

Solution to Problem 83-8: Gamma function expansion

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where

$$\eta = \tan^{-1} [ab / \{(\sqrt{a^2 + 1} - a + 1)a^2 + (\sqrt{a^2 + 1} - a - 1)b^2\}],$$

and

$$Cl_2(t) = \text{Im Li}_2(e^{it}) = \sum_1^{\infty} n^{-2} \sin(nt)$$

is Clausen's function.

Also solved by J. H. DAVENPORT (Emanuel College, Cambridge, England), W. B. JORDAN (Scotia, NY), O. P. LOSSERS (Eindhoven University of Technology, the Netherlands) and AVRAM SIDI (Technion, Haifa, Israel). Davenport showed that the integral cannot be expressed as an elementary closed form. Sidi expressed $I(a, b)$ as a rapidly convergent series of functions related to Chebyshev polynomials. Jordan found an effective series for $g(a, b)$.

Gamma Function Expansions

Problem 83-8, by W. B. JORDAN (Scotia, NY).

Prove that

$$(a) \quad \log \frac{\Gamma(z + 1/2)}{\sqrt{z}\Gamma(z)} = -\sum_{r=1}^k \frac{(1 - 2^{-2r})B_{2r}}{r(2r - 1)z^{2r-1}} + O(z^{-2k+1/2}),$$

$$(b) \quad \log \frac{\Gamma(z + 3/4)}{\sqrt{z}\Gamma(z + 1/4)} = -\sum_{r=1}^k \frac{E_{2r}}{4r(4z)^{2r}} + O(z^{-2k-1/2}),$$

$$(c)^* \quad \frac{1}{z} \left[\frac{\Gamma(z + 3/4)}{\Gamma(z + 1/4)} \right]^2 = 1 + \frac{2u}{1+} + \frac{9u}{1+} + \frac{25u}{1+} + \frac{49u}{1+} \cdot \dots,$$

where B_{2r} and E_{2r} are Bernoulli and Euler numbers, and $u = 1/64z^2$.

Solution by O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, the Netherlands).

(a), (b) The results immediately follow from Barnes' asymptotic expansion [1, form. 1.18(12)]

$$\begin{aligned} \log \Gamma(z + \alpha) &\sim (z + \alpha - 1/2) \log z - z + 1/2 \log(2\pi) \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{n+1}(\alpha)}{n(n+1)} z^{-n}, \quad z \rightarrow \infty, \quad |\arg z| < \pi, \end{aligned}$$

in which $B_{n+1}(\alpha)$ is the Bernoulli polynomial, and the known values [1, form. 1.14(7)]

$$B_{n+1}(1/2) - B_{n+1}(0) = -2^{-n}(2^{n+1} - 1)B_{n+1},$$

$$B_{n+1}(3/4) - B_{n+1}(1/4) = 2^{-2n-1}(n+1)E_n;$$

furthermore, recall that B_{n+1} vanishes for even n and E_n vanishes for odd n . The remainder terms may in fact be replaced by $O(z^{-2k-1})$ in case (a), and by $O(z^{-2k-2})$ in case (b).

In the same manner it can be shown that

$$\log \frac{\Gamma(z + \alpha + 1/2)}{\sqrt{z}\Gamma(z + \alpha)} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1} E_n(2\alpha)}{2^{n+1}n} z^{-n}, \quad z \rightarrow \infty, \quad |\arg z| < \pi,$$

where $E_n(2\alpha)$ is the Euler polynomial. This expansion includes the results (a) and (b) as special cases.

(c)* The continued fraction is known from Perron [2, p. 36, eq. (24)].

Also solved by V. BELEVITCH (Philips Research Laboratory, Brussels), BRUCE BERNDT (University of Illinois at Urbana-Champaign), M. L. GLASSER (Clarkson College), OTTO G. RUEHR (Michigan Technological University), ROBERT E. SHAFER (Berkeley, CA), JAMES A. WILSON (Iowa State University) and the proposer.

In their discussions of part (c), Belevitch mentioned Perron [2]; Berndt referred to Bauer [3], Ramanujan [4], and Stieltjes [5]; Ruehr appealed to Wall [7]; and Glasser referred to Belevitch [6].

Wilson employed the theory of orthogonal polynomials to prove (c) and obtained the generalization

$$\frac{1}{z} \frac{\Gamma(3/4 + p + z)}{\Gamma(1/4 + p + z)} \frac{\Gamma(3/4 - p + z)}{\Gamma(1/4 - p + z)} = 1 + \frac{(1 - 16p^2)/32z^2}{1} + \frac{c_1/z^2}{1} + \frac{c_2/z^2}{1} + \dots$$

for $\text{Re } z > 0$, with $c_n = [(2n + 1)^2 - 16p^2]/64$.

Shafer also referred to Ramanujan [4] and derived the following generalization:

$$\frac{\Gamma(z + \alpha + \beta + 3/4)\Gamma(z + \alpha - \beta + 3/4)\Gamma(z - \alpha + \beta + 3/4)\Gamma(z - \alpha - \beta + 3/4)}{\Gamma(z + \alpha + \beta + 1/4)\Gamma(z + \alpha - \beta + 1/4)\Gamma(z - \alpha + \beta + 1/4)\Gamma(z - \alpha - \beta + 1/4)}$$

$$= z^2 + 1/16 - \alpha^2 - \beta^2 - \frac{2(1/16 - \alpha^2)(1/16 - \beta^2)}{z^2 + 1/16 + 1^2 - \alpha^2 - \beta^2 - \frac{(1/16 - \alpha^2)(1/16 - \beta^2)}{z^2 + 1/16 + 2^2/2 - \alpha^2 - \beta^2 - \dots}}$$

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An Iteration Problem

Problem 83-9, by P. C. LIU (Great Lakes Environmental Research Laboratory, Ann Arbor, MI).

Given the integral

$$I = \int_0^\infty F^{-b} \exp\{x(1 - F^{-b/x})\} dF,$$

where $x > 0$, $0 < I \leq 1$, $2 < b < 30$, determine an efficient iteration scheme to solve for x given I and b .

Solution by L. W. FULLERTON (IMSL, Inc., Houston).

The integral may be reduced to simpler form by letting $y = x F^{-b/x}$. We find

$$(1) \quad I(x, b) = b^{-1} e^x x^{1 - ((b-1)/b)x} \Gamma\left(\frac{b-1}{b} x\right).$$

Obtaining an efficient iteration scheme depends on finding good initial approxima-