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A Finite Difference Discretization Method for Elliptic Problems on Composite Grids

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Abstract

In this paper we discuss a simple finite difference method for the discretization of elliptic boundary value problems on composite grids. For the model problem of the Poisson equation we prove stability of the discrete operator and bounds for the global discretization error. These bounds clearly show how the discretization error depends on the grid size of the coarse grid, on the grid size of the local fine grid and on the order of the interpolation used on the interface. Furthermore the constants in these bounds do not depend on the quotient of coarse grid size and fine grid size. We also discuss an efficient solution method for the resulting composite grid algebraic problem.

A.M.S. Classifications: 65N06, 65N15, 65N22
Keywords : finite difference scheme, local refinement, error estimates

1 Introduction

Many boundary value problems produce solutions that possess highly localized properties. In this paper we consider two-dimensional elliptic boundary value problems with one or a few small regions with high activity. In these regions the solution varies much more rapidly than in the remaining part of the domain. We are mainly interested in problems in which this behaviour is due to the source term (e.g. a strong well). In general, from the point of view of efficiency, it is not attractive to use a uniform grid for discretizing such a problem. Often the use of local grid refinement techniques will be advantageous.

In this paper we study a local grid refinement technique based on the combination of several uniform grids with different grid sizes that cover different parts of the domain. The continuous solution is then approximated on the composite grid which is the union of the uniform subgrids. Methods based on such a technique have been addressed by several authors. The finite volume element (FVE) method used in McCormick's fast adaptive composite grid (FAC) method is of this type and an analysis of this composite grid discretization is given in [3, 12]. This finite volume type of method uses vertex-centered approximations. A finite volume method for composite grids using special cell-centered approximations is analyzed in [5, 10]. The local defect correction (LDC) method introduced in [9] is a very general approach which can be used for discretization on a composite grid too. For discretization of parabolic problems on composite grids we refer to [6] and the references therein.

In this paper we analyze a very simple discretization technique based on standard finite differences on uniform grids and a suitable (linear or quadratic) interpolation on the interface between a coarse and a fine grid. The method is closely related to a special case of the LDC method. In fact, the idea to study this discretization method originated from an analysis of the LDC method in [7].

We consider a discretization in which all composite grid points on the interface are also part of a global coarse grid and we use the corresponding standard coarse grid stencils at these grid points. So we do not always use the nearest neighbours in the composite grid discretization on the interface. At the fine grid points adjacent to an interface we use the standard fine grid discretization stencil. Information needed on the interface is then provided by a suitable (piecewise linear or piecewise quadratic) interpolation. At all other grid points we use the standard finite difference discretization.

We will discuss how this approach results in a natural way from the LDC method. Two important issues in this discretization approach have to be adressed: the size of the global discretization error and a solution method for the resulting composite grid algebraic problem. We will discuss both issues although the emphasis lies on the first one. Using techniques on M-matrices and the discrete maximum principle we prove stability of the discrete operator and (optimal) estimates for the global discretization error. These estimates clearly show how the discretization error depends on the grid size of the coarse grid, on the grid size of the local fine grid and on the order of the interpolation used on the interface. Furthermore, the constants in our bounds do not depend on the refinement factor (i.e. the quotient of coarse grid size and fine grid size).

Nice features of the present discretization method are its simplicity, the optimal order discretization error and the fact that we can use an efficient solver for the resulting algebraic system. On the other hand, unlike the finite volume techniques we do not have a conservation property and in the analysis we need a high regularity of the solution (we use fourth order derivatives).

The remainder of this paper is organized as follows. In Section 2 we first consider a simple two-point boundary value problem. We discuss very elementary properties of discrete Greens functions corresponding to two types of composite grid discretizations. Most of these properties, which play an important role in the analysis of the discretization error, can be generalized to the two-dimensional case. This generalization and the resulting error estimates for a two-dimensional model problem are the topic of Section 3. In Section 4 we show how the composite grid discretization is related to the LDC method. Also we show how the composite grid algebraic problem can be solved using the LDC method. In Section 5 we present some numerical results and we discuss another seemingly rather natural finite difference discretization method on composite grids.

2 A One Dimensional Model Problem

In this section we consider a very elementary two-point boundary value problem. We introduce two different composite grid discretizations for this problem. The main issue is to show some interesting properties of the discrete Greens functions related to certain grid points on, or close to, the interface between the coarse and the fine grid. In the next sections we will show that these properties can be generalized to the two-dimensional case. The approach used in the analysis in this section is of interest, because a similar approach, with some technical complications, is used in the two-dimensional analysis in Section 3.

We consider the following two-point boundary value problem

$$\begin{aligned} &-U_{xx}(x) = f(x), \quad x \in \Omega := (0,1) \\ &U(0) = U(1) = 0. \end{aligned} \tag{2.1}$$

We use a composite grid based on a partitioning of Ω as

$$\Omega = (0,\Gamma) \cup [\Gamma, 1) =: \Omega_l \cup (\Omega \backslash \Omega_l) \quad (0 < \Gamma < 1).$$

We assume a "coarse" grid size H such that $1/H \in \mathbb{N}$ and $\Gamma/H \in \mathbb{N}$ and we introduce a "fine" grid size h given by

$$h := H/\sigma, \quad \sigma \in \mathbb{N}. \tag{2.2}$$

A fine grid Ω_c^h on Ω_l and a coarse grid Ω_c^H on $\Omega \backslash \Omega_l$ are defined as follows:

$$n_1 := \Gamma/h - 1, \quad \Omega_c^h := \{ih \mid 1 \le i \le n_1\}, \tag{2.3a}$$

$$n_2 := (1-\Gamma)/H, \quad \Omega_c^H := \{\Gamma + iH \mid 0 \le i \le n_2 - 1\}. \tag{2.3b}$$

The composite grid $\Omega_c^{h,H}$ is given by

$$\Omega_c^{h,H} := \Omega_c^h \cup \Omega_c^H. \tag{2.4}$$

The composite grid is illustrated in Figure 1.

We now discuss finite difference discretizations of (2.1) on this composite grid. At the grid points in Ω_c^h we use the standard stencil $h^{-2}[-1 \ \ 2 \ \ -1]$ for approximating $-d^2/dx^2$. At the points in $\Omega_c^H \backslash \{\Gamma\}$ we use the stencil $H^{-2}[-1 \ \ 2 \ \ -1]$. For the approximation at the interface point Γ we use two approaches, resulting in stiffness matrices $A_{h,H}$ and $\tilde{A}_{h,H}$. In Γ we consider the following two stencils ($u \in l^2(\Omega_c^{h,H})$, σ as in (2.2)):

$$[A_{h,H}]_\Gamma u = H^{-2}(-u(\Gamma - H) + 2u(\Gamma) - u(\Gamma + H)), \tag{2.5a}$$

$$[\tilde{A}_{h,H}]_\Gamma u = H^{-2}\left(-\frac{2\sigma^2}{\sigma+1}u(\Gamma - h) + 2\sigma u(\Gamma) - \frac{2\sigma}{\sigma+1}u(\Gamma + H)\right). \tag{2.5b}$$

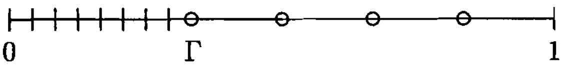

Figure 1: Composite grid $\Omega_c^{h,H}$, $H = 1/6$, $h = 1/24$.

