The maximum of a solution of a nonlinear differential equation

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ABSTRACT

Parameters occur in a second order nonlinear differential equation and in the initial values. The solution of this initial value problem has a maximum M. An asymptotic expression is derived for M as a function of the parameters.

1. INTRODUCTION

A colleague*) of the author has posed the following problem:

(1) $$\ddot{y} = -q\dot{y} + r(\dot{y})^2 - pe^y,$$

(2) $$y(0) = 0 \ , \ \dot{y}(0) = r^{-1}q,$$

where p, q, and r are positive real numbers, and $r < 1$. It is asked to determine

(3) $$M := \{\max y(t) \mid t \geq 0\} \ .$$

This problem arose in the study of the stress-strain behaviour of polymers that deform by crazing. In the special case under consideration the values of the parameters p and r are roughly $p = 0.01$, $r = 0.5$, and the values of q can be adjusted between 10 and 1000. It is unlikely that one can find an explicit solution. Therefore it is better to seek an expression $\tilde{M}(p,q,r)$ which approximates M with sufficient accuracy.

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2. RESULTS

The kind of formulas which we have derived are in fact asymptotic formulas. Instead of presenting them with the order symbols of Landau, we give explicit bounds for the error terms. We give two formulas for M; the first one is very simple, but not so precise as the second one.

The simple formula reads as follows:

$$(4) \qquad M = \log(q^2/(pr)) - C_1(r) + R_1 ,$$

where $C_1(r)$ depends only on r. The error R_1 satisfies the condition: if

$$(q^2/(pr))^{r-1} \le (3r)^{-1}$$

then

$$0 < R_1 < 8(q^2/(pr))^{r-1}.$$

Explanations about the determination of $C_1(r)$ will be given after formula (5).

A more complicated formula is

$$(5) \qquad M = \log(q^2/(pr)) - C_1(r) + C_2(r)(q^2/(pr))^{r-1} + R_2 ,$$

where $C_1(r)$ is the same function as in (4), and $C_2(r)$ also depends only on r. The error R_2 satisfies the condition:

if

$$(q^2/(pr))^{r-1} \le \min \{e^{1-1/r},(3r)^{-1}\}$$

then

$$|R_2| < 45r^{-1}(q^2/(pr))^{2r-2} .$$

The determination of $C_1(r)$ and $C_2(r)$ can be done in several ways:

(i) One can compute M numerically from (1), (2) for some large values of q^2/p and fixed r. Then (4) and (5) provide us some equations for $C_1(r)$ and $C_2(r)$.

(ii) One can compute m, defined by the boundary value problem (11), (12), for some large values of q^2/p, and then apply (9).

(iii) One can compute $C_1(r)$ and $C_2(r)$ using their definitions (27) and (37). Lemma's (16) and (38) provide some partial control.

3. THE BEHAVIOUR OF A SOLUTION

LEMMA. There are positive numbers T and T_1 with $0 < T < T_1$, such that y is increasing on $[0,T]$ with $\ddot{y} < 0$, y is decreasing on $[T,\infty)$ with $\ddot{y} < 0$ on $[T,T_1)$ and $\ddot{y} > 0$ on (T_1,∞). Moreover $y(t) \to -\infty$ if $t \to \infty$.

PROOF. Since $\ddot{y}(0) = -p < 0$ we have that $0 < \dot{y}(t) < r^{-1}q$ for $0 < t < \delta$, $\delta > 0$ and δ sufficiently small. Since $\ddot{y}(t) \le -p$ as long as $0 \le \dot{y}(t) \le r^{-1}q$, we see that $\dot{y}(t)$ decreases to zero in a finite time T for the first time after $t = 0$. From the fact that $\dot{y} = 0$ implies $\ddot{y} < 0$, we deduce that $\dot{y}(t) < 0$ on (T,∞). The supposition that $\ddot{y}(t) < 0$ on (T,∞) leads to a contradiction since the right hand side of (1) would become positive for t sufficiently large. The assumption that y has a lower bound leads also to a contradiction, for then y(t) would decrease to a limit, say L, for $t \to \infty$, and $\dot{y}(t)$ would increase to zero for $t \to \infty$, and hence $\ddot{y}(t)$ would tend to $-pe^{L}$ for $t \to \infty$. □

4. A FIRST APPROXIMATION OF THE MAXIMUM M.

Throughout the rest of this paper r is a fixed number between 0 and 1, and $\rho := 1/r$.

