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An infinite number of infinite hierarchies of conserved quantities of the Federbush model

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The construction of two Lie-Bäcklund transformations is given, which are Hamiltonian vector fields leading to an infinite number of hierarchies of conserved functionals and associated Lie-Bäcklund transformations.

I. INTRODUCTION AND GENERAL

In two recent papers^{1,2} we constructed eight [in effect four, $Y_i^+, Y_i^-, Z_i^+, Z_i^-$ ($i \in \mathbb{Z}$)] infinite hierarchies of Lie-Bäcklund transformations of the Federbush model.³ We conjectured that the hierarchies Y_i^+, Y_i^- ($i \in \mathbb{Z}$) are (x, t) independent, while the hierarchies Z_i^+, Z_i^- ($i \in \mathbb{Z}$) are linear in x and t . These Lie-Bäcklund transformations turned out to be Hamiltonian vector fields^{4,5} and the corresponding Hamiltonian densities were given. In this way we obtained t -independent and t -dependent conserved functionals for the Federbush model.

Now we shall construct two (x, t) -dependent Lie-Bäcklund transformations of degree 0, with respect to the grading, which are polynomial in x, t of degree 2 and from which we can obtain the creating and annihilating Lie-Bäcklund transformations $Z_{\pm 1}^{\pm}$, by taking the Lie bracket with the (x, t) -independent vector fields $Y_{\pm 1}^{\pm}$ (cf. the Appendix). Moreover these two vector fields turn out to be Hamiltonian vector fields and the associated Hamiltonian densities are given. This will be done in Sec. II. In Sec. III we prove a theorem from which we obtain an infinite number of infinite hierarchies of Hamiltonian vector fields, where the $Y_i^+, Y_i^-, Z_i^+, Z_i^-$ ($i \in \mathbb{Z}$) are just the first four of this infinite number of hierarchies. The Hamiltonian densities of the vector fields Z_i^{\pm} ($i = -1, 0, 1$), Y_j^{\pm} ($j = -2, -1, 0, 1, 2$) are surveyed in an Appendix at the end of this paper for reasons of completeness. In this section we shall introduce the notions needed in Secs. II and III. All computations have been carried through on a DEC-system 20 computer, using the symbolic language REDUCE⁶ and software packages^{7,8} to do the huge computations at hand.

Lie-Bäcklund transformations are vector fields V defined on the infinite jet bundle of $M, N, J^\infty(M, N)$, where M is the space of independent variables and N the space of the dependent variables. A Lie-Bäcklund transformation of a differential equation is a vector field V defined on $J^\infty(M, N)$ satisfying the condition

$$\mathcal{L}_V(D^\infty I) \subset D^\infty I, \quad (1.1)$$

where I denotes a differential ideal associated to the differential equation at hand, while $D^\infty I$ denotes its infinite prolongation to $J^\infty(M, N)$; \mathcal{L}_V is the Lie derivative with respect to

the vector field V . Since the vector field V is supposed to depend only on a finite number of variables, condition (1.1) reduces to

$$\mathcal{L}_V I \subset D^r I \quad \text{for some } r. \quad (1.2)$$

Using this method we computed Lie-Bäcklund transformations of the Federbush model.¹ It can be shown that the Lie-Bäcklund transformations in this setting are just symmetries in the works of Magri⁴ and Ten Eikelder⁵ where (generators of) symmetries of partial differential equations of evolutionary type are described as transformations on special types of infinite-dimensional spaces. Suppose that

$$\frac{du}{dt} = \Omega^{-1} dH \quad (1.3)$$

is an infinite-dimensional Hamiltonian system, where Ω is the symplectic operator, H is the Hamiltonian, and dH is the Fréchet derivative of H . Then to each Hamiltonian symmetry (also called canonical symmetry) Y there corresponds by definition a Hamiltonian $F(Y)$ such that

$$Y = \Omega^{-1} dF(Y) \quad (1.4)$$

and the Poisson bracket of F and H vanishes.^{4,5} Suppose that Y_1, Y_2 are two Hamiltonian symmetries, then $[Y_1, Y_2]$ is a Hamiltonian symmetry and

$$F([Y_2, Y_1]) = \{F(Y_1), F(Y_2)\}, \quad (1.5)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket defined by

$$\{F(Y_1), F(Y_2)\} = \langle dF(Y_1), Y_2 \rangle, \quad (1.6)$$

where $\langle \cdot, \cdot \rangle$ denotes the contraction of a one-form and a vector field.