Note that in (2.5a) the interface point Γ is treated as a coarse grid point; the corresponding local discretization error is $\mathcal{O}(H^2)$. In (2.5b) we have a nonsymmetric finite element type of stencil with local discretization error $\mathcal{O}(H)$. In both cases the constant in $\mathcal{O}(.)$ does not depend on $\sigma = H/h$.

First we analyze the discrete operator $A_{h,H}$. We introduce a block-partitioning corresponding to (2.4). By e_k we denote the k-th standard basis vector in $\mathbb{R}^m$ ($m = n_1$ or $m = n_2$). The matrix $A_{h,H}$ has the following block form:

$$A_{h,H} = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}, \tag{2.6a}$$

with

$$A_{11} = h^{-2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \tag{2.6b}$$

$$A_{22} = H^{-2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n_2 \times n_2}, \tag{2.6c}$$

$$A_{12} = h^{-2} e_{n_1} e_1^T \in \mathbb{R}^{n_1 \times n_2}, \quad A_{21} = H^{-2} e_1 e_{n_1+1-\sigma}^T \in \mathbb{R}^{n_2 \times n_1}. \tag{2.6d}$$

We recall that a matrix $B \in \mathbb{R}^{n \times n}$ is called *monotone* if B is regular and $B^{-1} \geq 0$ holds.

Theorem 2.1. $A_{h,H}$ is monotone and $\|A_{h,H}^{-1}\|_\infty \leq 1/8$ holds.
Proof. The result follows from a standard argument: $A_{h,H}$ is an M-matrix and

$$A_{h,H}((\frac{1}{2}x(1-x))_{|\Omega_c^{h,H}}) = (1, 1, \ldots, 1)^T$$

holds. □

By $\Gamma_h^\star$ we denote the fine grid point adjacent to the interface Γ, i.e. $\Gamma_h^\star := \Gamma - h$. The corresponding basis vector $(e_{n_1}^T\ \emptyset)^T$ (partitioning as in (2.6)) is denoted by $e_{\Gamma_h^\star}$.

Theorem 2.2. The following inequality holds:

$$\|A_{h,H}^{-1} e_{\Gamma_h^*}\|_\infty \leq (\Gamma + H)\frac{h^2}{H}. \tag{2.7}$$

Proof. We use the Schur complement $S := A_{22} - A_{21}A_{11}^{-1}A_{12}$ and the block LU-factorization

$$A_{h,H} = \begin{bmatrix} I & \emptyset \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & -A_{12} \\ \emptyset & S \end{bmatrix}.$$

Thus $A_{h,H}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e_{\Gamma_h^*} = \begin{pmatrix} e_{n_1} \\ \emptyset \end{pmatrix}$ results in

$$x_2 = S^{-1}A_{21}A_{11}^{-1}e_{n_1}, \quad x_1 = A_{11}^{-1}A_{12}x_2 + A_{11}^{-1}e_{n_1}. \tag{2.8}$$

The discrete maximum principle yields $\|A_{11}^{-1}A_{12}\|_\infty \leq 1$, and thus we get

$$\begin{aligned} \|x_1\|_\infty &= \|A_{11}^{-1}A_{12}x_2 + A_{11}^{-1}A_{12}(h^2e_1)\|_\infty \\ &\leq \|A_{11}^{-1}A_{12}\|_\infty(\|x_2\|_\infty + h^2) \\ &\leq \|x_2\|_\infty + h^2. \end{aligned} \tag{2.9}$$

Note that due to $S^{-1} = \begin{bmatrix} \emptyset & I \end{bmatrix} A_{h,H}^{-1} \begin{bmatrix} \emptyset \\ I \end{bmatrix}$ and the monotonicity of $A_{h,H}$ (Theorem 2.1) we have $S^{-1} \geq 0$. Also the inequalities $A_{21} \geq 0$, $A_{11}^{-1} \geq 0$ hold. Using this and $A_{11}^{-1}e_{n_1} = A_{11}^{-1}A_{12}(h^2e_1) \leq \|A_{11}^{-1}A_{12}\|_\infty h^2(1,1,\ldots,1)^T \leq h^2(1,1,\ldots,1)^T$ we get

$$0 \leq A_{21}A_{11}^{-1}e_{n_1} = H^{-2}e_1(e_{n_1+1-\sigma}^T A_{11}^{-1}e_{n_1}) \leq \sigma^{-2}e_1. \tag{2.10}$$

A simple calculation shows that the Schur complement S is given by

$$S = H^{-2}\begin{bmatrix} 1+\frac{H}{\Gamma} & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

From this we see that $S(1,1,\ldots,1)^T = H^{-2}(H/\Gamma,0,0,\ldots,0,1)^T \geq (H\Gamma)^{-1}e_1$ holds. This yields

$$0 \leq S^{-1}e_1 \leq H\Gamma(1,1,\ldots,1)^T. \tag{2.11}$$

Combination of (2.8), (2.10), (2.11) results in

$$\|x_2\|_\infty \leq \sigma^{-2}\|S^{-1}e_1\|_\infty \leq \Gamma H\sigma^{-2},$$

and using this in (2.9) completes the proof. □

Remark 2.3. Using a standard rank one perturbation argument one can derive an explicit expression for S^{-1} (the inverse of the Schur complement). Using this in a more detailed analysis then shows that the following equality holds:

$$\|A_{h,H}^{-1}e_{\Gamma_h^*}\|_\infty = (\Gamma - h)(1 - \Gamma + H)\frac{h^2}{H}. \tag{2.12}$$

So the bound in (2.7) is sharp in the sense that it shows the actual convergence rate for h and/or $H \downarrow 0$.

From the result in (2.12) we see that for H fixed the norm of the discrete Greens function corresponding to Γ_h^* decreases proportional to h^2 for $h \downarrow 0$. This behaviour is similar to the case of a discrete Greens function corresponding to a grid point next to the boundary in a global uniform grid with grid size h. In Section 3 we will see that a similar result holds in the two-dimensional case.

The situation is very different if we consider the discrete Greens function e_Γ corresponding to the interface point Γ (i.e. $e_\Gamma = (\emptyset \; e_1^T)^T$). Using an approach as in the derivation of (2.12) yields the following:

$$\|A_{h,H}^{-1} e_\Gamma\|_\infty = \Gamma(1-\Gamma)H. \tag{2.13}$$

So now there is a damping as if e_Γ is an interior point of a global uniform grid with grid size H.