Introducing

$$(6) \qquad \alpha := (p^{-1}q^2r^{-1})^r ,$$

and transforming according to

$$(7) \qquad \tau := qt \ , \ u := \alpha e^{-ry} ,$$

we get the initial value problem

$$\frac{d^2u}{d\tau^2} + \frac{du}{d\tau} = u^{1-\rho}, \ u(0) = \alpha \ , \ \frac{du}{d\tau}(0) = -\alpha \, .$$

The problem (3) is transformed into

(8) $\quad m := \min\{u(\tau) \mid \tau \geq 0\}.$

Clearly m depends only on α (and r). By (3), (7) and (8) we have

(9) $\quad M = r^{-1}\log(\alpha/m).$

Considering u a function of $v := -\frac{du}{d\tau}$ the problem becomes

$$\frac{du}{dv} = (1 + u^{1-\rho}v^{-1})^{-1},$$

$$u(\alpha) = \alpha \ , \ u(0) = m.$$

It is easily seen that $u \geq v$ for $0 \leq v \leq \alpha$. This suggests the substitution

(10) $\quad w := u - v \, ,$

which leads to

(11) $\quad \frac{dw}{dv} = -(1 + v(v + w)^{\rho-1})^{-1} =: F(v,w) \, ,$

(12) $\quad w(0) = m \, , \ w(\alpha) = 0.$

The problem (11),(12) is our starting point for finding approximations of m. To indicate that m depends on α we sometimes write $m(\alpha)$ instead of m. A solution of (11),(12) is denoted by $w(v,\alpha)$. Obviously, for fixed v, $w(v,\alpha)$ and, hence $m(\alpha) = w(0,\alpha)$, are increasing functions of α. Clearly $w(v) > 0$ on $[0,\alpha)$, hence $w'(v) \geq -(1 + v^{\rho})^{-1}$ on $[0,\alpha]$. Integrating over $[v,\alpha]$ we find

(13) $\quad w(v,\alpha) < \int_v^{\alpha} (1 + x^{\rho})^{-1}dx < \int_v^{\infty} (1 + x^{\rho})^{-1}dx < (\rho - 1)^{-1}v^{-\rho+1} \, .$

Formula (13) provides an upper bound for m when $v = 0$.

(14) $$m < \int_0^\infty (1 + x^\rho)^{-1}\, dx = \pi(\rho \sin(\pi/\rho))^{-1} < \rho(\rho - 1)^{-1}$$

It follows that $m(\alpha)$ increases to a limit, say $m(\infty)$, when $\alpha \to \infty$. We denote by $w(v,\infty)$ the solution w of (11) with initial value $w(0) = m(\infty)$. Some properties of $w(v,\infty)$ are summarized in the following

(15) LEMMA. The solution $w(v,\infty)$ is positive and decreasing, and $w(v,\infty) \to 0$ if $v \to \infty$.

PROOF. $w(v,\infty) > 0$ since $w(v,\infty) > w(v,\alpha)$ for all $\alpha > 0$. $w(v,\infty)$ is decreasing since it satisfies (11). Let $\varepsilon > 0$. Let $A := (2\gamma/\varepsilon)^\gamma$, where $\gamma = r/(1-r)$. Since a solution w of (11) depends continuously on the initial value $w(0)$, there exists a positive number, say α, such that $w(v,\infty) - w(v,\alpha) < \frac{1}{2}\varepsilon$ for $0 \le v \le A$. Hence, by (13), $w(A,\infty) < \varepsilon/2 + \gamma A^{-1/\gamma} = \varepsilon$. □

We sample some useful properties of $m(\infty)$ in the following

(16) LEMMA.

(17) $$\log 2 < m(\alpha) < m(\infty) \qquad (\alpha \ge 1,\ \rho > 1)$$

(18) $$m(\infty) = 1 - \rho^{-1}\log\rho + O(\rho^{-1}\log\log\rho) \qquad (\rho \to \infty)$$

(19) $$e^{-1}\rho(\rho - 1)^{-1} < m(\infty) < \rho(\rho - 1)^{-1} \qquad (\rho > 1)$$

PROOF OF (17). Since $u(v) = v + w(v,\alpha)$ is increasing in $v \in [0,\alpha]$ we have for all $v \in (0,\alpha]$