II. CONSTRUCTION OF TWO NEW LIE-BÄCKLUND TRANSFORMATIONS OF THE FEDERBUSH MODEL

We construct two Lie-Bäcklund transformations of the Federbush model. This model is described by

$$\begin{pmatrix} i(\partial_t + \partial_x) & -m(s) \\ -m(s) & i(\partial_t - \partial_x) \end{pmatrix} \begin{pmatrix} \psi_{s,1} \\ \psi_{s,2} \end{pmatrix} = 4s\pi\lambda \begin{pmatrix} |\psi_{-s,2}|^2 & \psi_{s,1} \\ -|\psi_{-s,1}|^2 & \psi_{s,2} \end{pmatrix} \quad (s = \pm 1), \quad (2.1)$$

where the $\psi_s(x, t)$ are two-component complex valued func-

tions. Suppressing the factor $4\pi(\lambda' = 4\pi\lambda)$ and introducing eight real variables $u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4$ by

$$\begin{aligned} \psi_{1,1} &= u_1 + iv_1, & \psi_{-1,1} &= u_3 + iv_3, & m(+1) &= m_1, \\ \psi_{1,2} &= u_2 + iv_2, & \psi_{-1,2} &= u_4 + iv_4, & m(-1) &= m_2, \end{aligned} \quad (2.2)$$

Eq. (2.1) is rewritten as a system of eight nonlinear partial differential equations for the functions u_1, \dots, v_4 , i.e.,

$$\begin{aligned} u_{1t} + u_{1x} - m_1 v_2 &= \lambda R_4 v_1, \\ -v_{1t} - v_{1x} - m_1 u_2 &= \lambda R_4 u_1, \\ u_{2t} - u_{2x} - m_1 v_1 &= -\lambda R_3 v_2, \\ -v_{2t} + v_{2x} - m_1 u_1 &= -\lambda R_3 u_2, \\ u_{3t} + u_{3x} - m_2 v_4 &= -\lambda R_2 v_3, \\ -v_{3t} - v_{3x} - m_2 u_4 &= -\lambda R_2 u_3, \\ u_{4t} - u_{4x} - m_2 v_3 &= \lambda R_1 v_4, \\ -v_{4t} + v_{4x} - m_2 u_3 &= \lambda R_1 u_4, \end{aligned} \quad (2.3)$$

where in (2.3)

$$\begin{aligned} R_1 &= u_1^2 + v_1^2, & R_2 &= u_2^2 + v_2^2, \\ R_3 &= u_3^2 + v_3^2, & R_4 &= u_4^2 + v_4^2, \end{aligned} \quad (2.4a)$$

and, for further use,

$$R_{12} = R_1 + R_2, \quad R_{34} = R_3 + R_4. \quad (2.4b)$$

In two recent papers we obtained Lie-Bäcklund transformations for this model; results that are surveyed in the Appendix for reasons of completeness. Motivated by the results obtained previously,² i.e., the existence of four infinite hierarchies of Hamiltonian vector fields, two hierarchies probably being independent of x and t [Y_i^+, Y_i^- ($i \in \mathbb{Z}$)] and two hierarchies probably being linear in x and t [Z_i^+, Z_i^- ($i \in \mathbb{Z}$)] ; we now want to search for a Lie-Bäcklund transformation that is polynomial in x, t of degree 2.

We require the vector field to be of degree 0 with respect to the grading of (2.3),

$$\begin{aligned} \deg(u_1) &= \dots = \deg(v_4) = 1, \\ \deg(x) &= \deg(t) = -2, \\ \deg(\partial_x) &= \deg(\partial_t) = 2, \\ \deg(m_1) &= \deg(m_2) = 2. \end{aligned} \quad (2.5)$$