We now discuss comparable results for the case with stiffness matrix $\tilde{A}_{h,H}$ (cf. (2.5b)). First we note that Theorem 2.1 (and the corresponding proof) also holds if $A_{h,H}$ is replaced by $\tilde{A}_{h,H}$. A straightforward analysis, using arguments similar to the case with stiffness matrix $A_{h,H}$, yields the following:

$$\|\tilde{A}_{h,H}^{-1} e_{\Gamma_h^*}\|_\infty = (\Gamma - h)(1 - \Gamma + h)h, \tag{2.14}$$

$$\|\tilde{A}_{h,H}^{-1} e_\Gamma\|_\infty = \frac{1}{2}\Gamma(1-\Gamma)(1+\frac{1}{\sigma})H. \tag{2.15}$$

Note that the result is (2.15) is very similar to the result in (2.13). However, there is a significant difference between the results in (2.12) and in (2.14). For H fixed we have a discrete Greens function of size $\mathcal{O}(h^2)$ in (2.12), whereas in (2.14) we have a discrete Greens function of size $\mathcal{O}(h)$. In Section 3 and Section 5 we will see that results similar to those in (2.12), (2.14) hold in the two-dimensional case.

Remark 2.4. Using standard techniques and the results in (2.13), (2.15) we can derive (sharp) bounds for the global discretization error. Define

$$e_{h,H}^{(1)} := U_{|\Omega_c^{h,H}} - \tilde{A}_{h,H}^{-1} f_{h,H}, \quad e_{h,H}^{(2)} := U_{|\Omega_c^{h,H}} - A_{h,H}^{-1} f_{h,H},$$

with U the continuous solution, $f_{h,H}(x) = f(x)$ for $x \in \Omega_c^{h,H}$. Then for $j = 1,2$ we obtain:

$$\|e_{h,H}^{(j)}\|_\infty \le C_1 h^2 + C_2 H^2 + C_3^{(j)} H^{j+1}. \tag{2.16}$$

The constants C_1, C_2 depend on $\max\{|\frac{d^4}{dx^4}U(x)| \mid x \in (0,\Gamma)\}$ and $\max\{|\frac{d^4}{dx^4}U(x)| \mid x \in (\Gamma,1)\}$ respectively, and $C_3^{(j)}$ depends on $|U^{(j+2)}(x)|$ with x in a small neighbourhood of Γ. From (2.16) we conclude that the difference between $A_{h,H}$ and $\tilde{A}_{h,H}$ as discussed above has only little influence on the global discretization error. In Section 5 we will see that a similar conclusion can not be drawn in the two-dimensional case.

Remark 2.5. Results very similar to those in Theorem 2.1 and Theorem 2.2 can be obtained if we consider a composite grid with two interface points, i.e. Ω_l is of the form (Γ_1, Γ_2) with $0 < \Gamma_1 < \Gamma_2 < 1$.

3 Finite Difference Discretization on Two-Dimensional Composite Grids

In this section we analyze a two-dimensional finite difference discretization method. Essentially we generalize the analysis of the previous section to obtain a result for the global discretization error on a composite grid. We will show what the effect is of the interpolation used on the interface. We consider a discretization method in which the interface points are treated as coarse grid points (cf. (2.5a)). In Secton 5 we will discuss a method that can be seen as a generalization of the one dimensional approach in (2.5b) (i.e. a nonsymmetric finite element type of stencil on the interface).

We take the following model problem

$$\begin{aligned} -\Delta U &= f \quad \text{in } \Omega := (0,1)\times(0,1) \\ U &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

and a composite grid that is composed of a global coarse grid that covers Ω and a local fine grid that covers the region $\Omega_l = (0,\gamma_1)\times(0,\gamma_2)$ (see Figure 2). We only consider coarse grid

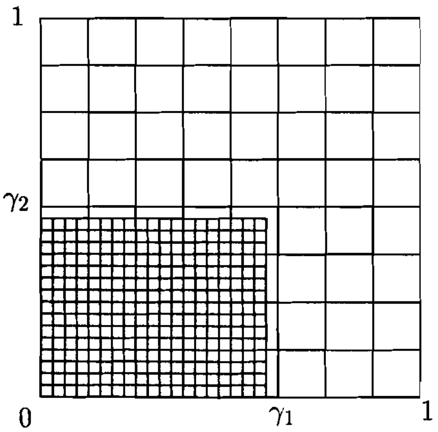

Figure 2: Composite grid $\Omega_c^{h,H}$, $H = 1/8$, $h = 1/32$.

sizes H such that $1/H \in I\!N$, $\gamma_1/H \in I\!N$, $\gamma_2/H \in I\!N$ and fine grid sizes h such that $h = H/\sigma$, $\sigma \in I\!N$.

We use the following notation (cf. Figure 2):

$$\begin{aligned}
&\Omega^h = \{(x,y)\in I\!R^2 \mid x/h,\ y/h \in I\!N\},\quad \Omega^H = \{(x,y)\in I\!R^2 \mid x/H,\ y/H \in I\!N\},\\
&\Omega_c^h = \Omega_l \cap \Omega^h,\ \Omega_c^H = (\Omega\backslash\Omega_l)\cap\Omega^H,\ \Omega_c^{h,H} = \Omega_c^h \cup \Omega_c^H,\\
&\Gamma_{vert} = \{(x,y)\in I\!R^2 \mid x=\gamma_1, 0<y\le\gamma_2\},\\
&\Gamma_{hor} = \{(x,y)\in I\!R^2 \mid y=\gamma_2, 0<x\le\gamma_1\}\\
&\Gamma = \Gamma_{vert}\cup\Gamma_{hor},\ \Gamma^h = \Gamma\cap\Omega^h,\ \Gamma^H = \Gamma\cap\Omega^H,\\
&\Gamma_{vert}^h = \Gamma_{vert}\cap\Omega^h,\ \Gamma_{vert}^H = \Gamma_{vert}\cap\Omega^H,\\
&\Gamma_{hor}^h = \Gamma_{hor}\cap\Omega^h, \Gamma_{hor}^H = \Gamma_{hor}\cap\Omega^H,\\
&\Gamma_h^* = \{(x,y)\in\Omega_c^h \mid \mathrm{dist}((x,y),\Gamma) = h\}.
\end{aligned} \tag{3.2}$$

The differential operator $-\Delta$ in (3.1) is replaced by the following stencils.

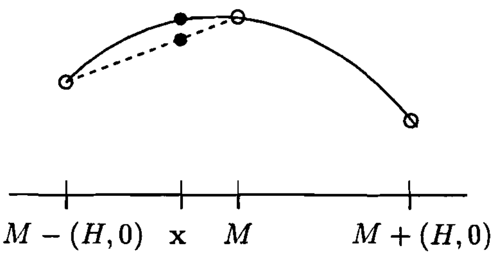

Figure 3: Interpolation p_Γ.

