^t \int_{\mathbb{R}^n} \Big(\nabla\xi\cdot\nabla u + \xi\Big(u - w + \gamma k u(1-w)\Big)\Big).$$

This shows that

$$\int_{\mathbb{R}^n} \xi(u(t,.) - \bar{u}_0) \to 0 \quad \text{as } t \to 0,$$

first for all $\xi \in H^{1,2}(\mathbb{R}^n)$ and then for all $\xi \in L^2(\mathbb{R}^n)$, due to the density of $H^{1,2}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$. Therefore we have proved that

$$u(t,.) \rightharpoonup \bar{u}_0 \quad \text{weakly in } L^2(\mathbb{R}^n) \tag{2.46}$$

as $t \to 0$ and this implies in particular that

$$\liminf_{t\to 0} \int_{\mathbb{R}^n} u^2(t,x)\,dx \geq \int_{\mathbb{R}^n} u_0^2(x)\,dx \tag{2.47}$$

holds. Together with (2.45) we obtain the convergence of the norms, that is

$$\lim_{t\to 0} \int_{\mathbb{R}^n} u^2(t,x)\,dx = \int_{\mathbb{R}^n} u_0^2(x)\,dx. \tag{2.48}$$

Recalling (2.46), this yields strong convergence in $L^2(\mathbb{R}^n)$. By a similar calculation one proves also the continuity of $w(t,.)$ at $t=0$ in $L^2(\mathbb{R}^n)$, so that we arrive at

$$u(t,.) \to \bar{u}_0 \quad \text{in } L^2(\mathbb{R}^n) \quad \text{as } t\to 0, \tag{2.49}$$
$$w(t,.) \to \bar{w}_0 \quad \text{in } L^2(\mathbb{R}^n) \quad \text{as } t\to 0. \tag{2.50}$$

To obtain also the continuity in $L^1(\mathbb{R}^n)$ we first observe that due to (2.30)

$$\limsup_{t\to 0} \int_{\mathbb{R}^n} \Big(u(t,x)+\gamma w(t,x)\Big)\,dx \leq \int_{\mathbb{R}^n} (\bar{u}_0(x)+\gamma\bar{w}_0(x))\,dx \tag{2.51}$$

holds. On the other hand, for any sequence $t_l \to 0$ $(l\to\infty)$ we deduce from (2.49) that there exist a subsequence $t_m \to 0$ $(m\to\infty)$ of $(t_l)_{l\in\mathbb{N}}$ such that $u(t_m,.), w(t_m,.)$ converge almost everywhere in $\mathbb{R}^n$ to $\bar{u}_0, \bar{w}_0$. Fatou's Lemma shows that

$$\liminf_{m\to\infty} \int_{\mathbb{R}^n} \Big(u(t_m,x)+\gamma w(t_m,x)\Big)\,dx \geq \int_{\mathbb{R}^n} (\bar{u}_0(x)+\gamma\bar{w}_0(x))\,dx. \tag{2.52}$$

From (2.51) and (2.52) we obtain the convergence of $u(t_m,.)+\gamma w(t_m,.)$ to $\bar{u}_0+\gamma\bar{w}_0$ in $L^1(\mathbb{R}^n)$ as $m\to\infty$. Therefore we have shown that for any sequence $t_l \to \infty$ there exists a subsequence $t_m \to 0$ such that $u(t_m,.)+\gamma w(t_m,.)$ converges to $\bar{u}_0+\gamma\bar{w}_0$. But then the whole sequence $t\to 0$ has to converge, so that

$$u(t,.)+\gamma w(t,.) \to \bar{u}_0+\gamma\bar{w}_0 \quad \text{in } L^1(\mathbb{R}^n) \tag{2.53}$$

as $t\to 0$. Finally $u(t,.)+\gamma w(t,.)$ gives a convergent dominator in $L^1(\mathbb{R}^n)$ for $u(t,.)$, $w(t,.)$ and a variant of Lebesgue's Dominated Convergence Theorem (see [EG92] §1.3 Theorem 4) yields the continuity of $u(t,.)$, $w(t,.)$ in $L^1(\mathbb{R}^n)$ at $t=0$. □

2.3 General initial data and $d = 0$

In this subsection we complete the proof of Theorem 1. We first obtain solutions for general initial data and eventually vanishing diffusion coefficient d, but still on the whole of $\mathbb{R}^n$.

Proposition 3. *Let $\bar{u}_0, \bar{w}_0$ satisfy (1.3), (1.4) and let $k > 0$, $d \geq 0$. Then there exists functions u_k, w_k, which satisfy for all $1 < p < \infty$*

$$\begin{aligned} &u_k, w_k \textit{ smooth in } (0,T) \times \mathbb{R}^n, \\ &u_k, w_k \in C^0([0,T]; L^1(\mathbb{R}^n)), \end{aligned}$$

in the case $d > 0$ and

$$u_k \in W^{1,2}_{p,loc}((0,T) \times \mathbb{R}^n) \cap C^0([0,T]; L^1(\mathbb{R}^n)), \tag{2.54}$$

$$w_k \in H^{1,\infty}(0,T; L^\infty(\mathbb{R}^n)) \cap C^\alpha([0,T]; L^1(\mathbb{R}^n)), \tag{2.55}$$

in the case $d = 0$, and which satisfy equations (1.5), (1.6) in $(0,T) \times \mathbb{R}^n$ and the initial conditions

$$u_k(0,.) \;=\; \bar{u}_0, \qquad\qquad w_k(0,.) \;=\; \bar{w}_0 \tag{2.56}$$

in $\mathbb{R}^n$. Moreover, the following estimates hold:

$$0 \;\leq\; u_k, w_k \;\leq\; 1, \tag{2.57}$$

$$\int_{\mathbb{R}^n} (u_k + \gamma w_k)(t,.) \;\leq\; e^{\frac{1}{\gamma}t} \int_{\mathbb{R}^n} (\bar{u}_0 + \gamma \bar{w}_0), \tag{2.58}$$

$$\int_0^T \int_{\mathbb{R}^n} k u_k (1 - w_k) \;\leq\; c(\gamma, T) \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)} \tag{2.59}$$

and

$$\frac{1}{2} \int_\Omega u_k^2(t,.) + \int_0^t \int_\Omega |\nabla u_k|^2 \;\leq\; \frac{1}{2} \int_\Omega \bar{u}_0^2 + t c(\gamma, T) \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)}, \tag{2.60}$$

$$\frac{1}{2} \int_\Omega w_k^2(t,.) + \int_0^t \int_\Omega d |\nabla w_k|^2 \;\leq\; \frac{1}{2} \int_\Omega \bar{w}_0^2 + \int_0^t \int_\Omega k u_k (1 - w_k) \tag{2.61}$$

for almost all $t \in (0,T)$ and

$$\|u_k\|_{W^{1,2}_p((\delta,T) \times B_L(0))} \leq C(\delta, L, p, \gamma, T, k) \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)} \tag{2.62}$$

for all $\delta, L > 0, 1 < p < \infty$. If $d > 0$ we also obtain the estimate

$$\|w_k\|_{W^{1,2}_p((\delta,T) \times B_L(0))} \leq C(\delta, L, d, p, \gamma, T, k) \|\bar{u}_0 + \gamma \bar{w}_0\|_{L^1(\mathbb{R}^n)}. \tag{2.63}$$

Before giving the proof we need some preparatory results. First we show that solutions of (1.5), (1.6) depend continuously with respect to $L^1(\mathbb{R}^n)$ on the initial data. This will be a crucial step to prove uniqueness for the reaction-diffusion system and to obtain convergence in the fast reaction limit. We include also the case that $d = 0$ in (1.6), that is equation (1.2).