The vector field has the following required structure:

$$\begin{aligned} Y^+(2,0) &= x^2(\alpha_1 Y_2^+ + \alpha_2 m_1 Y_1^+ + \alpha_3 m_1^2 Y_0^+ + \alpha_4 m_1 Y_{-1}^+ + \alpha_5 Y_{-2}^+) + 2xt(\beta_1 Y_2^+ + \beta_2 m_1 Y_1^+ + \beta_3 m_1^2 Y_0^+ \\ &\quad + \beta_4 m_1 Y_{-1}^+ + \beta_5 Y_{-2}^+) + t^2(\gamma_1 Y_2^+ + \gamma_2 m_1 Y_1^+ + \gamma_3 m_1^2 Y_0^+ + \gamma_4 m_1 Y_{-1}^+ + \gamma_5 Y_{-2}^+) \\ &\quad + xC_1^+ + tC_2^+ + C_0^+, \end{aligned} \quad (2.6)$$

where the Y_i^+ ($i = -2, -1, 0, 1, 2$) are the vector fields associated to the conserved functionals $F(Y_i^+)$ surveyed in the Appendix; $\alpha_i, \beta_i, \gamma_i$ ($i = 1, \dots, 5$) being constant, while C_1^+, C_2^+, C_0^+ are vector fields of degree 2, 2, and 0, respectively. Substituting (2.6) into the Lie-Bäcklund condition (1.2),

$$\mathcal{L}_{Y^+} I C D^2 I, \quad (2.7)$$

and solving the resulting overdetermined system of partial differential equations for the coefficients $\alpha_i, \beta_i, \gamma_i$ ($i = 1, \dots, 5$) and the vector fields C_0^+, C_1^+, C_2^+ using (2.4), we obtained the following result:

$$Y^+(2,0) = x^2(Y_2^+ - \frac{1}{2}m_1^2 Y_0^+ + Y_{-2}^+) + 2xt(Y_2^+ - Y_{-2}^+) + t^2(Y_2^+ + \frac{1}{2}m_1^2 Y_0^+ + Y_{-2}^+) + xC_1^+ + tC_2^+, \quad (2.8)$$

where in (2.8)

$$\begin{aligned} C_1^+ &= (-2v_{1x} - m_1 u_2 - \lambda R_{34} u_1) \partial_{u_1} + (+2u_{1x} - m_1 v_2 - \lambda R_{34} v_1) \partial_{v_1} + (-2v_{2x} + m_1 u_1 - \lambda R_{34} u_2) \partial_{u_2} \\ &\quad + (+2u_{2x} + m_1 v_1 - \lambda R_{34} v_2) \partial_{v_2}, \\ C_2^+ &= (+2v_{1x} + m_1 u_2 + \lambda R_{34} u_1) \partial_{u_1} + (-2u_{1x} + m_1 v_2 + \lambda R_{34} v_1) \partial_{v_1} + (-2v_{2x} + m_1 u_1 - \lambda R_{34} u_2) \partial_{u_2} \\ &\quad + (+2u_{2x} + m_1 v_1 - \lambda R_{34} v_2) \partial_{v_2}, \end{aligned} \quad (2.9)$$

while in (2.6)

$$C_0^+ = 0. \quad (2.10)$$

In a similar way,^{1,2} motivated by the structure of the Lie algebra, we obtain another Lie-Bäcklund transformation, i.e.,

$$Y^-(2,0) = x^2(Y_2^- - \frac{1}{2}m_2^2 Y_0^- + Y_{-2}^-) + 2xt(Y_2^- - Y_{-2}^-) + t^2(Y_2^- + \frac{1}{2}m_2^2 Y_0^- + Y_{-2}^-) + xC_1^- + tC_2^-, \quad (2.11)$$

where in (2.11)

$$\begin{aligned} C_1^- &= (-2v_{3x} - m_2 u_4 + \lambda R_{12} u_3) \partial_{u_3} + (+2u_{3x} - m_2 v_4 + \lambda R_{12} v_3) \partial_{v_3} + (-2v_{4x} + m_2 u_3 + \lambda R_{12} u_4) \partial_{u_4} \\ &\quad + (+2u_{4x} + m_2 v_3 + \lambda R_{12} v_4) \partial_{v_4}, \\ C_2^- &= (+2v_{3x} + m_2 u_4 - \lambda R_{12} u_3) \partial_{u_3} + (-2u_{3x} + m_2 v_4 - \lambda R_{12} v_3) \partial_{v_3} + (-2v_{4x} + m_2 u_3 + \lambda R_{12} u_4) \partial_{u_4} \\ &\quad + (+2u_{4x} + m_2 v_3 + \lambda R_{12} v_4) \partial_{v_4}. \end{aligned} \quad (2.12)$$