(20) $$u(v) < v + m(\alpha) - \int_0^v (1 + s(u(v))^{\rho-1})^{-1} ds =$$

$$= v + m(\alpha) - (u(v))^{1-\rho}\log(1 + v(u(v))^{\rho-1}).$$

Hence,

(21) $$m(\alpha) > u - v + u^{1-\rho}\log(1 + vu^{\rho-1}) \qquad (0 < v \le \alpha).$$

The righthand side of (21) is a decreasing function of v for fixed u; since $v < u(v)$, it follows by substitution of $v := u$, that

$$(22)\qquad m(\alpha) > u^{1-\rho}\log(1 + u^{\rho}) \qquad (m(\alpha) < u \le \alpha).$$

We will show that the inequality holds for all $u \in (0,\alpha]$. Let $f(u) := u^{1-\rho}\log(1 + u^{\rho})$ for $u > 0$. Let $f(u_0) = \max\{f(u) \mid 0 < u \le \alpha\}$ with $0 < u_0 \le \alpha$. Suppose $m(\alpha) \le f(u_0)$. Then $m(\alpha) < u_0$ since, trivially, $f(u) < u$ for all $u > 0$. But by (22) we would have $m(\alpha) > f(u_0)$, a contradiction. So

$$(23)\qquad m(\alpha) > u^{1-\rho}\log(1 + u^{\rho}) \qquad (0 < u \le \alpha).$$

Of course (23) implies

$$(24)\qquad m(\infty) > u^{1-\rho}\log(1 + u^{\rho}) \qquad (u > 0).$$

If $\alpha \ge 1$, then (17) follows by substitution of $u = 1$ in (23).

PROOF OF (18). Obviously, $v + w(v,\infty) \ge m(\infty)$ for $v \in [0,m(\infty)]$, $v + w(v,\infty) > v$ for $v \in (m(\infty),\infty)$. So

$$\begin{aligned} m(\infty) &< \int_0^{m(\infty)} (1 + v(m(\infty))^{\rho-1})^{-1}dv + \int_{m(\infty)}^{\infty} (1 + v^{\rho})^{-1}dv \\ &= (m(\infty))^{1-\rho}\log(1 + (m(\infty))^{\rho}) + m(\infty)\int_1^{\infty} (1 + v^{\rho}(m(\infty))^{\rho})^{-1}dv . \end{aligned}$$

Putting $x := (m(\infty))^{\rho}$, we derive

$$\begin{aligned} 1 &< x^{-1}\log(1 + x) + \frac{1}{\rho-1}\int_0^1 (t^{\rho/(\rho-1)} + x)^{-1}dt \\ &< x^{-1}\log(1 + x) + (\rho - 1)^{-1}\log(1 + x^{-1}). \end{aligned}$$

Now (18) can be derived, using this latter inequality and (24), by standard asymptotic methods.

PROOF OF (19). Substituting $u = \exp[(\rho - 1)^{-1}]$ in (24) we get the first inequality. The second inequality follows from (14). □

Since $\frac{d}{dv}(w(v,\infty) - w(v,\alpha)) > 0$ on $[0,\alpha]$, we have, for $v \in (0,\alpha)$,

(25) $$m(\infty) - m(\alpha) < w(v,\infty) - w(v,\alpha) < w(\alpha,\infty).$$

From (13) it follows that

(26) $$w(\alpha,\infty) < (\rho - 1)^{-1}\alpha^{1-\rho}.$$

We easily infer from (19), (25) and (26) that $m(\infty) - m(\alpha) \le 3^{-1}e\, m(\infty)$ if $\alpha > (3/\rho)^{1/(\rho-1)} =: \alpha_1$. Using the fact that $-\log(1 - x) < 2.7\,x$ if $x \le e/3$ we infer

$$R_1 := -\rho\log[1 - (m(\infty))^{-1}(m(\infty) - m(\alpha))] < 2.7\,\rho\,(m(\infty))^{-1}(m(\infty) - m(\alpha))$$

if $\alpha > \alpha_1$. By (9), (19), (25) and (26) we infer (4) where

(27) $$C_1(r) := r^{-1}\log m(\infty).$$

5. A SHARPER APPROXIMATION OF M.

We want to find a second term in the asymptotic expression for $m(\alpha)$, $\alpha \to \infty$. Therefore we need the following

(28) LEMMA.

