Permutations with given ups and downs

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PERMUTATIONS WITH GIVEN UPS AND DOWNS

BY

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1. *Introduction*

In a recent paper I. Niven [2] studied the following question. Let n be a positive integer, and let $Q = (q_1, \ldots, q_{n-1})$ be a vector whose components are all $+1$ or -1. A permutation $a_1, \ldots, a_n$ of $1, \ldots, n$ is said to have the signature Q if $q_1(a_2 - a_1), q_2(a_3 - a_2), \ldots, q_{n-1}(a_n - a_{n-1})$ are simultaneously positive.

The question is to determine $\psi(Q)$, i.e. the number of permutations with signature Q.

Niven succeeded in evaluating $\psi(Q)$ by expressing it as a determinant. Moreover he derived an inequality (see (3.1) below) from which he deduced that for any n the common value of $\psi(Q_0)$ and $\psi(-Q_0)$ is greater than any other $\psi(Q)$. Here Q_0 denotes the "alternating" vector $(1, -1, 1, -1, \ldots, (-1)^n)$. The value of $\psi(Q_0)$ was determined earlier by R. C. Entringer [1].

In this paper we present a simple algorithm for the evaluation of the $\psi(Q)$'s, and moreover we show how a short proof of Niven's inequality can be obtained by studying that algorithm.

2. *An algorithm for $\psi(Q)$ and $\theta(Q; j)$*

If $Q = (q_1, \ldots, q_{n-1})$ and if j is an integer, $1 \leq j \leq n$, then by $\theta(Q; j)$ we denote the number of permutations $a_1, \ldots, a_n$ of $1, \ldots, n$ with signature Q and with $a_n = j$. Thus

$$\Sigma_{j=1}^{n} \theta(Q, j) = \psi(Q). \tag{2.1}$$

LEMMA 1. *If* $Q = (q_1, \ldots, q_{n-1})$, $(Q, 1) = (q_1, \ldots, q_{n-1}, 1)$, *then*

$$\theta((Q, 1); j) = \Sigma_{1 \leqq h < j}\, \theta(Q; h) \tag{2.2}$$

if $1 \leq j \leq n+1$. *If* $j = 1$ *the sum is empty, and interpreted as zero.*

PROOF. If $\alpha = a_1, \ldots, a_{n+1}$ is a permutation of $1, \ldots, n+1$, then

$l(\alpha)$ will stand for a_{n+1}, and $\pi(\alpha)$ will denote the permutation of $1, \ldots, n$ that arises by taking $a_1, \ldots, a_n$ and subtracting 1 from each entry that exceeds a_{n+1}. (Example: if $\alpha = 2\,4\,5\,1\,6\,3$ then $l(\alpha) = 3$, $\pi(\alpha) = (2\,3\,4\,1\,5)$.

If α has signature $(Q, 1)$, then $\pi(\alpha)$ has signature Q. Moreover $1 \leq l(\pi(\alpha)) < l(\alpha)$. On the other hand, if β is a permutation of $1, \ldots, n$, with $l(\beta) < j$, and signature Q, then there is exactly one permutation α of $1, \ldots, n+1$ with $l(\alpha) = j$, $\pi(\alpha) = \beta$, and signature $(Q, 1)$. (This α arises from β by adding 1 to each entry exceeding $j - 1$, and putting an extra j at the end). Now (2.2) is obvious.

Lemma 2. *If* $Q = (q_1, \ldots, q_{n-1})$, $(Q, -1) = (q_1, \ldots, q_{n-1}, -1)$, *then*

$$\theta((Q, -1); i) = \textstyle\sum_{i-1<k\leqq n} \theta(Q; k) \tag{2.3}$$

if $1 \leq i \leq n + 1$. *The sum is empty, and thus zero, if* $i = n + 1$.

Proof. Obviously $\theta(Q; k) = \theta(-Q; n+1-k)$ and $\theta((Q, -1); i) = \theta((-Q, 1); n+2-i)$. Now (2.3) follows from (2.2).

Lemma 1 and 2 give a simple algorithm for the evaluation of the $\theta(Q, j)$'s. As an example we take $n = 9$, $Q = (1, -1, 1, 1, -1, 1, 1, 1)$. (Niven's determinant gave $\psi(Q) = 1099$ for this case.) We form n columns, headed by the entries of Q (apart from the first column). In the m-th column $(2 \leq m \leq n)$ we write the values of $\theta(Q_m, j)$ $(j = 1, \ldots, m$; j increasing from bottom to top), where Q_m is the vector consisting of the first $m - 1$ entries of Q. The first column consists of a single entry, viz. the number 1. Each further column contains the partial sums of the previous one, starting with 0, and working from bottom to top if the column head is $+1$, from top to bottom if it is -1. As an example we read from the list that $\theta((1, -1, 1, 1); 4) = 3$. As a check we mention that the three permutations with signature $(1, -1, 1, 1)$ and last entry 4 are 1 5 2 3 4, 2 5 1 3 4, 3 5 1 2 4. (see table on next page)

We obtain $\psi(Q) = 1099$ as the sum of the numbers in the last column, according to (2.1).