To give an idea of the action of the vector fields $Y^+(2,0)$ and $Y^-(2,0)$, we compute the Lie bracket with the vector fields $Y_1^+, Y_0^+, Y_{-1}^+, Y_1^-, Y_0^-, Y_{-1}^-$ yielding the following results:

$$\begin{aligned} [Y^+(2,0), Y_1^+] &= +2Z_1^+, & [Y^-(2,0), Y_1^-] &= +2Z_1^-, & [Y^+(2,0), Y_0^+] &= 0, & [Y^-(2,0), Y_0^-] &= 0, \\ [Y^+(2,0), Y_{-1}^+] &= -2Z_{-1}^+, & [Y^-(2,0), Y_{-1}^-] &= -2Z_{-1}^-, & [Y^+(2,0), Y_i^-] &= 0, \\ [Y^-(2,0), Y_i^+] &= 0 \quad (i = -1, 0, 1). \end{aligned} \quad (2.13)$$

These results suggest setting

$$Y^\pm(1,i) = Z_i^\pm \quad \text{and} \quad Y^\pm(0,i) = Y_i^\pm \quad (i \in \mathbb{Z}).$$

Now we arrive at the following remarkable fact: the vector fields $Y^+(2,0)$ and $Y^-(2,0)$ are again Hamiltonian vector fields, the corresponding Hamiltonian densities being given by

$$\begin{aligned} \tilde{F}(Y^-(2,0)) &= x^2(\tilde{F}(Y_2^-) - \frac{1}{2}m_2^2\tilde{F}(Y_0^-) + \tilde{F}(Y_{-2}^-)) + 2xt(\tilde{F}(Y_2^-) - \tilde{F}(Y_{-2}^-)) + t^2(\tilde{F}(Y_2^-) + \frac{1}{2}m_2^2\tilde{F}(Y_0^-) + \tilde{F}(Y_{-2}^-)) \\ &= (x+t)^2\tilde{F}(Y_2^-) - \frac{1}{2}m_2^2(x+t)(x-t)\tilde{F}(Y_0^-) + (x-t)^2\tilde{F}(Y_{-2}^-) \end{aligned} \quad (2.14a)$$

and

$$\tilde{F}(Y^+(2,0)) = (x+t)^2\tilde{F}(Y_2^+) - \frac{1}{2}m_1^2(x+t)(x-t)\tilde{F}(Y_0^+) + (x-t)^2\tilde{F}(Y_{-2}^+), \quad (2.14b)$$

where the densities $\tilde{F}(Y_i^\pm)$ ($i = -2, 0, 2$) are given in the Appendix.

This result shows a remarkable resemblance to the results for the Benjamin-Ono equation.⁹

III. PROOF OF THE EXISTENCE OF AN INFINITE NUMBER OF HIERARCHIES

In this section we shall first prove a generalization of a lemma proved in Ref. 2. The main theorem of this section is a direct application of Lemma 3.1 to the special cases at hand and leads to the existence of an infinite number of infinite hierarchies of algebraically independent conserved functionals for the Federbush model. The associated Lie-Bäcklund transformations are obtained from these results by application of formula (1.4).

We state the following lemma.

Lemma 3.1: Let $H_n^r(u,v)$, $K_n^r(u,v)$ be defined by

$$H_n^r(u,v) = \int_{-\infty}^{\infty} x^r(u_n^2 + v_n^2) \quad (r, n = 0, 1, \dots), \quad (3.1)$$

$$K_n^r(u,v) = \int_{-\infty}^{\infty} x^r(u_{n+1}v_n - v_{n+1}u_n) \quad (r, n = 0, 1, \dots),$$

where in (3.1)

$$u_n = \left(\frac{d}{dx}\right)^n u, \quad v_n = \left(\frac{d}{dx}\right)^n v,$$

and r, n such that the degree of H_n^r, K_n^r is positive. Define the Poisson bracket of functionals F, L by

$$\{F, L\} = \int_{-\infty}^{\infty} \left(+ \frac{\delta F}{\delta v} \frac{\delta L}{\delta u} - \frac{\delta F}{\delta u} \frac{\delta L}{\delta v} \right), \quad (3.2)$$

then

$$\{H_1^1, H_n^r\} = 4(n-r)K_n^r, \quad (3.3a)$$

$$\{H_1^1, K_n^r\} = (4(n-r) + 2)H_{n+1}^r + r(r-1)(r-n-1)H_n^{r-2}, \quad (3.3b)$$

$$\{H_1^2, H_n^r\} = 4(2n-r)K_n^{r+1}, \quad (3.3c)$$

$$\{H_1^2, K_n^r\} = (2n+1-r)(4H_{n+1}^{r+1} - r^2H_n^{r-1}) \quad (r, n = 0, 1, \dots). \quad (3.3d)$$