Proposition 4. *Let $k > 0$, $d \geq 0$ and (u_k, w_k), $(\tilde{u}_k, \tilde{w}_k)$ be two solutions of* (1.5), (1.6) *with initial data $(\bar{u}_0, \bar{w}_0)$, $(\tilde{u}_0, \tilde{w}_0)$, respectively, where $(\bar{u}_0, \bar{w}_0)$, $(\tilde{u}_0, \tilde{w}_0)$ both satisfy* (1.3), (1.4) *on $(0,T) \times \mathbb{R}^n$. Moreover, let the estimates* (2.29)-(2.31) *hold for (u_k, w_k) and $(\tilde{u}_k, \tilde{w}_k)$. Then we obtain for all $t \in (0,T)$*

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(|u_k(t,.) - \tilde{u}_k(t,.)| + \gamma |w_k(t,.) - \tilde{w}_k(t,.)| \Big) \\
\leq\, & e^{\frac{1}{\gamma} t} \int_{\mathbb{R}^n} \Big(|\bar{u}_0 - \tilde{u}_0| + \gamma |\bar{w}_0 - \tilde{w}_0| \Big). \qquad (2.64)
\end{aligned}$$

Proof. Define

$$\begin{aligned}
U &:= u_k - \tilde{u}_k, & W &:= w_k - \tilde{w}_k, \\
U_0 &:= \bar{u}_0 - \tilde{u}_0, & W_0 &:= \bar{w}_0 - \tilde{w}_0.
\end{aligned}$$

From equations (1.5) and (1.6) we obtain that

$$\partial_t U = \Delta U - U + W - \gamma k u_k (1 - w_k) + \gamma k \tilde{u}_k (1 - \tilde{w}_k), \qquad (2.65)$$
$$\partial_t W = d \Delta W + k u_k (1 - w_k) - k \tilde{u}_k (1 - \tilde{w}_k), \qquad (2.66)$$

and

$$U(0,.) = U_0, \qquad W(0,.) = W_0.$$

The main idea now is to multiply equations (2.65), (2.66) by the sign-functions of U and W respectively, to integrate in time and space and obtain, using partial integration formulas, the desired estimate. Since the sign-function is not differentiable and $\partial_t u_k, \Delta u_k$ and $\partial_t w_k, \Delta w_k$ are not necessarily integrable over the whole space some additional effort is needed.
Take a smooth convex function $m : \mathbb{R} \to \mathbb{R}$ with

$$m \geq 0, \qquad m(0) = 0, \qquad m(r) = |r| - \frac{1}{2} \quad \text{for } |r| > 1,$$

and define for $\alpha > 0$ approximations of the modulus function by

$$m_\alpha(r) := \alpha m(\frac{r}{\alpha}).$$

Let $(\eta_L)_{L \in \mathbb{N}}$ be a monotone sequence of cut-off functions,

$$\eta_L \in C_c^\infty(B_{L+1}(0)), \qquad 0 \leq \eta_L \leq 1, \qquad \eta_L = 1 \text{ in } B_L(0),$$

with

$$|\nabla\eta_L|,\, |\Delta\eta_L| \,\le\, C$$

uniformly in L. For an arbitrary fixed $t_0 \in (0,T)$ we multiply (2.65) by $\eta_L m'_\alpha(U)$ and integrate this equation to obtain after a partial integration

$$\begin{aligned}
&\int_0^{t_0}\int_{\mathbb{R}^n} \eta_L \partial_t m_\alpha(U(t,.))\,dt \\
&= \int_0^{t_0}\int_{\mathbb{R}^n} -\eta_L m''_\alpha(U)|\nabla U|^2 - \nabla\eta_L\cdot m'_\alpha(U)\nabla U - \eta_L m'_\alpha(U)(U-W) \\
&\quad - \int_0^{t_0}\int_{\mathbb{R}^n} \gamma k m'_\alpha(U)\eta_L\Big(u_k(1-w_k) - \tilde{u}_k(1-\tilde{w}_k)\Big)\,.
\end{aligned} \tag{2.67}$$

Evaluating the left-hand side, using $m''_\alpha \ge 0$, and integrating by parts again we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \eta_L m_\alpha(U(t_0,.)) - \int_{\mathbb{R}^n} \eta_L m_\alpha(U_0) \\
&\le \int_0^{t_0}\int_{\mathbb{R}^n} m_\alpha(U)\Delta\eta_L - \eta_L m'_\alpha(U)(U-W) \\
&\quad - \int_0^{t_0}\int_{\mathbb{R}^n} \gamma k m'_\alpha(U)\eta_L\Big(u_k(1-w_k) - \tilde{u}_k(1-\tilde{w}_k)\Big)\,.
\end{aligned}$$

Now we let $\alpha \to 0$ and observe that $m_\alpha(r) \nearrow |r|$ and $m'_\alpha(r) \to \operatorname{sgn}(r)$. The Dominated Convergence Theorem allows us to take the limit $\alpha \to 0$ in the last inequality and we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \eta_L |U(t_0,.)| \\
&\le \int_{\mathbb{R}^n} \eta_L|U_0| + \int_0^{t_0}\int_{\mathbb{R}^n} |U|\Delta\eta_L - \eta_L \operatorname{sgn}(U)(U-W) \\
&\quad - \int_0^{t_0}\int_{\mathbb{R}^n} \gamma k \operatorname{sgn}(U)\eta_L\Big(u_k(1-w_k) - \tilde{u}_k(1-\tilde{w}_k)\Big)\,.
\end{aligned}$$

Now observe that $U \in L^\infty(0,T;L^1(\mathbb{R}^n))$ and let L tend to infinity. This gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} |U(t_0,.)| - \int_{\mathbb{R}^n} |U_0| \\
&\le \int_0^{t_0}\int_{\mathbb{R}^n} W\operatorname{sgn}(U) - \gamma k \operatorname{sgn}(U)\Big(u_k(1-w_k) - \tilde{u}_k(1-\tilde{w}_k)\Big)\,.
\end{aligned} \tag{2.68}$$

Analogously we obtain from (2.66) the inequality

$$\begin{aligned}
&\int_{\mathbb{R}^n} |W(t_0,.)| \\
&\le \int_{\mathbb{R}^n} |W_0| + \int_0^{t_0}\int_{\mathbb{R}^n} k\operatorname{sgn}(W)\Big(u_k(1-w_k) - \tilde{u}_k(1-\tilde{w}_k)\Big)\,.
\end{aligned} \tag{2.69}$$

If we multiply (2.69) by γ and add (2.68) we find

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(|U(t_0,.)| + \gamma|W(t_0,.)|\Big) - \int_{\mathbb{R}^n} \Big(|U_0| + \gamma|W_0|\Big) \\
\leq &\int_0^{t_0} \int_{\mathbb{R}^n} W \operatorname{sgn}(U) \\
&- \gamma k \int_0^{t_0} \int_{\mathbb{R}^n} \Big(\operatorname{sgn}(U) - \operatorname{sgn}(W)\Big)\Big(u_k(1-w_k) - \tilde{u}_k(1-\tilde{w}_k)\Big).
\end{aligned}$$

We observe that the last term on the right-hand side is nonpositive,

$$\begin{aligned}
&-\Big(\operatorname{sgn}(U) - \operatorname{sgn}(W)\Big)\Big(u_k(1-w_k) - \tilde{u}_k(1-\tilde{w}_k)\Big) \\
= &-(1-w_k)(|U| - \operatorname{sgn}(W)U) - \tilde{u}_k(|W| - \operatorname{sgn}(U)W) \quad \leq 0,
\end{aligned}$$

since $0 \leq \tilde{u}_k, w_k \leq 1$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(|U| + \gamma|W|\Big)(t_0,.) \\
\leq &\int_{\mathbb{R}^n} \Big(|U_0| + \gamma|W_0|\Big) + \frac{1}{\gamma}\int_0^{t_0} \int_{\mathbb{R}^n} (|U| + \gamma|W|)
\end{aligned}$$

for all $t_0 \in (0,T)$. Applying the Gronwall Lemma the desired inequality (2.64) follows. $\square$

An immediate consequence of Proposition 4 is an estimate for space differences of solutions to (1.5), (1.6).

Lemma 2. *Let $k > 0, d \geq 0$ and (u_k, w_k) be a solution of* (1.5), (1.6) *on $(0,T) \times \mathbb{R}^n$ with initial data $\bar{u}_0, \bar{w}_0$ satisfying* (1.3), (1.4). *Then for all $\xi \in \mathbb{R}^n$ and almost all $t \in (0,T)$*

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u_k(t,x) - u_k(t,x-\xi)| + \gamma|w_k(t,x) - w_k(t,x-\xi)|\,dx \\
\leq &e^{\frac{1}{\gamma}T}\int_{\mathbb{R}^n} |\bar{u}_0(x) - \bar{u}_0(x-\xi)| + \gamma|\bar{w}_0(x) - \bar{w}_0(x+\xi)|\,dx
\end{aligned} \tag{2.70}$$

holds.

Proof. We observe that $(t,x) \mapsto (u_k(t,x-\xi), w_k(t,x-\xi))$ is a solution of (1.5), (1.6) with initial data $(\bar{u}_0(.-\xi), \bar{w}_0(.-\xi))$. Then the claim follows from Proposition 4. $\square$

To obtain the compactness of the approximations as d tends to zero and to prepare for the study of the fast degradation rate limit $k \to \infty$ we also give an estimate for differences in time.