3. *Proof of Niven's inequality*

Let Q_1, Q_2 be vectors of ± 1's, possibly empty. We build new vectors $(Q_1, -1, -1, Q_2)$ and $(-Q_1, 1, -1, Q_2)$. We shall show that

$$\psi((Q_1, -1, -1, Q_2)) < \psi(-Q_1, 1, -1, Q_2). \tag{3.1}$$

	1	−1	1	1	−1	1	1	1
								477
							169	
						40		308
					0		129	
				5		40		179
			2		5		89	
		0		3		35		90
	1		2		8		54	
1		1		1		27		36
	0		1		9		27	
		1		0		18		9
			0		9		9	
				0		9		0
					9		0	
						0		0
							0	
								0

This is one of the equivalent forms of Niven's inequality (other forms can be obtained by noting that $\psi(Q)$ does not change if the order in Q is reversed or if Q is replaced by $-Q$.

In order to prove (3.1) we carry out the algorithm of the previous section both for $(Q_1, -1, -1, Q_2)$ and for $(-Q_1, 1, -1, Q_2)$. We refer to the columns headed by Q_1 or $-Q_1$ as yellow columns, those headed by Q_2 as green columns. The two remaining columns will be referred to as the red one and the blue one. So in both cases we have a group (possibly empty) of yellow columns, followed by a single red column, a single blue column, and a group (possibly empty) of green columns.

The yellow and red columns for $(Q_1, -1, -1, Q_2)$ are obtained from the corresponding ones for $(-Q_1, 1, -1, Q_2)$ by putting them upside down. It is obvious from our construction that the red column for $(Q_1, -1, -1, Q_2)$ is monotonic, at least in the weak sense, with a zero at the top and a positive number at the bottom.

Denoting this column by $c_1, \ldots, c_k$ we have $0 = c_1 \leq c_2 \leq \ldots \leq c_k$ and $0 < c_k$. It follows that $c_1 < c_k$, $c_1 + c_2 < c_k + c_{k-1}$, $c_1 + c_2 + c_3 < c_k + c_{k-1} + c_{k-2}, \ldots, c_1 + \ldots + c_{k-1} < c_k + \ldots + c_2$. Therefore, the entries of the blue column for $(Q_1, -1, -1, Q_2)$ are strictly smaller than the corresponding ones for $(-Q_1, 1, -1, Q_2)$

(apart from bottom and top entry, where we have equality). This inequality does not get lost through the green columns, since the operations to be carried out for deriving the green columns from the blue column are the same for both vectors. Since the ψ's are obtained by forming the sum in the last column, we get strict inequality for the ψ's. This finishes the proof.

4. *Appendix*

The following program, written in ALGOL 60, produces $\psi(Q)$ for all Q.

```
begin integer m, n, h, j, total; m := READ;
   begin integer array a[0:m + 1, 0:m + 1], q[1:m + 1];
                    a[0, 0] := 1;
     for n := 1 step 1 until m do
     begin NLCR; NLCR; PRINTTEXT (≮ permutations of ≯);
         ABSFIXT (2, 0, n + 1);
         PRINTTEXT (≮ elements ≯); NLCR; total := 0;
         for h := 1 step 1 until n do q[h] := −1;
         q[n + 1] := 1;
    again: for h := 1 step 1 until n + 1 do
         begin if q[h] = 1 then begin a[h, 0] := 0; for j := 1 step
                                    1 until h do
                                    a[h, j] := a[h, j − 1] +
                                    + a[h − 1, j − 1]
                                end
                           else begin a[h, h] := 0; for
                                    j := h − 1 step − 1
                                    until 0 do
                                    a[h, j] := a[h, j + 1] +
                                    + a[h − 1, j]
                                end
         end; NLCR;
         for h := 1 step 1 until n do FIXT (1, 0, q[h]);
         SPACE (4); ABSFIXT (8, 0, a[n + 1, n + 1]);
         total := total + a[n + 1, n + 1];
         for h := n step − 1 until 1 do
         if q[h] = 1 then q[h] := −1
                  else begin q[h] := 1; goto again end;
         NLCR; NLCR; PRINTTEXT (≮ total ≯);
```

```
            ABSFIXT (8, 0, total); NLCR;
        end
    end
end
```

The program was tested with $m = 9$ on the Electrologica-X8 computer of the Technological University, Eindhoven. The only input instruction is READ (integer to be read from tape). The output instructions are instructions for printing: NLCR (for new line and carriage return), SPACE(k) for spacing k places, PRINT-TEXT($\nless$ $\ngtr$) for printing the text stated between $\nless$ $\ngtr$, ABSFIXT(8, 0, Z) for printing the absolute value of the integer Z up to 8 places, and FIXT(1, 0, Z) is used for printing $+1$ or -1 according to Z having the value 1 or -1. Needless to say, the output instruction ABSFIXT(8, 0, Z) has to be revised if the ψ's get values beyond $10^8 - 1$.

The effect of the program is that for $n = 1, \ldots, m$ (m to be given on input tape) a table is printed of Q followed by $\psi(Q)$, for all 2^n possible signatures Q consisting of n entries $+1$ or -1. For each n, the table is symmetric: Q and $-Q$ (which have the same value of ψ) have the same distance to the centre of the table.

REFERENCES

[1] Entringer, R. C., *A combinatorial interpretation of the Euler and Bernoulli numbers*. Nieuw Archief Wiskunde (3) **14** (1966) 241–246.

[2] Niven, I., *A combinatorial problem on finite sequences*. Nieuw Archief Wiskunde (3) **16** (1968) 116–123.

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