Proof: Relations (3.3a) and (3.3b) are generalizations of formulas given in Ref. 2 and can be proved in a similar way. We now prove (3.3c) and (3.3d). Calculation of the Fréchet derivatives of H_n^r, K_n^r yields

$$\frac{\delta H_n^r}{\delta u} = \left(-\frac{d}{dx}\right)^n (2x^r u_n), \quad \frac{\delta H_n^r}{\delta v} = \left(-\frac{d}{dx}\right)^n (2x^r v_n), \quad (3.4a)$$

$$\frac{\delta K_n^r}{\delta u} = \left(-\frac{d}{dx}\right)^{n+1} (x^r v_n) - \left(-\frac{d}{dx}\right)^n (x^r v_{n+1}), \quad \frac{\delta K_n^r}{\delta v} = -\left(-\frac{d}{dx}\right)^{n+1} (x^r u_n) + \left(-\frac{d}{dx}\right)^n (x^r u_{n+1}). \quad (3.4b)$$

Substitution of (3.4a) into (3.2) results in

$$\begin{aligned}
\{H_1^2, H_n^r\} &= \int_{-\infty}^{\infty} -\frac{d}{dx} (2x^2 v_1) \cdot (-1)^n \left(\frac{d}{dx}\right)^n (2x^r u_n) + \frac{d}{dx} (2x^2 u_1) \cdot (-1)^n \left(\frac{d}{dx}\right)^n (2x^r v_n) \\
&= (-1)^n (-1)^{n-1} \int_{-\infty}^{\infty} \left(\frac{d}{dx}\right)^n (2x^2 u_1) \frac{d}{dx} (2x^r v_n) - \left(\frac{d}{dx}\right)^n (2x^2 v_1) \frac{d}{dx} (2x^r u_n) \\
&= -4 \int_{-\infty}^{\infty} (x^2 u_{n+1} + 2nxu_n + n(n-1)u_{n-1})(x^r v_{n+1} + rx^{r-1}v_n) \\
&\quad - (x^2 v_{n+1} + 2nxv_n + n(n-1)v_{n-1})(x^r u_{n+1} + rx^{r+1}u_n) \\
&= -4 \int_{-\infty}^{\infty} rx^{r+1}(u_{n+1}v_n - v_{n+1}u_n) - 2nx^{r+1}(u_{n+1}v_n - v_{n+1}u_n) \\
&\quad + n(n-1)x^r(v_{n+1}u_{n-1} - u_{n+1}v_{n-1}) + n(n-1)rx^{r-1}(v_n u_{n-1} - u_n v_{n-1}) = 4(2n-r)K_n^{r+1}, \quad (3.5)
\end{aligned}$$

which proves relation (3.3c). The last equality in (3.5) results from the fact that the last two terms are just a total derivative of $n(n-1)x^r(v_n u_{n-1} - u_n v_{n-1})$.

In order to prove (3.3d) we substitute (3.4a) and (3.4b) into (3.2), which results in

$$\begin{aligned}
\{H_1^2, K_n^r\} &= \int_{-\infty}^{\infty} -\frac{d}{dx} (2x^2 v_1) \cdot \left[(-1)^{n+1} \left(\frac{d}{dx}\right)^{n+1} (x^r v_n) - (-1)^n \left(\frac{d}{dx}\right)^n (x^r v_{n+1}) \right] \\
&\quad + \frac{d}{dx} (2x^2 u_1) \cdot \left[(-1)^n \left(\frac{d}{dx}\right)^{n+1} (x^r u_n) + (-1)^n \left(\frac{d}{dx}\right)^n (x^r u_{n+1}) \right]. \quad (3.6)
\end{aligned}$$

Integration, n times, of the terms in brackets leads to

$$\begin{aligned}
\{H_1^2, K_n^r\} &= 2 \int_{-\infty}^{\infty} \left(\frac{d}{dx}\right)^{n+1} (x^2 v_1) \cdot \left(\frac{d}{dx} (x^r v_n) + x^r v_{n+1}\right) + \left(\frac{d}{dx}\right)^{n+1} (x^2 u_1) \cdot \left(\frac{d}{dx} (x^r u_n) + x^r u_{n+1}\right) \\
&= 2 \int_{-\infty}^{\infty} (x^2 v_{n+2} + 2(n+1)xv_{n+1} + n(n+1)v_n)(2x^r v_{n+1} + rx^{r-1}v_n) \\
&\quad + (x^2 u_{n+2} + 2(n+1)xu_{n+1} + n(n+1)u_n)(2x^r u_{n+1} + rx^{r-1}u_n). \quad (3.7)
\end{aligned}$$

Expanding the expressions in (3.7), we arrive after a short computation at

$$\{H_1^2, K_n^r\} = (2n+1-r)(4H_{n+1}^{r+1} - r^2 H_n^{r-1}), \quad (3.8)$$

which proves (3.3d).