Lemma 3. *Let $k > 0$, $d \geq 0$ and (u_k, w_k) be a solutions of* (1.5), (1.6), *as in Lemma 2. For any $\tau \in (0,T)$ we obtain*

$$\int_0^{T-\tau}\int_{\mathbb{R}^n} |u_k(t+\tau,.)-u_k(t,.)|^2\,dt \le \tau c(\gamma,T)\|\bar{u}_0+\gamma\bar{w}_0\|_{L^1(\mathbb{R}^n)}, \tag{2.71}$$
$$\int_0^{T-\tau}\int_{\mathbb{R}^n} |w_k(t+\tau,.)-w_k(t,.)|^2\,dt \le \tau c(\gamma,T)\|\bar{u}_0+\gamma\bar{w}_0\|_{L^1(\mathbb{R}^n)}, \tag{2.72}$$

with a constant $c(\gamma,T)$ *which is independent of* d,k.

Proof. Let $(\eta_L)_{L\in\mathbb{N}}$,$\eta_L \in C_c^\infty(B_{L+1}(0))$, be a sequence of cut-off functions,

$$0 \le \eta_L \le 1, \quad \eta_L = 1 \text{ in } B_L(0), \quad |\nabla\eta_L| \le 2.$$

We obtain by using equation (1.6)

$$\begin{aligned}
&\int_0^{T-\tau}\int_{\mathbb{R}^n}\eta_L(w_k(t+\tau)-w_k(t))^2\,dx\,dt\\
&=\int_0^{T-\tau}\int_{\mathbb{R}^n}\eta_L(w_k(t+\tau)-w_k(t))\Big(\int_0^\tau\frac{\partial}{\partial s}w_k(t+s)\,ds\Big)\,dx\,dt\\
&=\int_0^{T-\tau}\int_{\mathbb{R}^n}\int_0^\tau\eta_L(w_k(t+\tau)-w_k(t))\Big(d\Delta w_k+ku_k(1-w_k)\Big)(t+s)\,ds\,dx\,dt\\
&=\int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n}-(w_k(t+\tau)-w_k(t))d\nabla\eta_L\cdot\nabla w_k(t+s)\,dx\,dt\,ds\\
&\quad-\int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n}\eta_L\nabla(w_k(t+\tau)-w_k(t))\cdot d\nabla w_k(t+s)\,dx\,dt\,ds\\
&\quad+\int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n}\eta_L(w_k(t+\tau)-w_k(t))ku_k(t+s)(1-w_k(t+s))\,dx\,dt\,ds.
\end{aligned} \tag{2.73}$$

Due to our assumptions on η_L we can estimate the first term on the right-hand side by

$$\begin{aligned}
&\Big|\int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n}-(w_k(t+\tau)-w_k(t))d\nabla\eta_L\cdot\nabla w_k(t+s)\,dx\,dt\,ds\Big|\\
&\le\int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n}2d\mathcal{X}_{B_{L+1}(0)\setminus B_L(0)}|w_k(t+\tau)-w_k(t)|\cdot|\nabla w_k(t+s)|\,dx\,dt\,ds.
\end{aligned}$$

We observe that the integrand on the right-hand side of this inequality converges pointwise almost everywhere in $(0,T)\times\mathbb{R}^n$ to zero as $L\to\infty$. Moreover this integrand is dominated by the function

$$(t,x)\mapsto 2d|w_k(t+\tau,x)-w_k(t,x)||\nabla w_k(t+s,x)|,$$

which is integrable since

$$
\begin{aligned}
&\Big(\int_0^{T-\tau}\int_{\mathbb{R}^n} d|w_k(t+\tau,x)-w_k(t,x)||\nabla w_k(t+s,x)|\,dx\,dt\Big)^2\\
&\leq \Big(\int\limits_0^{T-\tau}\int\limits_{\mathbb{R}^n} d|w_k(t+\tau,x)-w_k(t,x)|^2 dx\,dt\Big)\Big(\int\limits_0^{T-\tau}\int\limits_{\mathbb{R}^n} d|\nabla w_k(t+s,x)|^2 dx\,dt\Big)\\
&\leq 2d\Big(\int_0^T\int_{\mathbb{R}^n} w_k(t,x)\,dx\,dt\Big)\Big(\int_0^T\int_{\mathbb{R}^n} d|\nabla w_k(t,x)|^2\,dx\,dt\Big).
\end{aligned}
$$

With these computations we obtain that

$$
\int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n} -(w_k(t+\tau)-w_k(t))d\nabla\eta_L\cdot\nabla w_k(t+s)\,dx\,dt\,ds \to 0
$$

as $L\to\infty$. For the second term on the right-hand side of (2.73) we deduce that

$$
\begin{aligned}
&\Big|\int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n} \eta_L\nabla(w_k(t+\tau)-w_k(t))\cdot d\nabla w_k(t+s)\,dx\,dt\,ds\Big|\\
&\leq \int_0^\tau d\Big(\int_0^{T-\tau}\int_{R^n} |\nabla w_k(t+\tau)-\nabla w_k(t)|^2\Big)^{\frac{1}{2}}\Big(\int_0^{T-\tau}\int_{R^n} |\nabla w_k(t+s)|^2\Big)^{\frac{1}{2}} ds\\
&\leq 2d\tau\|\nabla w_k\|^2_{L^2((0,T)\times\mathbb{R}^n)},
\end{aligned}
$$

and for the third term

$$
\begin{aligned}
&\Big|\int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n} \eta_L(w_k(t+\tau)-w_k(t))ku_k(t+s)(1-w_k(t+s))\,ds\,dx\,dt\Big|\\
&\leq \int_0^\tau\int_0^{T-\tau}\int_{\mathbb{R}^n} ku_k(t+s)(1-w_k(t+s))\,ds\,dx\,dt\\
&\leq \tau\int_0^T\int_{\mathbb{R}^n} ku_k(1-w_k)\,dx\,dt.
\end{aligned}
$$

Using these estimates and letting $L\to\infty$ in (2.73) we obtain

$$
\begin{aligned}
&\int_0^{T-\tau}\int_{\mathbb{R}^n} (w_k(t+\tau)-w_k(t))^2\,dx\,dt\\
&\leq \tau\Big(2d\|\nabla w_k\|^2_{L^2((0,T)\times\mathbb{R}^n)}+\int_0^T\int_{\mathbb{R}^n} ku_k(1-w_k)\,dx\,dt\Big)
\end{aligned}
$$

and, due to the uniform bounds (2.31) and (2.33), we obtain (2.72).
The estimate of time-differences for u is derived in a similar way. □

Next we prove Proposition 3 by allowing general initial data and, in the case $d=0$, by letting the diffusion coefficient tend to zero.

Proof of Proposition 3. We approximate $\bar{u}_0, \bar{w}_0$ by a sequence $\bar{u}_0^i, \bar{w}_0^i$, $i \in \mathbb{N}$, of smooth functions such that

$$\begin{gathered} \bar{u}_0^i \to u_0, \quad \bar{w}_0^i \to w_0, \qquad \text{in } L^1(\mathbb{R}^n) \text{ as } i \to \infty, \\ \bar{u}_0^i, \bar{w}_0^i \quad \text{satisfy (2.2), (2.3).} \end{gathered} \tag{2.74}$$

Moreover define a sequence of positive diffusion coefficients by

$$d_i := d + \frac{1}{i}.$$

From Proposition 2 we get the existence of solutions u^i, w^i of equations (1.5), (1.6) in $\mathbb{R}^n$ with diffusion coefficient d_i in (1.5) and initial data $\bar{u}_0^i, \bar{w}_0^i$. The estimates (2.29), (2.30) show that u^i, w^i are bounded in $L^p((0,T) \times \mathbb{R}^n)$ for all $1 \le p \le \infty$, uniformly in $i \in \mathbb{N}$.
Due to (2.74) and the Fréchet-Kolmogorov-Riesz Theorem (see for example [DS88], IV.8 Theorem 21) we deduce that differences of space-shifts for $\bar{u}_0^i, \bar{w}_0^i$, and therefore the right-hand side of (2.70), decay uniformly in $i \in \mathbb{N}$ in $L^1(\mathbb{R}^n)$. From Lemma 2 we thus obtain the uniform decay of space shifts for u^i, w^i in $L^1((0,T) \times \mathbb{R}^n)$ as well as in $L^2((0,T) \times \mathbb{R}^n)$ due to the boundedness of the functions by unity. Lemma 3 gives the uniform decay in $L^2((0,T) \times \mathbb{R}^n)$ of shifts in time. Using again the Theorem of Fréchet-Kolmogorov-Riesz, the uniform decay of time and space shifts, together with the uniform bounds in $L^2((0,T) \times \mathbb{R}^n)$, proves the existence of a subsequence $i_l \to \infty$ $(l \to \infty)$ and of functions u_k, w_k such that