In $\Omega_c^H \backslash \Gamma^H$ we use

$$H^{-2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}. \tag{3.3a}$$

In $\Omega_c^h \backslash \Gamma_h^\star$ we use

$$h^{-2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}. \tag{3.3b}$$

At grid points $M \in \Gamma^H$ we use the difference given by ($u \in l^2(\Omega^H)$):

$$H^{-2}(4u(M) - u(M-(H,0)) - u(M+(H,0)) - u(M-(0,H)) - u(M+(0,H))) \tag{3.3c}$$

(i.e. M is treated as a coarse grid point, cf. (2.5a)).
In points $M \in \Gamma_h^\star$ we use the following discretization. We assume a given interpolation operator $p_\Gamma : l^2(\Gamma^H) \to C(\Gamma)$. Now in M we discretize by applying the standard 5-point fine grid stencil as in (3.3b); unknowns corresponding to grid points in $\Gamma^h \backslash \Gamma^H$ are eliminated using p_Γ.

The usual modifications are used at grid points close to the boundary $\partial\Omega$. The discretization above is fully determined if $p_\Gamma : l^2(\Gamma^H) \to C(\Gamma)$ is given. In this paper we consider a piecewise linear interpolation and a piecewise quadratic interpolation, denoted by $p_\Gamma^{(1)}$ and $p_\Gamma^{(2)}$ respectively.
If $u^H \in l^2(\Gamma^H)$ is given ($u^H \equiv 0$ on $\partial\Omega$), then at $\mathbf{x} \in \Gamma^h \backslash \Gamma^H$ we use an interpolated value $(p_\Gamma u^H)(\mathbf{x})$ as shown in Figure 3. Note that in case of quadratic interpolation there is some freedom: one may apply a shift of the interpolation points by a factor H (in Figure 3: use $M-(2H,0)$, $M-(H,0)$, M as interpolation points).

Corresponding to $\Omega_c^{h,H} = \Omega_c^h \cup \Omega_c^H$ we partition the discrete operator, resulting in

$$A_{h,H} = \begin{bmatrix} A_{11} & -A_{1\Gamma}p_\Gamma \\ -A_{21} & A_{22} \end{bmatrix}. \tag{3.4}$$

In (3.4) the operator $p_\Gamma : l^2(\Omega_c^H) \to C(\Gamma)$ is defined by linear ($p_\Gamma^{(1)}$) or quadratic ($p_\Gamma^{(2)}$) interpolation $\Gamma^H \to \Gamma$ and $p_\Gamma \equiv 0$ on $\Omega_c^H \backslash \Gamma^H$. The matrix $[A_{11} \;\; -A_{1\Gamma}]$ corresponds to the standard 5-point stencil on the local fine grid (Ω_c^h) and $[-A_{21}\ A_{22}]$ corresponds to the standard 5-point stencil on the coarse grid (cf. (3.3b), (3.3c)).

Below we use the following notation. For a subset V of grid points in $\Omega_c^{h,H}$ we denote by $\mathbb{I}_V$ the grid function (vector) with value 1 at all grid points of V and 0 at all other grid points.

Lemma 3.1. Both for linear and quadratic interpolation, the operator $A_{h,H}$ satisfies

$$A_{h,H}((\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega_c^{h,H}}) \geq \mathbb{I}_{\Omega_c^{h,H}}. \tag{3.5}$$

Proof. For $M \in \Omega_c^H$ we have

$$[A_{h,H}]_M(\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega_c^{h,H}} \geq H^{-2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_M (\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega^H} = 1.$$

Similarly, for $M \in \Omega_c^h \backslash \Gamma_h^\star$ we have

$$[A_{h,H}]_M(\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega_c^{h,H}} \geq h^{-2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_M (\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega^h} = 1.$$

Finally, take $M \in \Gamma_h^\star$. Note that both for piecewise linear and piecewise quadratic interpolation we have

$$p_\Gamma((\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Gamma^H}) \leq (\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Gamma}.$$

Using this we get, with e_M the standard basis vector corresponding to M:

$$\begin{aligned}
[A_{h,H}]_M(\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega_c^{h,H}} &= e_M^T[A_{11} - A_{1\Gamma}p_\Gamma](\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega_c^{h,H}} \\
&= e_M^T[A_{11}(\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega_c^h} - A_{1\Gamma}p_\Gamma(\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega_c^H}] \\
&\geq e_M^T[A_{11}(\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega_c^h} - A_{1\Gamma}(\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Gamma})] \\
&\geq h^{-2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_M (\frac{1}{2}\mathrm{x}(1-\mathrm{x}))_{|\Omega^h} = 1.
\end{aligned}$$

□

In the following theorem we prove monotonicity of $A_{h,H}$ (cf. Theorem 2.1.). For the case with quadratic interpolation some technical tools are needed. This is due to the fact that then $A_{h,H}$ is not an M-matrix.

Theorem 3.2. Both for linear and quadratic interpolation, the operator $A_{h,H}$ is monotone, i.e. $A_{h,H}$ is nonsingular and $A_{h,H}^{-1} \geq 0$ holds.
Proof. First we consider the case with linear interpolation.

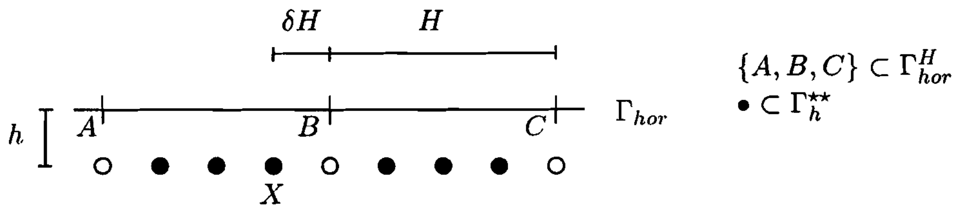

Figure 4: Example of $X \in \Gamma_h^{\star\star}$.