(29) $$\frac{d}{d\alpha} m(\alpha) = -F(\alpha,0)\exp\left[-\int_0^\alpha \frac{\partial F}{\partial w}(v,w(v,\alpha))dv \right].$$

PROOF. Let α be a given positive number. Then, using (11), we have for every $0 \le v \le \alpha$ and every $h > 0$ that there is a number $\eta = \eta(v,h)$ between $w(v,\alpha)$ and $w(v,\alpha + h)$ such that

$$\frac{d}{dv}(w(v,\alpha + h) - w(v,\alpha)) = (w(v,\alpha + h) - w(v,\alpha))\frac{\partial F}{\partial w}(v,\eta).$$

Dividing both sides by $w(v,\alpha + h) - w(v,\alpha)$, integrating over $[0,\alpha]$, and exponentiating we find

$$(30) \qquad m(\alpha + h) - m(\alpha) = w(\alpha,\alpha + h) \exp[-\int_0^\alpha F_w(v,\eta)dv].$$

Furthermore we have, for all $h > 0$,

$$w(\alpha,\alpha + h) = -\int_\alpha^{\alpha+h} F(v,w(v,\alpha + h)dv .$$

Since $-F(v,w)$ is decreasing in both v and w, we have

$$-hF(\alpha + h,w(\alpha,\alpha + h)) < w(\alpha,\alpha + h) < -hF(\alpha,0).$$

It follows that

$$\lim_{h\downarrow 0} h^{-1}w(\alpha,\alpha + h) = -F(\alpha,0).$$

Dividing both sides of (30) by h and taking limits for $h \downarrow 0$ we arrive at (29) for the righthand derivative of $m(\alpha)$. In a similar way we can prove (29) for the lefthand derivative. □

We define a function g by

$$(31) \qquad g(\alpha) := \int_0^\alpha \frac{\partial F}{\partial w} (v,w(v,\alpha))dv \qquad (0 \leq \alpha < \infty).$$

As we shall see

$$(32) \qquad g(\infty) := \int_0^\infty \frac{\partial F}{\partial w} (v,w(v,\infty))dv$$

is the limit value of $g(\alpha)$ for $\alpha \to \infty$. The integral in (32) exists since the integrand is continuous on $[0,\infty)$ and $\frac{\partial F}{\partial w}(v,w(v,\infty)) =$ $= O(v^{-\rho-1}) \quad (v \to \infty)$. We need an estimate of $g(\infty) - g(\alpha)$. We have

$$g(\infty) - g(\alpha) = I_1 + I_2,$$

where

$$0 < I_1 := \int_\alpha^\infty \frac{\partial F}{\partial w}(v,w(v,\infty))dv \le \int_\alpha^\infty (\rho - 1)v^{-1}(1 + v^\rho)^{-1}dv$$

$$\le (1 - r)\alpha^{-\rho},$$

and

$$I_2 := \int_0^\alpha [\frac{\partial F}{\partial w}(v,w(v,\infty)) - \frac{\partial F}{\partial w}(v,w(v,\alpha))]dv.$$

Denoting the integrand of I_2 by A, we have

$$|A| \le (w(v,\infty) - w(v,\alpha)) \left| \frac{\partial^2 F}{\partial w^2}(v,\eta) \right|$$

$$\le \rho\alpha^{1-\rho}(1 + v^\rho)^{-1}(v + w(v,\alpha))^{-2},$$

where, besides (25) and (26), we used that

$$\frac{\partial^2 F}{\partial w^2} = (\rho - 1)(v + w)^{-2}F(1 + F)(\rho + 2(\rho - 1)F),$$

and

$$|1 + F| \; |\rho + 2(\rho - 1)F| \le \rho .$$

It follows that

$$\int_0^\alpha |A|dv = \int_0^1 + \int_1^\alpha \le [(m(\alpha))^{-2} + \int_1^\alpha v^{-2-\rho}dv]\rho\alpha^{1-\rho}$$

$$\le \rho[(m(\alpha)^{-2} + (\rho + 1)^{-1}]\alpha^{1-\rho}.$$

So, using (17), we have, for $\alpha \ge 1$,

$$| g(\infty) - g(\alpha) | \le (1 - \rho^{-1})\alpha^{-\rho} + [\rho(\log 2)^{-2} + \rho(\rho + 1)^{-1}]\alpha^{1-\rho} <$$

$$< 3\rho\alpha^{1-\rho}.$$

Integrating both sides of (29) over the interval $[\alpha,\infty)$ we get

$$m(\infty) - m(\alpha) = -\int_\alpha^\infty F(\beta,0)e^{-g(\beta)}d\beta = \int_\alpha^\infty \beta^{-\rho}e^{-g(\infty)}d\beta + R ,$$

where R is given by

$$e^{g(\infty)}R = \int_\alpha^\infty (1 + \beta^\rho)^{-1}(e^{g(\infty)-g(\beta)} - 1)d\beta - \int_\alpha^\infty \beta^{-\rho}(1 + \beta^\rho)^{-1}d\beta .$$