$$u^{i_l} \to u_k \quad \text{in } L^2((0,T) \times B_L(0)), \tag{2.75}$$

$$w^{i_l} \to w_k \quad \text{in } L^2((0,T) \times B_L(0)), \tag{2.76}$$

as $l \to \infty$, for all $L > 0$. This implies in particular pointwise convergence in $\mathbb{R}^n$ and convergence in $L^p((0,T) \times B_L(0))$ for all $L > 0$, $1 \le p < \infty$. From the weak precompactness of reflexive spaces and (2.32), (2.33) we moreover obtain

$$u^{i_l} \rightharpoonup u_k \quad \text{weakly in } L^2(0,T; H^{1,2}(\mathbb{R}^n)) \tag{2.77}$$

as $l \to \infty$, where we have used that any limit point can be identified with u_k due to (2.75). As in the proof of Proposition 2, we deduce from the uniform bound (2.34) that

$$u_k \in W^{1,2}_{p,loc}((0,T) \times \mathbb{R}^n), \tag{2.78}$$

and from (2.29), (2.30), (2.31) we obtain by the pointwise convergence of u^{i_l}, w^{i_l} and Fatou's Lemma that the estimates (2.57), (2.58), (2.59) hold. The

energy estimate (2.32) and the local Sobolev estimate (2.34) yield (2.60) and (2.62). In the case that $d > 0$ we also obtain (2.63).
If we now multiply equation (1.5) for u^i, w^i with a function $\zeta \in C_c^\infty([0,T)\times\mathbb{R}^n)$ and integrate this equation we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}\partial_t\zeta(u^i-\bar{u}_0^i)\\ =&\int_0^T\int_{\mathbb{R}^n}\nabla\zeta\cdot\nabla u^i+\zeta\Big(u^i-w^i+\gamma k u^i(1-w^i)\Big).\end{aligned}\tag{2.79}$$

By (2.75), (2.76), (2.77) and (2.74) we can pass to the limit $i\to\infty$ and obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}\partial_t\zeta(u_k-\bar{u}_0)\\ =&\int_0^T\int_{\mathbb{R}^n}\nabla\zeta\cdot\nabla u_k+\zeta\Big(u_k-w_k+\gamma k u_k(1-w_k)\Big).\end{aligned}\tag{2.80}$$

Choosing in (2.80) a test function with compact support in $(0,T)\times\mathbb{R}^n$ and using the regularity (2.78) we deduce as in the proof of Proposition 2 that equation (1.5) holds almost everywhere in $(0,T)\times\mathbb{R}^n$.
Again by repeating the arguments from Proposition 2 the inequalities (2.58), (2.60) and (2.80) yield that u_k is continuous in $L^p((0,T)\times\mathbb{R}^n)$ at $t=0$ for all $1\le p<\infty$ and attains the initial datum $\bar{u}_0$.
From (1.6) for u^i, w^i we obtain for $\zeta\in C_c^\infty([0,T)\times\mathbb{R}^n)$ the integral equality

$$\int_0^T\int_{\mathbb{R}^n}\partial_t\zeta(w^i-\bar{w}_0)=\int_0^T\int_{\mathbb{R}^n}d_i\nabla\zeta\cdot\nabla w^i-k\zeta u^i(1-w^i).\tag{2.81}$$

By (2.33) we observe that $\sqrt{d_i}\nabla w^i$ is weakly precompact in $L^2((0,T)\times\mathbb{R}^n)$. This gives for a subsequence $i_l\to\infty$ $(l\to\infty)$

$$\nabla w^{i_l}\rightharpoonup\nabla w_k\qquad\text{weakly in } L^2((0,T)\times\mathbb{R}^n)\quad\text{as } l\to\infty\tag{2.82}$$

if $d>0$, and

$$d_{i_l}\nabla w^{i_l}\rightharpoonup 0\qquad\text{weakly in } L^2((0,T)\times\mathbb{R}^n)\quad\text{as } l\to\infty\tag{2.83}$$

if $d=0$, since in this case

$$\int_0^T\int_{\mathbb{R}^n}d_{i_l}\nabla w^{i_l}\cdot\nabla\zeta\le\|\nabla\zeta\|_{L^2((0,T)\times\mathbb{R}^n)}\sqrt{\frac{1}{i_l}}\|\sqrt{d_{i_l}}\nabla w^{i_l}\|_{L^2((0,T)\times\mathbb{R}^n)},$$

which tends to zero as $l \to \infty$. From (2.82), (2.83) and (2.33) we obtain also that (2.61) holds. By (2.75), (2.76), (2.82), (2.83) we can pass to the limit $l \to \infty$ in (2.81) and obtain

$$\int_0^T \int_{\mathbb{R}^n} \partial_t \zeta (w_k - \bar{w}_0) = \int_0^T \int_{\mathbb{R}^n} d\nabla\zeta \cdot \nabla w_k - k\zeta u_k (1 - w_k)$$

in the case $d > 0$ and

$$\int_0^T \int_{\mathbb{R}^n} \partial_t \zeta (w_k - \bar{w}_0) = \int_0^T \int_{\mathbb{R}^n} -k\zeta u_k (1 - w_k)$$

in the case $d = 0$. In the latter case it follows that $w_k \in H^{1,\infty}(0, T; L^\infty(B_L(0)))$ for all $L > 0$ with

$$\partial_t w_k = k u_k (1 - w_k),$$

which gives (1.2). In this case, for all $L > 0$ and all $0 \leq \alpha < 1$ the Sobolev embedding theorem shows that $w_k \in C^\alpha\Big(0, T; L^p(B_L(0))\Big)$ holds, and the initial datum $\bar{w}_0$ is attained.
In the case $d > 0$ the equality (1.6) and the continuity of w_k at $t = 0$ is proven in the same way as we have proved the corresponding results for u_k.
The smoothness of solutions in $(0, T) \times \mathbb{R}^n$ in the case $d > 0$ follows as in the proof of Proposition 2 from interior L^2-regularity results for parabolic equations and a bootstrapping argument.
The uniqueness of solutions on $(0, T) \times \mathbb{R}^n$ is immediate from Proposition 4, which completes the proof of Proposition 3. □

Finally we show that u_k, w_k, as in Proposition 3, is in fact a solution of Problem (P_k).

Proof of Theorem 1. Let u_k, w_k be the unique solution of Problem (P_k) on the whole of $\mathbb{R}^n$, as obtained above. It remains to prove that u_k satisfies a zero Neumann boundary condition on $\mathbb{R}^{n-1} \times \{0\}$, as does w_k if $d > 0$. We define the pair of functions $(\tilde{u}_k, \tilde{w}_k)$ such that

$$\tilde{u}_k = u_k, \quad \tilde{w}_k = w_k \quad \text{on } Q_T,$$

and such that $(\tilde{u}_k, \tilde{w}_k)$ extends (u_k, w_k) symmetrically to $(0, T) \times \mathbb{R}^n$, that is

$$\tilde{u}_k(t, x_1, ..., x_n) := u_k(t, x_1, ..., -x_n), \quad \tilde{w}_k(t, x_1, ..., x_n) := w_k(t, x_1, ..., -x_n)$$

for $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Due to the homogeneity of the coefficients in the equations and to the absence of first-order space derivatives, $(\tilde{u}_k, \tilde{w}_k)$ is also a solution of (1.5), (1.6). Moreover, due to (2.1) we find that $\tilde{u}_k, \tilde{w}_k$ have the same initial data as u_k, w_k. Thus, by the uniqueness of solutions stated in Proposition 3,

$$\tilde{u}_k = u_k, \quad \tilde{w}_k = w_k. \tag{2.84}$$

This shows that u_k, w_k are symmetric with respect to $\mathbb{R}^{n-1} \times \{0\}$, and in particular that $\nabla u_k \cdot \vec{e}_n = 0$ and $\nabla w_k \cdot \vec{e}_n = 0$ in the case that $d > 0$ holds on S_T, where the traces exist for almost all $t > 0$ since $u_k \in L^p(\delta, T; H^{2,p}(B_L(0)))$ for all $\delta, L > 0$, as does w_k if $d > 0$.
To prove the uniqueness of solutions, we assume that there are two solutions of Problem (P_k) in the upper half-space and extend them symmetrically to the whole of $\mathbb{R}^n$ to give solutions as in Proposition 3. But, in view of Proposition 3, these solutions have to be identical. $\square$

3 The fast degradation rate limit

In this section we prove the convergence of the solutions (u_k, w_k) of Problem (P_k) to the unique solution of Problem (P_∞) as k tends to infinity. Moreover we show the equivalence of Problem (P_∞) to Problem $(\tilde{P}_\infty)$ and we give the classical formulation of the limit problem.