For every line segment $[M - (H,0), M] =: l_M$ on Γ_{hor} (cf. Figure 3) the linear interpolation $p_\Gamma^{(1)}$ of a grid function $u \in l^2(\Gamma^H)$ on l_M results in

$$(p_\Gamma^{(1)}u)(\mathbf{y}) = \alpha_1(\mathbf{y})u(M - (H,0)) + \alpha_2(\mathbf{y})u(M)$$

with weights $\alpha_1(\mathbf{y}) \geq 0$, $\alpha_2(\mathbf{y}) \geq 0$, $\alpha_1(\mathbf{y}) + \alpha_2(\mathbf{y}) = 1$ for $\mathbf{y} \in l_M$.
A similar result holds on Γ_{vert}. Using this, it follows that $A_{h,H}$ is an irreducibly diagonally dominant matrix with $(A_{h,H})_{i,j} \leq 0$ for $i \neq j$. Hence $A_{h,H}$ is an M-matrix and thus $A_{h,H}$ is monotone.
We now consider the case with quadratic interpolation, which is more involved. We will show that $A_{h,H}$ (which is not an M-matrix) can be written as the product of two M-matrices. The technique is based on ideas from [2, 11].
A special role is played by the equations in which the quadratic interpolation is used. So we introduce the set

$$\Gamma_h^{\star\star} := \{X \in \Gamma_h^\star \mid (X + (h,0)) \notin \Gamma^H \wedge (X + (0,h)) \notin \Gamma^H\}.$$

As an example we take $X \in \Gamma_h^{\star\star}$ as shown in Figure 4. The equation at X is as follows:

$$\begin{aligned}[A_{h,H}]_X u &= h^{-2}\{4u(X) - u(X-(h,0)) - u(X+(h,0)) - u(X-(0,h)) \\ &\quad - \alpha_3 u(A) - \alpha_2 u(B) - \alpha_1 u(C)\},\end{aligned} \qquad (3.6)$$

with $\alpha_1 = \frac{1}{2}\delta(\delta - 1)$, $\alpha_2 = (1-\delta)(1+\delta)$, $\alpha_3 = \frac{1}{2}\delta(1+\delta)$, $0 < \delta < 1$.
Note that $0 < \delta < 1$ implies $\alpha_1 < 0$, $0 < \alpha_2 < 1$, $0 < \alpha_3$. Also we have

$$\frac{-\alpha_1}{\alpha_2} = \frac{1}{2}\frac{\delta}{1+\delta} \leq \frac{1}{4}. \qquad (3.7)$$

We decompose $A_{h,H}$ as $A_{h,H} = D + N + P$ such that D diagonal and $diag(D) = diag(A_{h,H})$, $diag(N) = 0$, $N_{ij} \leq 0$ for all $i \neq j$, $diag(P) = 0$, $P_{ij} \geq 0$ for all $i \neq j$.
Now introduce N_1, N_2 with stencils $[N_i]_X$ $(i = 1,2)$ defined as follows.
For $X \notin (\Gamma^H \cup \Gamma_h^{\star\star})$ we take $[N_1]_X = [N]_X$, $[N_2]_X = [\,\emptyset\,]$. Also at the corner point $X = (\gamma_1, \gamma_2)$ we take $[N_1]_X = [N]_X$, $[N_2]_X = [\,\emptyset\,]$.
For $X \in \Gamma_{hor}^H \backslash (\gamma_1, \gamma_2)$ we define

$$[N_1]_X = H^{-2}\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad [N_2]_X = H^{-2}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (3.8a)$$

Similarly, for $X \in \Gamma_{vert}^H \backslash (\gamma_1, \gamma_2)$ we define

$$[N_1]_X = H^{-2}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad [N_2]_X = H^{-2}\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}. \qquad (3.8b)$$

(Note that obvious modifications are used if X is close to the boundary $\partial\Omega$).
Finally, we consider $X \in \Gamma_h^{**}$. As an example we take X as in Figure 4; then we define (cf. (3.6)):

$$\begin{aligned} [N_1]_X u &= h^{-2}\{-u(X-(h,0)) - u(X+(h,0)) - u(X-(0,h)) - \alpha_2 u(B)\}, && (3.9a) \\ [N_2]_X &= [\ \emptyset\ ]. && (3.9b) \end{aligned}$$

Note that $[N_2]_X \neq [\ \emptyset\ ]$ only for $X \in \Gamma^H \backslash (\gamma_1, \gamma_2)$. From the definitions of D and N_2 it immediately follows that $I + D^{-1}N_2$ is an M-matrix.
It is easy to check that $D + N_1$ is an irreducibly diagonally dominant matrix (use $0 < \alpha_2 < 1$) with $(D + N_1)_{ij} \leq 0$ for all $i \neq j$, and thus $D + N_1$ is an M-matrix.
From the definitions of N_1, N_2 it follows that

$$N \leq N_1 + N_2 \tag{3.10}$$

holds.
We now consider the nonnegative matrix P. First note that $[P]_X \neq [\ \emptyset\ ]$ only for points $X \in \Gamma_h^{**}$. Again, as a model situation we take X as in Figure 4, in which case we have (cf. (3.6)):

$$[P]_X u = -h^{-2}\alpha_1 u(C). \tag{3.11}$$

For this X we also have

$$[N_1 D^{-1} N_2]_X u = \frac{1}{4} h^{-2}\alpha_2 (u(A) + u(C)). \tag{3.12}$$

Combination of the results in (3.7), (3.11), (3.12) and using $N_1 D^{-1} N_2 \geq 0$ yields the inequality

$$P \leq N_1 D^{-1} N_2. \tag{3.13}$$

From (3.10), (3.13) we get the following:

$$A_{h,H} = D + N + P \leq D + N_1 + N_2 + N_1 D^{-1} N_2 = (D + N_1)(I + D^{-1}N_2). \tag{3.14}$$

Since both $D + N_1$ and $I + D^{-1}N_2$ are M-matrices we conclude that $((D + N_1)^{-1}A_{h,H})_{ij} \leq (I + D^{-1}N_2)_{ij} \leq 0$ for all $i \neq j$. From Lemma 3.1 we see that there exists a vector $v > 0$ such that $A_{h,H}v > 0$. Due to $(D + N_1)^{-1}$ nonsingular and $(D + N_1)^{-1} \geq 0$ this yields $(D + N_1)^{-1}A_{h,H}v > 0$. Thus we obtain (cf. [8]) that $(D + N_1)^{-1}A_{h,H}$ is an M-matrix. Thus we see that $A_{h,H} = (D + N_1)((D + N_1)^{-1}A_{h,H})$ is the product of two M-matrices and consequently we have that $A_{h,H}$ is nonsingular and $A_{h,H}^{-1} \geq 0$ holds. □

Stability of the discretization is proved in the following theorem.

Theorem 3.3. Both for linear and quadratic interpolation we have the following stability result:

$$\|A_{h,H}^{-1}\|_\infty \leq \frac{1}{8}. \tag{3.15}$$

Proof. Follows directly from Lemma 3.1 and Theorem 3.2. □

We now consider, as in the one dimensional case in Section 2 (cf. Theorem 2.2) a problem where the source term has nonzero values only in $\Gamma_h^\star$. More precisely, we will derive bounds for $\|A_{h,H}^{-1}\mathbb{I}_{\Gamma_h^\star}\|_\infty$. The analysis is based on the same approach as used in the proof of Theorem 2.2.