We are now in a position to prove the main theorem of this section.

Theorem 3.1: The conserved functionals $F(Y^\pm(2,0))$ associated to the Lie-Bäcklund transformations $Y^\pm(2,0)$ generate an infinite number of hierarchies, starting at the $F(Y_j^+)$, $F(Y_j^-)$ hierarchies by repeated action of the Poisson bracket.

The $F(Z_j^+)$, $F(Z_j^-)$ hierarchies are obtained by the first step of this procedure [cf. (2.13)]. Moreover the $F(Y_j^+)$, $F(Y_j^-)$ hierarchies are obtained from $F(Y_{\pm 1}^\pm)$ by repeated action of the conserved functional

$$F(Z_{\pm 1}^\pm) = \pm \frac{1}{2} F([Y^\pm(2,0), Y_{\pm 1}^\pm]) \quad (3.9)$$

(cf. Table I).

Proof: The proof of theorem 3.1 is a straightforward application of Lemma 3.1 and the observation that the (λ, m_1, m_2) -independent parts of the conserved densities associated to $Y_{\pm 1}^\pm$, $Y(+2,0)$, $Y(-2,0)$, (A3), (A4), (2.14a), and (2.14b) are given by

$$Y_1^+ \rightarrow -\frac{1}{2}(u_{2x}v_2 - v_{2x}u_2), \quad Y_{-1}^+ \rightarrow -\frac{1}{2}(u_{1x}v_1 - v_{1x}u_1), \quad (3.10)$$

$$\begin{aligned}
Y_1^- &\rightarrow -\frac{1}{2}(u_{4x}v_4 - v_{4x}u_4), \quad Y_{-1}^- \rightarrow -\frac{1}{2}(u_{3x}v_3 - v_{3x}u_3), \\
Y^+(2,0) &\rightarrow -\frac{1}{2}(x+t)^2(u_{2x}^2 + v_{2x}^2) - \frac{1}{2}(x-t)^2(u_{1x}^2 + v_{1x}^2), \quad (3.11)
\end{aligned}$$

$$Y^-(2,0)$$

$$\rightarrow -\frac{1}{2}(x+t)^2(u_{4x}^2 + v_{4x}^2) - \frac{1}{2}(x-t)^2(u_{3x}^2 + v_{3x}^2).$$

Note that in applying Lemma 3.1 we have to choose $(u, v) \equiv (u_2, v_2)$, $(u, v) = (u_1, v_1), \dots$, where now (u_i, v_i) ($i = 1, \dots, 4$) refer to (2.2)!