Proof of Theorem 2. The solutions u_k, w_k of Problem (P_k) with initial data $\bar{u}_0, \bar{w}_0$ are bounded in $L^2(Q_T)$ uniformly in $k \in \mathbb{N}$ according to the estimates (2.57) and (2.58). Recalling the estimates for space differences (2.70) (or rather its counterpart in $L^2(Q_T)$ which holds as well) and for time differences (2.71), (2.72), and invoking the Theorem of Fréchet-Kolmogorov-Riesz (see [DS88], IV.8 Theorem 21), we deduce that a subsequence $k_l \to \infty$ $(l \to \infty)$ and functions $u_\infty, w_\infty \in L^2(Q_T)$ exist, such that

$$u_{k_l} \to u_\infty, \quad w_{k_l} \to w_\infty \qquad \text{in } L^2_{loc}(Q_T), \tag{3.1}$$

$$u_{k_l} \to u_\infty, \quad w_{k_l} \to w_\infty \qquad \text{almost everywhere in } Q_T \tag{3.2}$$

as $l \to \infty$. Since u_k, w_k are uniformly bounded in $L^1(Q_T) \cap L^\infty(Q_T)$ we also obtain the convergence in $L^p_{loc}(Q_T)$ for all $1 \leq p < \infty$. The uniform $L^1(Q_T)$-bound for $ku_k(1 - w_k)$ stated in (2.59) and the Lemma of Fatou imply

$$\begin{aligned}\int_{Q_T} u_\infty(1 - w_\infty) &\leq \liminf_{l\to\infty} \int_{Q_T} u_{k_l}(1 - w_{k_l}) \\ &\leq \liminf_{l\to\infty} \frac{1}{k_l} c(\gamma, T) \;=\; 0,\end{aligned}$$

which proves (1.10).
From the estimates (2.60), (2.61) the weak compactness in $L^2(0,T;H^{1,2}(\mathbb{R}^n_+))$ of $(u_{k_l})_{l\in\mathbb{N}}$ and of $(w_{k_l})_{l\in\mathbb{N}}$ if $d>0$ follow. Therefore, eventually restricting ourselves to another subsequence, we obtain

$$u_{k_l} \rightharpoonup u_\infty \quad \text{weakly in } L^2(0,T;H^{1,2}(\mathbb{R}^n_+)), \tag{3.3}$$
$$w_{k_l} \rightharpoonup w_\infty \quad \text{weakly in } L^2(0,T;H^{1,2}(\mathbb{R}^n_+)) \quad \text{if } d>0, \tag{3.4}$$

as $l\to\infty$. Multiplying (1.6) by γ and adding (1.5) we deduce that u_k, w_k satisfy for $d\geq 0$

$$\partial_t(u_k+\gamma w_k) = \Delta(u_k+\gamma d w_k) - u_k + w_k, \tag{3.5}$$

which yields the integral identity

$$\begin{aligned}&\int_0^T\!\!\int_{\mathbb{R}^n_+} \partial_t\zeta(u_k+\gamma w_k-(\bar{u}_0+\gamma\bar{w}_0))\\ =&\int_0^T\!\!\int_{\mathbb{R}^n_+} \nabla\zeta\cdot\nabla(u_k+\gamma d w_k)+\zeta(u_k-w_k)\end{aligned} \tag{3.6}$$

for all $\zeta\in C_c^\infty([0,T)\times\mathbb{R}^n)$. For the subsequence $(k_l)_{l\in\mathbb{N}}$ for which (3.1)- (3.4) hold we can pass to the limit $l\to\infty$ in (3.6) and obtain that u_∞, w_∞ satisfy (1.14) for $d\geq 0$. Therefore, (u_∞,w_∞) is a solution of (P_∞) with initial datum $\bar{u}_0+\gamma\bar{w}_0$ for $u_\infty+\gamma w_\infty$.
The uniqueness of solutions follows from the equivalence between the problems (P_∞), $(\tilde{P}_\infty)$, as stated in Theorem 3, and the unique solvability of $(\tilde{P}_\infty)$ which we will prove in Proposition 5. Moreover, from the uniqueness of solutions of (P_∞), we deduce that

$$u_k \to u_\infty, \quad w_k \to w_\infty \quad \text{in } L^p_{\mathrm{loc}}(Q_T) \tag{3.7}$$

holds for the whole sequence $k\to\infty$.
Let us finally prove that we also obtain the convergence in $L^p(Q_T), 1\leq p<\infty$. With this aim in mind, we first assume that

$$\int_{\mathbb{R}^n_+}\Big(\bar{u}_0(x)+\gamma\bar{w}_0(x)\Big)|x|^2\,dx \;<\;\infty, \tag{3.8}$$

and we let $(\eta_L)_{L\in\mathbb{N}}$ be a sequence of cut-off functions,

$$\eta_L(x)=\eta(\frac{x}{L}),$$

where

$$\eta \in C_c^\infty(B_1(0)), \qquad 0 \le \eta \le 1, \qquad \eta = 1 \text{ in } B_{\frac{1}{2}}(0),$$
$$\nabla\eta \cdot \vec{e}_n \;=\; 0 \quad \text{ on } \{x_n = 0\}.$$

Multiplying (3.5) by $x \mapsto \eta_L(x)|x|^2$ and integrating over $\mathbb{R}^n_+$ we obtain after two partial integrations that

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^n_+} \eta_L(x)|x|^2\Big(u_k + \gamma w_k\Big)(t,x)\,dx \\
&= \int_{\mathbb{R}^n_+} \Big(u_k(t,x) + \gamma d w_k(t,x)\Big)\Big(\Delta\eta_L|x|^2 + 4\nabla\eta_L \cdot x + 2n\eta_L(x)\Big)\,dx \\
&\quad - \int_{\mathbb{R}^n_+} \eta_L(x)|x|^2(u_k(t,x) - w_k(t,x))\,dx
\end{aligned} \tag{3.9}$$

holds. We observe that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n_+} \Big(u_k(t,x) + \gamma d w_k(t,x)\Big)\Big(\Delta\eta_L|x|^2 + 4\nabla\eta_L \cdot x\Big)\,dx\Big| \\
&= \Big|\int_{\mathbb{R}^n_+} \Big(u_k(t,x) + \gamma d w_k(t,x)\Big)\Big(\Delta\eta\Big(\frac{x}{L}\Big)\frac{|x|^2}{L^2} + 4\nabla\eta\Big(\frac{x}{L}\Big)\cdot\frac{x}{L}\Big)\,dx\Big| \\
&\le \int_{\mathbb{R}^n_+} \Big(u_k(t,x) + \gamma d w_k(t,x)\Big)\Big(|\Delta\eta\Big(\frac{x}{L}\Big)| + 4|\nabla\eta\Big(\frac{x}{L}\Big)|\Big)\,dx \\
&\le \Big(\|\Delta\eta\|_{L^\infty(\mathbb{R}^n)} + \|\nabla\eta\|_{L^\infty(\mathbb{R}^n)}\Big)\int_{\mathbb{R}^n_+} \Big(u_k(t,x) + \gamma d w_k(t,x)\Big)\,dx \\
&\le C(\eta)c(\gamma,T)(1+d)\|\bar{u}_0 + \gamma\bar{w}_0\|_{L^1(\mathbb{R}^n_+)},
\end{aligned}$$

where we have used (2.58). Therefore we deduce from (3.9) that

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^n_+} \eta_L(x)|x|^2\Big(u_k + \gamma w_k\Big)(t,x)\,dx \\
&\le C(\eta,\gamma,T,d)\|\bar{u}_0 + \gamma\bar{w}_0\|_{L^1(\mathbb{R}^n_+)} + 2n(1+d)\int_{\mathbb{R}^n_+} \eta_L(x)|x|^2(u_k + \gamma w_k)(t,x)\,dx \\
&\quad + 2n(1+\gamma d)|B_1(0)| + \int_{\mathbb{R}^n_+} \eta_L(x)|x|^2 w_k(t,x)\,dx. \\
&\le C(\eta,\gamma,T,d)\Big(1 + \|\bar{u}_0 + \gamma\bar{w}_0\|_{L^1(\mathbb{R}^n_+)}\Big) \\
&\quad + C(n,\gamma,d)\int_{\mathbb{R}^n_+} \eta_L(x)|x|^2\Big(u_k + \gamma w_k\Big)(t,x)\,dx
\end{aligned}$$

and Gronwall's Lemma yields that

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} \eta_L(x)|x|^2\Big(u_k + \gamma w_k\Big)(t,x)\,dx \\
&\le e^{tC(\eta,\gamma,T,d)(1+\|\bar{u}_0+\gamma\bar{w}_0\|_{L^1(\mathbb{R}^n_+)})}\Big[\int_{\mathbb{R}^n_+} \eta_L(x)|x|^2\Big(\bar{u}_0 + \gamma\bar{w}_0\Big)(x)\,dx + tC(n,\gamma,d)\Big] \\
&\le C(\eta,\gamma,T,n,d,\|\bar{u}_0 + \gamma\bar{w}_0\|_{L^1(\mathbb{R}^n_+)})\Big[1 + \int_{\mathbb{R}^n_+} |x|^2\Big(\bar{u}_0 + \gamma\bar{w}_0\Big)(x)\,dx\Big]
\end{aligned}$$