Theorem 3.4. The following inequality holds:

$$\|A_{h,H}^{-1}\mathbb{I}_{\Gamma_h^\star}\|_\infty \;\le\; (C_{p_\Gamma}C_\Gamma + H)\frac{h^2}{H}, \tag{3.16a}$$

with

$$C_\Gamma := 2-\gamma_1-\gamma_2 \le 2 \tag{3.16b}$$

and

$$C_{p_\Gamma} := \begin{cases} 1 & \text{for linear interpolation,} \\ \frac{5}{4} & \text{for quadratic interpolation.} \end{cases} \tag{3.16c}$$

Proof. With $v := A_{h,H}^{-1}\mathbb{I}_{\Gamma_h^\star}$ and using the partitioning as in (3.4) we get

$$\begin{bmatrix} A_{11} & -A_{1\Gamma}p_\Gamma \\ -A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathbb{I}_{\Gamma_h^\star} \\ \emptyset \end{bmatrix}.$$

Here $\mathbb{I}_{\Gamma_h^\star}$ is used as an element in $l^2(\Omega_c^h)$. Using the block LU-factorization of $A_{h,H}$ (as in the proof of Theorem 2.2) results in

$$v_1 = A_{11}^{-1}A_{1\Gamma}p_\Gamma v_2 + A_{11}^{-1}\mathbb{I}_{\Gamma_h^\star} \tag{3.17a}$$

$$v_2 = S^{-1}A_{21}A_{11}^{-1}\mathbb{I}_{\Gamma_h^\star} \tag{3.17b}$$

with

$$S := A_{22} - A_{21}A_{11}^{-1}A_{1\Gamma}p_\Gamma. \tag{3.17c}$$

Note that we can represent $\mathbb{I}_{\Gamma_h^\star}$ as

$$\mathbb{I}_{\Gamma_h^\star} = h^2 A_{1\Gamma}w_{\Gamma^h}, \tag{3.18}$$

with w_{Γ^h} a grid function on Γ^h with values $\frac{1}{2}$ (at grid points $M \in \Gamma^h$ with $dist(M,(\gamma_1,\gamma_2)) = h$) or 1 (elsewhere). So for v_1 we have

$$A_{11}v_1 - A_{1\Gamma}(p_\Gamma v_2 + h^2 w_{\Gamma^h}) = 0.$$

The discrete maximum principle yields $\|v_1\|_\infty \le \|p_\Gamma v_2\|_\infty + h^2$. For piecewise linear interpolation (i.e. $p_\Gamma = p_\Gamma^{(1)}$) we have $\|p_\Gamma v_2\|_\infty \le \|v_2\|_\infty$ and for piecewise quadratic interpolation ($p_\Gamma = p_\Gamma^{(2)}$) we have $\|p_\Gamma v_2\|_\infty \le \frac{5}{4}\|v_2\|_\infty$. This yields

$$\|v_1\|_\infty \le C_{p_\Gamma}\|v_2\|_\infty + h^2, \tag{3.19}$$

with $C_{p_\Gamma} = 1$ if $p_\Gamma = p_\Gamma^{(1)}$ and $C_{p_\Gamma} = \frac{5}{4}$ if $p_\Gamma = p_\Gamma^{(2)}$.
It remains to obtain a bound for $\|v_2\|_\infty = \|S^{-1}A_{21}A_{11}^{-1}\mathbb{I}_{\Gamma_h^\star}\|_\infty$.

We introduce $w := A_{11}^{-1}\mathbb{1}_{\Gamma_h^\star}$. From (3.18) we obtain that $A_{11}w - A_{1\Gamma}(h^2 w_{\Gamma^h}) = 0$ holds. The discrete maximum principle yields that $0 \le w \le h^2\mathbb{1}_{\Omega_c^h}$ holds. So for $\hat{w} := A_{21}w \in l^2(\Omega_c^H)$, which has nonzero values on $\Gamma^H\backslash(\gamma_1,\gamma_2)$ only, we have $0 \le \hat{w}(M) \le H^{-2}h^2 = \sigma^{-2}$ for $M \in \Gamma^H\backslash(\gamma_1,\gamma_2)$. We define $e_{hor}^H \in l^2(\Omega_c^H)$ as the grid function with value 1 at all points of $\Gamma_{hor}^H\backslash(\gamma_1,\gamma_2)$ and value 0 at all other points of Ω_c^H. Similarly we define e_{vert}^H (cf. Figure 5). Note that $\hat{w} = A_{21}A_{11}^{-1}\mathbb{1}_{\Gamma_h^\star}$ and that the characteristic function in Ω_c^H corresponding to

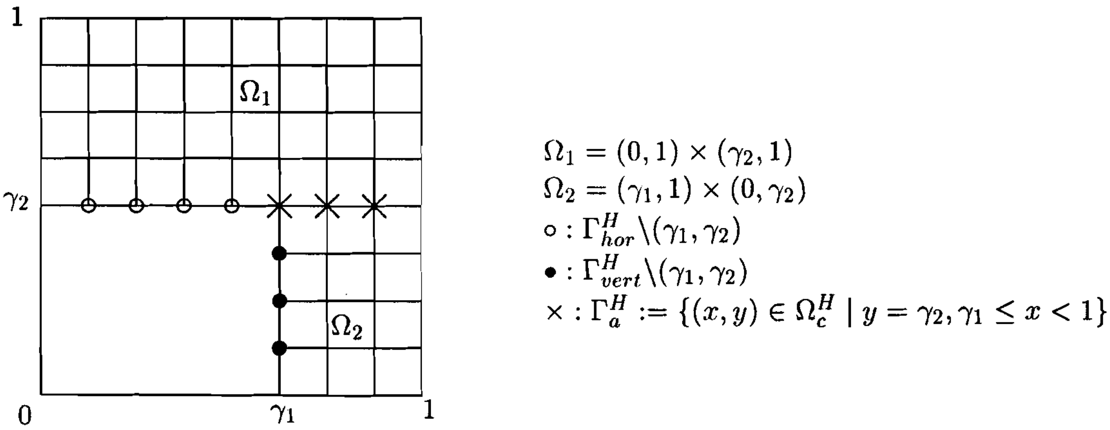

Figure 5: Partitioning of Ω_c^H.

$\Gamma^H\backslash(\gamma_1,\gamma_2)$ is given by $e_{hor}^H + e_{vert}^H$. Hence we have the following result

$$0 \le A_{21}A_{11}^{-1}\mathbb{1}_{\Gamma_h^\star} \le \sigma^{-2}(e_{hor}^H + e_{vert}^H). \tag{3.20}$$

Due to $S^{-1} = \begin{bmatrix} \emptyset & I \end{bmatrix} A_{h,H}^{-1} \begin{bmatrix} \emptyset \\ I \end{bmatrix}$ and the monotonicity of $A_{h,H}$ (Theorem 3.2) we have $S^{-1} \ge 0$. Combination with the result in (3.20) yields

$$\|v_2\|_\infty = \|S^{-1}A_{21}A_{11}^{-1}\mathbb{1}_{\Gamma_h^\star}\|_\infty \le \sigma^{-2}(\|S^{-1}e_{hor}^H\|_\infty + \|S^{-1}e_{vert}^H\|_\infty). \tag{3.21}$$