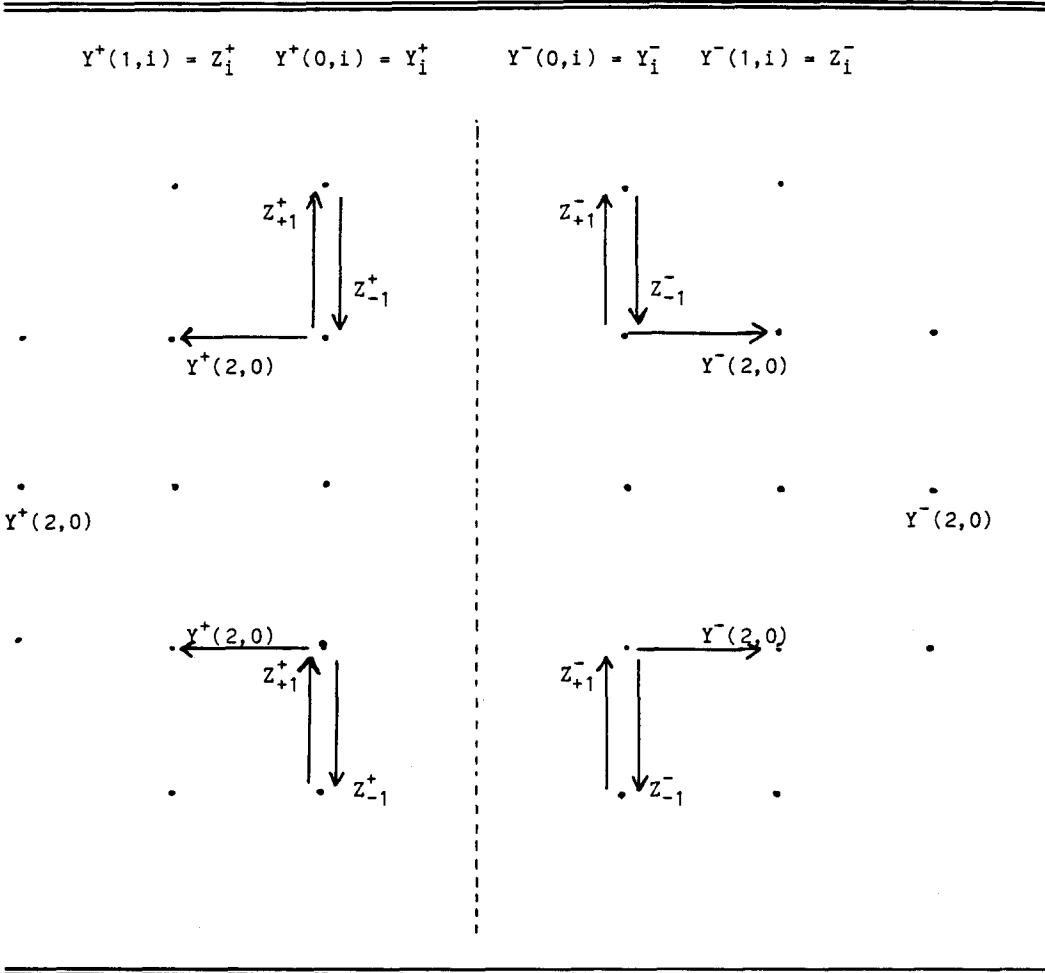
Remark: The Lie-Bäcklund transformations of degree 0, $Y_0^+ = Y^+(0,0)$, $Z_0^+ = Y^+(1,0)$, $Y^+(2,0)$ and $Y_0^- = Y^-(0,0)$, $Z_0^- = Y^-(1,0)$, $Y^-(2,0)$ being just the first few of them, can probably be obtained by the action of $Z_{\pm 1}^\pm$ on the vector fields of degree 1 (cf. Ref. 1), i.e.,

$$Y^\pm(k,0) = \alpha_k [Z_{\pm 1}^\pm, Y^\pm(k, \pm 1)].$$

IV. CONCLUSION

By the construction of two Hamiltonian vector fields $Y^+(2,0)$ and $Y^-(2,0)$ we construct an infinite number of infinite hierarchies, the elements of which are all Hamiltonian vector fields. The associated conserved functionals are obtained by the action of the Poisson bracket.

TABLE I. The Lie algebraic picture of the Lie-Bäcklund transformations.



APPENDIX: CONSERVED FUNCTIONALS FOR THE FEDERBUSH MODEL

We summarize here some of the results obtained in Ref. 2 that are of interest in Sec. II. We derived the following conserved functionals:

$$F(*) = \int_{-\infty}^{\infty} \tilde{F}(*) dx, \tag{A1}$$

where the densities $\tilde{F}(*)$ are given by

$$\tilde{F}(Y_0^+) = \frac{1}{2}(R_1 + R_2), \quad \tilde{F}(Y_0^-) = \frac{1}{2}(R_3 + R_4), \tag{A2}$$

$$\begin{aligned} \tilde{F}(Y_1^+) &= -\frac{1}{2}(u_{2x}v_2 - u_2v_{2x}) + (\lambda/4)R_{34}R_2 \\ &\quad - \frac{1}{2}m_1(u_1u_2 + v_1v_2), \\ \tilde{F}(Y_{-1}^+) &= -\frac{1}{2}(u_{1x}v_1 - u_1v_{1x}) + (\lambda/4)R_{34}R_1 \\ &\quad + \frac{1}{2}m_1(u_1u_2 + v_1v_2), \end{aligned} \tag{A3}$$