uniformly in $k \in \mathbb{N}$. Letting $L \to \infty$ we obtain from the Monotone Convergence Theorem that

$$\begin{aligned}&\int_{\mathbb{R}^n_+} |x|^2 \Big(u_k + \gamma w_k\Big)(t,x)\,dx \\ &\leq C(\eta,\gamma,T,n,d,\|\bar{u}_0+\gamma\bar{w}_0\|_{L^1(\mathbb{R}^n_+)})\Big[1+\int_{\mathbb{R}^n_+}|x|^2\Big(\bar{u}_0+\gamma\bar{w}_0\Big)(x)\,dx\Big]\end{aligned}$$

uniformly in $k \in \mathbb{N}$. This decay property together with (3.7) yields

$$u_k \to u_\infty, \quad w_k \to w_\infty \quad \text{ in } L^p(Q_T) \tag{3.10}$$

for all $1 \leq p < \infty$.
Let us now allow general initial data $\bar{u}_0, \bar{w}_0$. Taking a sequence of cut-off functions $(\eta_L)_{L\in\mathbb{N}}$ as above we denote by $u_{k,L}$ and $w_{k,L}$ the solution of Problem (P_k) with initial data $\eta_L\bar{u}_0$ and $\eta_L\bar{w}_0$ respectively. Since these modified initial data satisfy (3.8) we deduce from the already proven parts of Theorem 2 that there exist $u_{\infty,L}$ and $w_{\infty,L}$ such that for all $1 \leq p < \infty$

$$u_{k,L} \to u_{\infty,L}, \quad w_{k,L} \to w_{\infty,L} \qquad \text{in } L^p(Q_T) \tag{3.11}$$

as k tends to infinity. Moreover we deduce from Proposition 4 that

$$\begin{aligned}&\int_{Q_T}\Big(|u_k-u_{k,L}|+\gamma|w_k-w_{k,L}|\Big) \\ &\leq c(\gamma,T)\int_{R^n_+}\Big(|\bar{u}_0-\eta_L\bar{u}_0|+\gamma|\bar{w}_0-\eta_L\bar{w}_0|\Big)\end{aligned} \tag{3.12}$$

is satisfied. Using (3.7) and (3.11) we deduce from (3.12) and Fatou's Lemma that

$$\begin{aligned}&\int_{Q_T}\Big(|u_\infty-u_{\infty,L}|+\gamma|w_\infty-w_{k,\infty}|\Big) \\ &\leq c(\gamma,T)\int_{\mathbb{R}^n_+}\Big(|\bar{u}_0-\eta_L\bar{u}_0|+\gamma|\bar{w}_0-\eta_L\bar{w}_0|\Big)\end{aligned} \tag{3.13}$$

holds. Now putting (3.11)-(3.13) together we obtain

$$\begin{aligned}&\limsup_{k\to\infty}\int_{Q_T}|u_k-u_\infty| \\ &\leq \limsup_{k\to\infty}\int_{Q_T}\Big(|u_k-u_{k,L}|+|u_{k,L}-u_{\infty,L}|+|u_{\infty,L}-u_\infty|\Big) \\ &\leq 2c(\gamma,T)\int_{\mathbb{R}^n_+}\Big(|\bar{u}_0-\eta_L\bar{u}_0|+\gamma|\bar{w}_0-\eta_L\bar{w}_0|\Big) \\ &\quad +\limsup_{k\to\infty}\int_{Q_T}|u_{k,L}-u_{\infty,L}| \\ &= 2c(\gamma,T)\int_{\mathbb{R}^n_+}\Big(|\bar{u}_0-\eta_L\bar{u}_0|+\gamma|\bar{w}_0-\eta_L\bar{w}_0|\Big).\end{aligned}$$

Since the Dominated Convergence Theorem implies that the right-hand side of this inequality becomes arbitrarily small as L tends to infinity, this proves the convergence of u_k to u_∞ in $L^1(Q_T)$. Finally, due to the uniform bounds in $L^1(Q_T) \cap L^\infty(Q_T)$, we obtain the convergence of u_k to u_∞ in $L^p(Q_T)$ for all $1 \leq p < \infty$. The proof that w_k converges to w_∞ in $L^p(Q_T)$ is analogous. □

Next we will prove that solutions of Problem $(\tilde{P}_\infty)$ depend continuously on the initial data and are unique.

Proposition 5. *Let $d \geq 0$ and consider two solutions $z, \tilde{z}$ of Problem $(\tilde{P}_\infty)$ with initial data $z_0, \tilde{z}_0$ respectively. Then*

$$\int_0^T \int_{\mathbb{R}^n_+} |z - \tilde{z}| \leq C(\gamma, T) \int_{\mathbb{R}^n_+} |z_0 - \tilde{z}_0| \tag{3.14}$$

holds. In particular, for a given initial function z_0 there exists at most one solution of Problem $(\tilde{P}_\infty)$.

Proof. To simplify the notations we present the proof only for the more difficult case that the diffusion coefficient d vanishes and thus assume $d = 0$ in what follows.

Using (1.19) we obtain that the difference $z - \tilde{z}$ satisfies, for all $\zeta \in H^{1,2}(Q_T)$ with $\zeta(T, .) = 0$ in $\mathbb{R}^n_+$, the identity

$$\begin{aligned} 0 = & \int_0^T \int_{\mathbb{R}^n_+} \Big(\big(z - \tilde{z} - (z_0 - \tilde{z}_0) \big) \partial_t \zeta - \nabla(\varphi(z) - \varphi(\tilde{z})) \cdot \nabla \zeta \Big) \\ & + \int_0^T \int_{\mathbb{R}^n_+} (h(z) - h(\tilde{z}))\zeta. \end{aligned}$$

In particular this implies, for all $\zeta \in W^{1,2}_2(Q_T)$ with $\zeta(T, .) = 0$ in $\mathbb{R}^n_+$ and $\nabla \zeta \cdot \vec{e}_n = 0$ on S_T, that

$$\begin{aligned} 0 = & \int_0^T \int_{\mathbb{R}^n_+} \big(z - \tilde{z} \big) \Big(\partial_t \zeta + q \Delta \zeta - (1 + \frac{1}{\gamma}) q \zeta + \frac{1}{\gamma} \zeta \Big) \\ & + \int_{\mathbb{R}^n_+} (z_0 - \tilde{z}_0) \zeta(0, .) \end{aligned} \tag{3.15}$$

holds, where we have set

$$q := \frac{z_+ - \tilde{z}_+}{z - \tilde{z}} \quad \text{ if } z \neq \tilde{z},$$

and $q := 0$ if $z = \tilde{z}$. Observe that $q \in L^\infty(Q_T)$ with $0 \leq q \leq 1$. Now let $\xi \in C^\infty_c([0, T) \times \mathbb{R}^n_+)$ be given and consider a sequence $(q_i)_{i \in \mathbb{N}}$, $q_i \in C^0(Q_T)$

with

$$\inf_{Q_T} q_i \, > \, 0, \qquad \sup_{Q_T} q_i \, \leq \, 2,$$

such that $q_i \geq q$ and

$$\frac{q_i - q}{\sqrt{q_i}} \leq 2 \qquad \text{in } Q_T, \tag{3.16}$$

$$\frac{q_i - q}{\sqrt{q_i}} \to 0 \quad \text{pointwise in } Q_T \text{ as } i \to \infty. \tag{3.17}$$

One can obtain such a sequence $(q_i)_{i\in\mathbb{N}}$ by adding $1/i$ to q and smoothing the result. Let $\zeta_i \in W_2^{1,2}(Q_T)$ be the solution of

$$\begin{aligned} \xi &= \partial_t \zeta_i + q_i \Delta \zeta_i - (1 + \frac{1}{\gamma}) q_i \zeta_i + \frac{1}{\gamma} \zeta_i && \text{in } Q_T, \\ 0 &= \nabla \zeta_i \cdot \vec{e}_n && \text{on } S_T, \\ 0 &= \zeta_i(T, .) && \text{in } \mathbb{R}^n_+. \end{aligned} \tag{3.18}$$