We now consider the term $\|S^{-1}e_{hor}^H\|_\infty$.
We use notation as explained in Figure 5.
The piecewise linear function g is defined as follows

$$g(x,y) := \begin{cases} \frac{1}{1-\gamma_2}(1-y) & \text{if } y \ge \gamma_2 \\ 1 & \text{if } y < \gamma_2 \end{cases}. \tag{3.22}$$

We use the notation $g^H := g_{|\Omega^H}$, $g_c^H := g_{|\Omega_c^H}$.
Now consider $Sg_c^H = (A_{22} - A_{21}A_{11}^{-1}A_{1\Gamma}p_\Gamma)g_c^H \in l^2(\Omega_c^H)$.
For $M \notin (\Gamma_{hor}^H \cup \Gamma_{vert}^H \cup \Gamma_a^H)$ we have

$$(Sg_c^H)(M) \quad = \quad [A_{22}]_M g_c^H \ge H^{-2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_M g^H \ge 0. \tag{3.23a}$$

For $M \in \Gamma_a^H$ we get

$$\begin{aligned}(Sg_c^H)(M) &= [A_{22}]_M g_c^H \geq H^{-2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_M g^H \\ &= \frac{\partial^2}{\partial x^2} g_{|y=\gamma_2} - \frac{1}{H}\frac{\partial}{\partial y} g_{|\Omega_1} + \frac{1}{H}\frac{\partial}{\partial y} g_{|\Omega_2} \\ &= 0 + \frac{1}{H}\frac{1}{1-\gamma_2} + 0 \geq 0. \end{aligned} \tag{3.23b}$$

With respect to the result on $(\Gamma_{vert}^H \cup \Gamma_{hor}^H)\backslash(\gamma_1, \gamma_2)$ we first note the following. Define $w_h := A_{11}^{-1} A_{1\Gamma} p_\Gamma g_c^H$. Because g is constant on Γ_{hor} and on Γ_{vert} we have $p_\Gamma g_c^H = g_{|\Gamma}$, and w_h satisfies $A_{11} w_h - A_{1\Gamma} g_{|\Gamma} = 0$. The discrete maximum principle yields $0 \leq w_h \leq \mathbb{1}_{\Omega_c^h}$. Thus we get

$$0 \leq A_{21} A_{11}^{-1} A_{1\Gamma} p_\Gamma g_c^H \leq H^{-2}(e_{hor}^H + e_{vert}^H).$$

Using this we have for $M \in \Gamma_{vert}^H\backslash(\gamma_1, \gamma_2)$:

$$\begin{aligned}(Sg_c^H)(M) &= ((A_{22} - A_{21}A_{11}^{-1}A_{1\Gamma}p_\Gamma)g_c^H)_{|M} \geq H^{-2}\begin{bmatrix} & -1 & \\ 0 & 4 & -1 \\ & -1 & \end{bmatrix}_M g^H - H^{-2} \\ &= \frac{\partial^2}{\partial y^2} g_{|\Gamma_{vert}} - \frac{1}{H}\frac{\partial}{\partial x} g_{|\Omega_2} + \frac{1}{H^2} - \frac{1}{H^2} = 0. \end{aligned} \tag{3.23c}$$

Finally, for $M \in \Gamma_{hor}^H\backslash(\gamma_1, \gamma_2)$ we get:

$$\begin{aligned}(Sg_c^H)(M) &= ((A_{22} - A_{21}A_{11}^{-1}A_{1\Gamma}p_\Gamma)g_c^H)_{|M} \geq H^{-2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & 0 & \end{bmatrix}_M g^H - H^{-2} \\ &= \frac{\partial^2}{\partial x^2} g_{|\Gamma_{hor}} - \frac{1}{H}\frac{\partial}{\partial y} g_{|\Omega_1} + \frac{1}{H^2} - \frac{1}{H^2} = \frac{1}{H}\frac{1}{1-\gamma_2}. \end{aligned} \tag{3.23d}$$

Combination of (3.23a-d) yields

$$Sg_c^H \geq \frac{1}{H}\frac{1}{1-\gamma_2} e_{hor}^H,$$

and thus

$$\|S^{-1}e_{hor}^H\|_\infty \leq H(1-\gamma_2)\ \|g_c^H\|_\infty = H(1-\gamma_2).$$

The term $\|S^{-1}e_{vert}^H\|_\infty$ can be treated similarly. Using these results in (3.21) we get

$$\|v_2\|_\infty \leq \sigma^{-2}(H(1-\gamma_2) + H(1-\gamma_1)). \tag{3.24}$$

Using (3.24) in (3.19) completes the proof of the theorem. □

Remark 3.5. Note that the result in Theorem 3.4 is very similar to the one dimensional result in Theorem 2.2.

It is well-known (cf. e.g. [1, 4]) that in case of a global uniform grid with grid size h relatively large (e.g. $\mathcal{O}(1)$) local discretization errors at grid points close to the boundary may still result in acceptable (e.g. $\mathcal{O}(h^2)$) global discretization errors. In Theorem 3.4. we have a very similar effect with H fixed and $h \downarrow 0$, but now with respect to local discretization errors at grid points of $\Gamma_h^\star$ (i.e. close to the interface). Below we will see that this effect (i.e. the result of Theorem 3.4) plays an important role in the analysis of the global discretization error.

We discretize the right hand side of (3.1) as usual, i.e. $f_{h,H} \in l^2(\Omega_c^{h,H})$ is given by $f_{h,H}(M) = f(M)$ for all $M \in \Omega_c^{h,H}$. The local discretization error at $\mathbf{y} \in \Omega_c^{h,H}$ corresponding to the discretization $A_{h,H}u_{h,H} = f_{h,H}$ is denoted by $d_{h,H}(\mathbf{y})$. As usual in a finite difference setting we assume $U \in C^4(\Omega)$. Then for the local discretization errors we have the following:

$$\max_{\mathbf{y}\in\Omega_c^h\backslash\Gamma_h^\star} |d_{h,H}(\mathbf{y})| \leq C_1 h^2 \tag{3.25a}$$

$$\max_{\mathbf{y}\in\Omega_c^H} |d_{h,H}(\mathbf{y})| \leq C_2 H^2 \tag{3.25b}$$

$$\max_{\mathbf{y}\in\Gamma_h^\star} |d_{h,H}(\mathbf{y})| \leq C_3\sigma^2 H^{j-1} + C_1 h^2 \tag{3.25c}$$

with $j = 1$ for linear interpolation ($p_\Gamma^{(1)}$) and $j = 2$ for quadratic interpolation ($p_\Gamma^{(2)}$). The constants C_i are of the form

$$C_1 = c_1 \max\{|U^{(4)}(\mathbf{x})| \mid \mathbf{x} \in \Omega_l = (0,\gamma_1)\times(0,\gamma_2)\} \tag{3.26a}$$

$$C_2 = c_2 \max\{|U^{(4)}(\mathbf{x})| \mid \mathbf{x} \in \Omega\backslash((0,\gamma_1 - H)\times(0,\gamma_2 - H))\} \tag{3.26b}$$

$$C_3 = c_3 \max\{|U^{(1+j)}(\mathbf{x})| \mid \mathbf{x} \in \Gamma\}, \tag{3.26c}$$

with c_1, c_2, c_3 independent of h, H, U.