If $\beta \geq \alpha \geq e$, then $\exp(3\rho\beta^{1-\rho}) - 1 \leq 20\rho\beta^{1-\rho}$. Hence, if $\alpha \geq e$,

$$e^{g(\infty)}|R| \leq 20(2\rho - 2)^{-1}\rho\alpha^{2-2\rho} + (2\rho - 1)^{-1}\alpha^{1-2\rho} < 11\rho(\rho - 1)^{-1}\alpha^{2-2\rho},$$

where we used that $|e^x - 1| \leq e^{|x|} - 1$ for all $x \in \mathbb{R}$
It follows that

$$(33) \qquad m(\infty) - m(\alpha) = (\rho - 1)^{-1}e^{-g(\infty)}\alpha^{1-\rho} + R,$$

where

$$(34) \qquad |R| < 11e^{-g(\infty)}\rho(\rho - 1)^{-1}\alpha^{2-2\rho} \qquad (\alpha \geq e).$$

As before, we easily infer from (19), (25) and (26) that $(m(\infty))^{-1}(m(\alpha) - m(\alpha)) \leq e/3$ if $\alpha \geq (3/\rho)^{1/(\rho-1)}$. Further, using that $|x^{-2}\log(1 + x) + x^{-1}| < 2$ if $x \leq e/3$, and (19), we deduce that, for $\alpha \geq (3/\rho)^{1/(\rho-1)}$,

$$0 < \log m(\alpha) - \log m(\infty) + (m(\infty))^{-1}(m(\infty) - m(\alpha))$$

$$< 2(m(\infty))^{-2}(m(\infty) - m(\alpha))^2 < 2e^2\rho^{-2}\alpha^{2-2\rho}.$$

By means of (33) and (34) we derive

$$(35) \qquad \rho\log m(\alpha) = \rho\log m(\infty) - \rho(m(\infty))^{-1}(\rho - 1)^{-1}e^{-g(\infty)}\alpha^{1-\rho} + R_2,$$

where

(36) $\quad |R_2| < 45\rho\alpha^{2-2\rho} \qquad (\alpha \geq \max\{e,(3/\rho)^{1/(\rho-1)}\}$.

Using (35) and (36) in (9) we arrive at (5) with

(37) $\quad C_2(r) = (1 - r)^{-1}(m(\infty))^{-1}e^{-g(\infty)}$.

Finally, we sample some properties of $g(\infty)$ and $C_2(r)$ in the following lemma.

(38) LEMMA.

(39) $\quad e^{-g(\infty)} < (1 + (m(\infty))^{-\rho})^{-1+r}$.

(40) $\quad e^{-g(\infty)} \to 0 \quad (r \downarrow 0)$

(41) $\quad e^{-g(\infty)} = O(1)(r \uparrow 1)$

(42) $\quad C_2(r) \to 0 \quad (r \downarrow 0)$

(43) $\quad C_2(r) = O((r - 1)^2) \quad (r \uparrow 1)$

PROOF OF (39). We have $\frac{\partial F}{\partial w} = -(\rho - 1)u^{-1}(1 + F)F$, where u is defined by (10). Furthermore, $1 + F(v,w) = \frac{du}{dv} < u^{\rho}(1 + u^{\rho})^{-1}$ and $-F(v,w) = 1 - \frac{du}{dv}$. Hence,

$$g(\infty) = (\rho - 1)\int_0^{\infty} \left[\frac{du}{dv} - \left(\frac{du}{dv}\right)^2\right]u^{-1} dv .$$

Since $u^{-1}(\frac{du}{dv})^2 < u^{\rho-1}(1 + u^{\rho})^{-1} \frac{du}{dv}$ we have

$$g(\infty) > (\rho - 1) \int_0^{\infty} \left[u^{-1} \frac{du}{dv} - u^{\rho-1}(1 + u\ )^{-1} \frac{du}{dv}\right]dv$$

$$= (\rho - 1) \log [(m(\infty))^{-1}(1 + (m(\infty))^{\rho})^{1/\rho}].$$

PROOF OF (40), (41), (42) and (43). Now using (18) and (19) it is a routine matter to prove the rest of lemma (38). □