$$\begin{aligned} \tilde{F}(Y_1^-) &= -\frac{1}{2}(u_{4x}v_4 - u_4v_{4x}) - (\lambda/4)R_{12}R_4 \\ &\quad - \frac{1}{2}m_2(u_3u_4 + v_3v_4), \\ \tilde{F}(Y_{-1}^-) &= -\frac{1}{2}(u_{3x}v_3 - u_3v_{3x}) - (\lambda/4)R_{12}R_3 \\ &\quad + \frac{1}{2}m_2(u_3u_4 + v_3v_4), \end{aligned}$$

and

$$\begin{aligned} \tilde{F}(Y_2^+) &= -\frac{1}{2}(u_{2x}^2 + v_{2x}^2) + (\lambda/2)R_{34}(u_{2x}v_2 - u_2v_{2x}) \\ &\quad - \frac{1}{2}m_1(u_{2x}v_1 - u_1v_{2x}) - \frac{1}{8}\lambda^2 R_{34}^2 R_2 \\ &\quad + \frac{1}{4}m_1\lambda R_{34}(u_1u_2 + v_1v_2) - \frac{1}{8}m_1^2 R_{12}, \\ \tilde{F}(Y_{-2}^+) &= -\frac{1}{2}(u_{1x}^2 + v_{1x}^2) + (\lambda/2)R_{34}(u_{1x}v_1 - u_1v_{1x}) \\ &\quad + \frac{1}{2}m_1(u_{1x}v_2 - u_2v_{1x}) - \frac{1}{8}\lambda^2 R_{34}^2 R_1 \\ &\quad - \frac{1}{4}m_1\lambda R_{34}(u_1u_2 + v_1v_2) - \frac{1}{8}m_1^2 R_{12}, \\ \tilde{F}(Y_2^-) &= -\frac{1}{2}(u_{4x}^2 + v_{4x}^2) - (\lambda/2)R_{12}(u_{4x}v_4 - u_4v_{4x}) \\ &\quad - \frac{1}{2}m_2(u_{4x}v_3 - u_3v_{4x}) - \frac{1}{8}\lambda^2 R_{12}^2 R_4 \\ &\quad - \frac{1}{4}m_2\lambda R_{12}(u_3u_4 + v_3v_4) - \frac{1}{8}m_2^2 R_{34}, \\ \tilde{F}(Y_{-2}^-) &= -\frac{1}{2}(u_{3x}^2 + v_{3x}^2) - (\lambda/2)R_{12}(u_{3x}v_3 - u_3v_{3x}) \\ &\quad + \frac{1}{2}m_2(u_{3x}v_4 - u_4v_{3x}) - \frac{1}{8}\lambda^2 R_{12}^2 R_3 \\ &\quad + \frac{1}{4}m_2\lambda R_{12}(u_3u_4 + v_3v_4) - \frac{1}{8}m_2^2 R_{34}. \end{aligned} \tag{A4}$$

The t -dependent conserved functionals are

$$\begin{aligned}\tilde{F}(Z_0^+) &= (x+t)\tilde{F}(Y_1^+) - (x-t)\tilde{F}(Y_{-1}^+), \\ \tilde{F}(Z_0^-) &= (x+t)\tilde{F}(Y_1^-) - (x-t)\tilde{F}(Y_{-1}^-),\end{aligned}\quad (\text{A5})$$

and

$$\begin{aligned}\tilde{F}(Z_1^+) &= (x+t)\tilde{F}(Y_2^+) - \frac{1}{4}m_1^2(x-t)\tilde{F}(Y_0^+), \\ \tilde{F}(Z_{-1}^+) &= -(x-t)\tilde{F}(Y_{-2}^+) + \frac{1}{4}m_1^2(x+t)\tilde{F}(Y_0^+),\end{aligned}\quad (\text{A6})$$

$$\tilde{F}(Z_1^-) = (x+t)\tilde{F}(Y_2^-) - \frac{1}{4}m_2^2(x-t)\tilde{F}(Y_0^-),$$

$$\tilde{F}(Z_{-1}^-) = -(x-t)\tilde{F}(Y_{-2}^-) + \frac{1}{4}m_2^2(x+t)\tilde{F}(Y_0^-).$$

The vector fields Y_i^\pm ($i = -2, -1, 0, 1, 2$) and Z_j^\pm ($j = -1, 0, 1$) obtained from (A2)–(A6) by

$$Y = \Omega^{-1} dF(Y). \quad (\text{A7}).$$

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