The existence of a solution to this problem follows from standard parabolic theory, see for example [LSU68], IV Theorem 9.1. Moreover we claim that the estimates

$$\int_0^T \int_{\mathbb{R}^n_+} q_i |\Delta \zeta_i|^2 \leq C(\xi, \gamma, T), \tag{3.19}$$

$$\int_0^T \int_{\mathbb{R}^n_+} |\zeta_i|^2 \leq C(\xi, \gamma, T), \tag{3.20}$$

$$\|\zeta_i\|_{L^\infty(Q_T)} \leq C(\|\xi\|_{L^\infty(Q_T)}, \gamma, T). \tag{3.21}$$

hold. To prove these estimates we observe that $\tilde{\zeta} := e^{\frac{t}{\gamma}} \zeta_i$ satisfies the equation

$$e^{\frac{t}{\gamma}} \xi = \partial_t \tilde{\zeta} + q_i \Delta \tilde{\zeta} - (1 + \frac{1}{\gamma}) q_i \tilde{\zeta} \qquad \text{in } Q_T, \tag{3.22}$$

which becomes a standard linear parabolic equation if one performs the change of variables $\tau = T - t$. The maximum principle and a comparison of $\tilde{\zeta}$ with the functions v_+, v_-,

$$v_+(t) \, = \, e^{\alpha(T-t)}, \qquad v_-(t) \, = \, -e^{\alpha(T-t)},$$

where $\alpha = e^{\frac{T}{\gamma}} \|\xi\|_{L^\infty(Q_T)}$, gives the bound (3.21).
If we multiply equation (3.22) by $\Delta\tilde{\zeta}$ and integrate over $(t,T) \times \mathbb{R}^n_+$ we obtain

$$\int_{\mathbb{R}^n_+}\frac{1}{2}|\nabla\tilde{\zeta}|^2(t,.)+\int_t^T\int_{\mathbb{R}^n_+}\Big(q_i(\Delta\tilde{\zeta})^2+(1+\frac{1}{\gamma})q_i|\nabla\tilde{\zeta}|^2\Big)=\int_t^T\int_{\mathbb{R}^n_+}\tilde{\zeta}e^{\frac{t}{\gamma}}\Delta\xi,$$

from which we deduce that

$$\int_0^T\int_{\mathbb{R}^n_+}q_i(\Delta\zeta_i)^2\le e^{\frac{2T}{\gamma}}\|\zeta_i\|_{L^\infty(Q_T)}\|\Delta\xi\|_{L^1(Q_T)}$$

holds; recalling (3.21) we then obtain (3.19).
Next we multiply (3.22) by $-\tilde{\zeta}$ and integrate over $(t,T)\times\mathbb{R}^n_+$. This yields

$$\int_{\mathbb{R}^n_+}\frac{1}{2}\tilde{\zeta}^2(t,.)+\int_t^T\int_{\mathbb{R}^n_+}\Big(q_i|\nabla\tilde{\zeta}|^2+(1+\frac{1}{\gamma})q_i\tilde{\zeta}^2\Big)=\int_t^T\int_{\mathbb{R}^n_+}-\tilde{\zeta}e^{\frac{t}{\gamma}}\xi,$$

and the estimate

$$\int_{\mathbb{R}^n_+}\frac{1}{2}\tilde{\zeta}^2(t,.)\le\int_t^T\int_{\mathbb{R}^n_+}\Big(\frac{1}{2}\tilde{\zeta}^2+\frac{1}{2}e^{\frac{2T}{\gamma}}\xi^2\Big).$$

Applying the Gronwall Lemma we deduce (3.20).
Using ζ_i as a test function in (3.15) we obtain

$$\begin{aligned}0=&\int_0^T\int_{\mathbb{R}^n_+}(z-\tilde{z})\Big(\xi+(q-q_i)\Delta\zeta_i-(1+\frac{1}{\gamma})(q-q_i)\zeta_i\Big)\\&+\int_{\mathbb{R}^n_+}(z_0-\tilde{z}_0)\zeta_i(0,.).\end{aligned}\tag{3.23}$$

We deduce from Hölder's inequality and (3.17), (3.19) that

$$\begin{aligned}&\limsup_{i\to\infty}\Big|\int_0^T\int_{\mathbb{R}^n_+}(z-\tilde{z})(q-q_i)\Delta\zeta_i\Big|\\&\le\limsup_{i\to\infty}\|\frac{q-q_i}{\sqrt{q_i}}(z-\tilde{z})\|_{L^2(Q_T)}\|\sqrt{q_i}\Delta\zeta_i\|_{L^2(Q_T)}\\&\le C(\xi,\gamma,T)^{1/2}\limsup_{i\to\infty}\|\frac{q-q_i}{\sqrt{q_i}}(z-\tilde{z})\|_{L^2(Q_T)}\\&=0,\end{aligned}$$

where we have used (3.16), (3.17) and Lebesgue's Dominated Convergence

Theorem. Similarly we obtain from (3.17) and (3.20) that

$$
\begin{aligned}
&\limsup_{i\to\infty}\Big|\int_0^T\int_{\mathbb{R}^n_+}(z-\tilde{z})(1+\frac{1}{\gamma})(q-q_i)\zeta_i\Big|\\
&\le \limsup_{i\to\infty}(1+\frac{1}{\gamma})\|(q-q_i)(z-\tilde{z})\|_{L^2(Q_T)}\|\zeta_i\|_{L^2(Q_T)}\\
&\le C(\xi,\gamma,T)^{1/2}(1+\frac{1}{\gamma})\limsup_{i\to\infty}\|(q-q_i)(z-\tilde{z})\|_{L^2(Q_T)}\\
&=0.
\end{aligned}
$$

Thus, in the limit $i\to\infty$, we get from (3.21) and (3.23) that

$$
\int_0^T\int_{\mathbb{R}^n_+}(z-\tilde{z})\xi\le C(\|\xi\|_{L^\infty(Q_T)},\gamma,T)\int_{\mathbb{R}^n_+}|z_0-\tilde{z}_0| \tag{3.24}
$$

holds. Taking a sequence $(\xi_j)_{j\in\mathbb{N}}$, $\xi_j\in C_c^\infty(Q_T)$, with $\|\xi_j\|_{L^\infty(Q_T)}\le 2$ and $\xi_j\to\operatorname{sgn}(z-\tilde{z})$ almost everywhere, we obtain from (3.24) in the limit $j\to\infty$

$$
\int_0^T\int_{\mathbb{R}^n_+}|z-\tilde{z}|\le C(T,\gamma)\int_{\mathbb{R}^n_+}|z_0-\tilde{z}_0|,
$$

which gives (3.14). □

We next prove the equivalence of the problems (P_∞), $(\tilde{P}_\infty)$, as stated in Theorem 3.

Proof of Theorem 3. Let $\bar{u}_0,\bar{w}_0$ satisfy (1.3), (1.4) and u_∞,w_∞ be a solution of (P_∞). We define z by (1.20) and set

$$
z_0:=\bar{u}_0-\gamma(1-\bar{w}_0).
$$

Due to (1.10) we obtain

$$
u_\infty\;=\;z_+,\qquad w_\infty\;=\;1+\frac{1}{\gamma}(z-z_+), \tag{3.25}
$$

and we calculate that

$$
\begin{aligned}
u_\infty+\gamma w_\infty-\Big(\bar{u}_0+\gamma\bar{w}_0\Big)&=z-z_0,\\
u_\infty+\gamma d w_\infty&=\varphi(z),\\
u_\infty-w_\infty&=-h(z),
\end{aligned}
$$

such that (1.14) yields that

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^n_+} \partial_t \zeta \Big(u_\infty + \gamma w_\infty - (\bar{u}_0 + \gamma \bar{w}_0) \Big) \\
&\quad - \int_0^T \int_{\mathbb{R}^n_+} \Big(\nabla \zeta \cdot \nabla (u_\infty + \gamma d w_\infty) + \zeta (u_\infty - w_\infty) \Big) \\
&= \int_0^T \int_{\mathbb{R}^n_+} \Big(\partial_t \zeta \Big(z - z_0 \Big) - \nabla \zeta \cdot \nabla \varphi(z) + \zeta h(z) \Big)
\end{aligned}
$$

holds, which proves (1.19).
Conversely, from the definition of $\hat{u}, \hat{w}$ and $\hat{u}_0, \hat{w}_0$ in (1.21), (1.22) we obtain that

$$
\begin{aligned}
\varphi(\hat{z}) &= \hat{u} + \gamma d \hat{w}, \\
h(\hat{z}) &= -\hat{u} + \hat{w}, \\
\hat{z} - \hat{z}_0 &= \Big(\hat{u} + \gamma \hat{w} \Big) - \Big(\hat{u}_0 + \gamma \hat{w}_0 \Big).
\end{aligned}
$$

With these calculations we obtain from (1.19) that

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^n_+} \partial_t \zeta (z - z_0) - \nabla \zeta \cdot \nabla \varphi(z) + \zeta h(z) \\
&= \int_0^T \int_{\mathbb{R}^n_+} \partial_t \zeta \Big(\hat{u} + \gamma \hat{w} \Big) - (\hat{u}_0 + \gamma \hat{w}_0) \Big) - \nabla \zeta \cdot \nabla \Big(\hat{u} + \gamma d \hat{w} \Big) \\
&\quad + \int_0^T \int_{\mathbb{R}^n_+} \zeta \Big(- \hat{u} + \hat{w} \Big)
\end{aligned}
$$

which shows that (1.14) hold for $\hat{u}, \hat{w}$ with initial datum $\hat{u}_0 + \gamma \hat{w}_0$ for $\hat{u} + \gamma \hat{w}$. □

Let us finally show that the *classical formulation* of the Problem (P_∞) takes the form of variants of a Stefan problem.

Proposition 6. *Let u_∞, w_∞ be a solution of Problem (P_∞) and define the sets*

$$
\begin{aligned}
Q^0 &:= \text{ interior of } \{u_\infty = 0\}, \qquad & Q^1 &:= \text{ interior of } \{u_\infty > 0\}, \\
Q^0_t &:= \text{ interior of } \{u_\infty(t,.) = 0\}, \qquad & Q^1_t &:= \text{ interior of } \{u_\infty(t,.) > 0\}, \\
\Gamma(t) &:= \partial Q^0_t \cap \partial Q^1_t.
\end{aligned}
$$

Assume that u_∞ is smooth in $\overline{Q_1}$, that w_∞ is smooth in $\overline{Q_0}$, and that $\Gamma(t)$ is for each t a smooth hypersurfaces in $\mathbb{R}^n$, varying smoothly with t. Then we have

$$
\partial_t u_\infty = \Delta u_\infty - u_\infty + 1 \quad \text{in } Q_1, \tag{3.26}
$$
$$
\partial_t w_\infty = \gamma d \Delta w_\infty + w_\infty \quad \text{in } Q_0, \tag{3.27}
$$

and $\Gamma(t)$ moves according to

$$\gamma[w_\infty]V(t) = -[\nabla(u_\infty + \gamma d w_\infty)\cdot\nu(t)], \tag{3.28}$$

where $\nu(t)$ denotes the unit normal of $\Gamma(t)$ pointing into Q^0_t and $V(t)$ the velocity of $\Gamma(t)$ in direction of $\nu(t)$, and where $[.]$ denotes the jump across the interface $\Gamma(t)$ from Q^1_t to Q^0_t.

Proof. Let us define the restrictions of u_∞, w_∞ to Q_1, Q_0 by

$$\begin{aligned} u^{(1)}_\infty &= u_\infty|_{Q_1}, & u^{(0)}_\infty &= u_\infty|_{Q_0} = 0,\\ w^{(0)}_\infty &= w_\infty|_{Q_0}, & w^{(1)}_\infty &= w_\infty|_{Q_1} = 1. \end{aligned}$$

For an arbitrary $\zeta \in C^\infty_c(Q_T)$ we obtain from (1.14) the identity

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^n_+} \partial_t\zeta(u_\infty + \gamma w_\infty) - \nabla\zeta\cdot\nabla(u_\infty + \gamma d w_\infty) - \zeta(u_\infty - w_\infty),\\ &= \int_{Q^1} (u^{(1)}_\infty + \gamma)\partial_t\zeta - \nabla\zeta\cdot\nabla u^{(1)}_\infty - \zeta(u^{(1)}_\infty - 1)\\ &\quad + \int_{Q^0} \gamma w^{(0)}_\infty \partial_t\zeta - \gamma\nabla\zeta\cdot d\nabla w^{(0)}_\infty + \zeta w^{(0)}_\infty. \end{aligned} \tag{3.29}$$

Now we observe that

$$\begin{aligned} &\frac{d}{dt}\int_{Q^1_t} \zeta(u^{(1)}_\infty + \gamma)\\ &= \int_{Q^1_t} \Big(\partial_t\zeta(u^{(1)}_\infty + \gamma) + \zeta\partial_t u^{(1)}_\infty\Big) + \int_{\Gamma(t)} \zeta(u^{(1)}_\infty + \gamma)V(t)\,d\mathcal{H}^{n-1},\\ &= \int_{Q^1_t} \Big(\partial_t\zeta(u^{(1)}_\infty + \gamma) + \zeta\partial_t u^{(1)}_\infty\Big) + \int_{\Gamma(t)} \zeta\gamma V(t)\,d\mathcal{H}^{n-1}, \end{aligned}$$

for $t \in (0,T)$, since on $\Gamma(t)$ we find that $u_\infty = u^{(1)}_\infty = u^{(0)}_\infty$ vanishes. By similar calculations we obtain

$$\frac{d}{dt}\int_{Q^0_t} \gamma\zeta w^{(0)}_\infty = \int_{Q^0_t} \gamma(\partial_t\zeta w^{(0)}_\infty + \zeta\partial_t w^{(0)}_\infty) - \int_{\Gamma(t)} \zeta\gamma w^{(0)}_\infty V(t)\,d\mathcal{H}^{n-1}.$$

Together with (3.29) and a partial integration in Q^1_t and Q^0_t respectively this gives

$$\begin{aligned} 0 &= \int\int_{Q^1} \zeta(-\partial_t u^{(1)}_\infty + \Delta u^{(1)}_\infty - u^{(1)}_\infty + 1)\\ &\quad + \int\int_{Q^0} \zeta(-\gamma\partial_t w^{(0)}_\infty + \gamma d\Delta w^{(0)}_\infty + w^{(0)}_\infty)\\ &\quad - \int_0^T \int_{\Gamma(t)} \zeta\Big(\gamma(1 - w^{(0)}_\infty(t,.))V(t) + \nabla(u^{(1)}_\infty - \gamma d w^{(0)}_\infty)\cdot\nu(t)\Big) d\mathcal{H}^{n-1}\,dt. \end{aligned}$$

Taking now any function ζ with compact support in Q^1 we obtain (3.26), while taking any ζ with compact support in Q^0 yields (3.27). We deduce that

$$
\begin{aligned}
0 = & -\int_0^T \int_{\Gamma(t)} \zeta\Big(\gamma(1 - w_\infty^{(0)})V(t) + \nabla(u_\infty^{(1)} - \gamma d w_\infty^{(0)}) \cdot \nu(t)\Big) d\mathcal{H}^{n-1}\, dt, \\
= & -\int_0^T \int_{\Gamma(t)} \zeta\gamma(w_\infty^{(1)} - w_\infty^{(0)})V(t)\, d\mathcal{H}^{n-1}\, dt \\
& -\int_0^T \int_{\Gamma(t)} \zeta\nu(t) \cdot \Big(\nabla\Big(u_\infty^{(1)} - u_\infty^{(0)}\Big) + \gamma d \nabla\Big((w_\infty^{(1)} - w_\infty^{(0)})\Big)\Big) d\mathcal{H}^{n-1}\, dt,
\end{aligned}
$$

which gives the Rankine-Hugoniot condition (3.28). □

In what follows we drop again the upper indices (0) and (1).

Remark 1. *Let us consider the set of initial data for which the equations in [KKC+ 03] were analysed. There, it was assumed that $d = 0$, that $\bar{w}_0$ is a characteristic function with compact support and that $\bar{u}_0 = 0$. Under these assumptions the Ranking-Hugoniot condition reduces to*

$$\gamma[w_\infty]V(t) = -[\nabla u_\infty \cdot \nu(t)].$$

We observe, since $w_\infty = 1$ in Q^1 and $w_\infty \le 1$, that

$$[w_\infty] \ge 0.$$

Similarly, using $u_\infty > 0$ in Q^1 and $u_\infty = 0$ in Q^0, we find that

$$[\nabla u_\infty \cdot \nu(t)] \le 0,$$

which shows $V(t) \ge 0$. This means that the region $\{u_\infty > 0, w_\infty = 1\}$, representing the destroyed tissue, is growing. From the special choice of the initial data $\bar{u}_0, \bar{w}_0$ and (3.27) we further obtain

$$w_\infty = 0 \qquad \text{in } Q^0,$$

which yields that w_∞ is a characteristic function and $[w_\infty] = 1$. This finally reduces (3.28) to

$$\gamma V(t) = -[\nabla u_\infty \cdot \nu(t)]$$

and the problem takes the form of a classical one-phase Stefan problem.

Remark 2. *If $d > 0$ we find that $w \in L^2(0, T; H^{1,2}(\mathbb{R}^n_+))$ and $[w(t, .)] = 0$. This case corresponds to a Stefan problem with zero latent heat. The Rankine-Hugoniot condition (3.28) gives no explicit formula for the velocity of $\Gamma(t)$ but states the continuity of the normal derivative of $(u_\infty + \gamma d w_\infty)$ across the interface, that is*

$$[\nabla(u_\infty + \gamma d w_\infty) \cdot \nu(t)] = 0.$$

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