Remark 3.6. The bound in (3.25c) is not sharp for the (less interesting) case $\sigma = 1$. A composite grid as in Figure 2 only makes sense for problems in which the solution U varies much more rapidly in Ω_l than in $\Omega\backslash\Omega_l$. Thus we assume $C_1 \gg C_2$, $C_1 \gg C_3$. Clearly, then one would use a composite grid with $h \ll H$, i.e. $\sigma \gg 1$. In that case the local discretization error on $\Gamma_h^\star$ may be large compared to the local discretization error on $\Omega_c^{h,H}\backslash\Gamma_h^\star$ (cf. (3.25)). A strong damping of these large local discretization errors is a necessity for obtaining an acceptable global discretization error.

Theorem 3.7. For the global discretization error the following holds

$$\begin{aligned}\|u_{h,H} - U_{|\Omega_c^{h,H}}\|_\infty &\leq C_1(\tfrac{1}{8} + C_{p_\Gamma}C_\Gamma\tfrac{h}{\sigma} + h^2)h^2 + \tfrac{1}{8}C_2H^2 + C_3(C_{p_\Gamma}C_\Gamma + H)H^j \\ &\leq \tfrac{13}{8}C_1h^2 + \tfrac{1}{8}C_2H^2 + 3C_3H^j,\end{aligned} \tag{3.27}$$

with C_i as in (3.26), C_{p_Γ} and C_Γ as in (3.16), $j = 1$ for linear interpolation and $j = 2$ for quadratic interpolation.

Proof. Using Theorems 3.2-3.4 and (3.25) we get

$$\|u_{h,H} - U_{|\Omega_c^{h,H}}\|_\infty \leq \frac{1}{8}C_1h^2 + \frac{1}{8}C_2H^2 + (C_{p_\Gamma}C_\Gamma + H)\frac{h^2}{H}(C_3\sigma^2H^{j-1} + C_1h^2).$$

The first inequality in (3.27) follows from rearranging the terms on the right hand side. The second inequality in (3.27) is a consequence of $h \leq H \leq \frac{1}{2}$, $C_{p\Gamma} \leq \frac{5}{4}$ and $C_\Gamma \leq 2$. □

Remark 3.8. We comment on the main result of this paper given in Theorem 3.7. As usual in finite difference estimates, the result in (3.27) has the disadvantage that high (fourth order) derivatives are involved. A nice feature is that the constants in (3.27) do not depend on $\sigma = H/h$. Furthermore, the bounds in (3.27) nicely separate the influence of the high activity region (C_1h^2), the low activity region (C_2H^2), and the interpolation on the interface (C_3H^j). Comparing this with related results in the literature we note the following. The analyses in [3, 5] use weaker assumptions concerning the regularity of the solution. On the other hand, the analysis for the finite volume element method in [3] only treats the case with $\sigma = 2$. In the schemes in [5] larger values of σ are allowed, but it is not clear how the discretization error (bound) depends on σ.
The sharpness of the bounds in (3.27) will be discussed in Section 5.

Remark 3.9. Results very similar to those in Theorem 3.4 and Theorem 3.7 can be obtained if we consider a composite grid with Ω_l of the form $(\gamma_{11}, \gamma_{12}) \times (\gamma_{21}, \gamma_{22})$ with $0 < \gamma_{11} < \gamma_{12} < 1$, $0 < \gamma_{21} < \gamma_{22} < 1$.

4 Connection with the Local Defect Correction Method

In this section we will discuss a close connection between the composite grid discretization analyzed in Section 3 and the Local Defect Correction method (LDC) introduced in [9]. The results in this section are based on [7]. This connection can be used to solve efficiently the composite grid system of Section 3. Below we explain the LDC method applied to the problem in (3.1). For a more general discussion of the LDC method we refer to [9].

In Section 3 we introduced the local fine grid Ω_c^h and the coarse grid Ω_c^H (both part of the composite grid, cf. (3.2)). To make the notation in this section more transparant, we will write Ω_l^h instead of Ω_c^h. We now introduce the global coarse grid

$$\Omega_g^H := \Omega^H \cup \Omega, \tag{4.1}$$

and the standard 5-point discretization on this grid denoted by

$$A_H u_H = f_H. \tag{4.2}$$

Below we also use the local coarse grid

$$\Omega_l^H := \Omega_l \cap \Omega^H, \tag{4.3}$$

and we define the trivial injection $r_l : l^2(\Omega_l^h) \to l^2(\Omega_l^H)$ by

$$(r_l v)(\mathbf{x}) := v(\mathbf{x}), \quad v \in l^2(\Omega_l^h), \;\; \mathbf{x} \in \Omega_l^H. \tag{4.4}$$

Furthermore, we introduce the characteristic function $\chi : l^2(\Omega_l^H) \to l^2(\Omega_g^H)$ given by

$$(\chi w)(\mathbf{x}) := \begin{cases} w(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ 0 & \mathbf{x} \in \Omega_g^H \backslash \Omega_l^H \end{cases}. \tag{4.5}$$

For a given $v_H \in l^2(\Omega_g^H)$ we consider a corresponding local fine grid problem defined as follows. We use the standard 5-point stencil on Ω_l^h and *artificial* boundary values on Γ^h given by $p_\Gamma v_H$, where p_Γ is an interpolation as in Section 3 ($p_\Gamma^{(1)}$: linear interpolation; $p_\Gamma^{(2)}$: quadratic interpolation). Using the notation as in (3.4) this yields a local fine grid system

$$A_{11}^h v_h - A_{1\Gamma}^h p_\Gamma v_H = f_h. \tag{4.6}$$

In LDC one starts with solving the basic coarse grid problem (4.2). The resulting u_H is used to define boundary values for a local fine grid problem, i.e. we solve (4.6) with $v_H = u_H$, resulting in a local fine grid approximation u_h. By solving the local fine grid problem we aim at improving the approximation of the continuous solution U in the region Ω_l. However, the Dirichlet boundary conditions on Γ^h result from the basic global coarse grid problem and the approximation u_h can be no more accurate than the approximation u_H at the interface, which in general will be rather inaccurate. Therefore the results of this simple two step process usually do not achieve an accuracy that is in agreement with the added resolution (see e.g. [9],). In the LDC iteration coarse and fine grid processing steps are reused to obtain (quickly) such accuracy.

In the next step of the LDC iteration the approximation u_h is used to update the global coarse grid problem (4.2). The right hand side of (4.2) is updated at grid points that are part of Ω_l^H. The updated global coarse grid problem is given by

$$A_H \bar{u}_H = \bar{f}_H, \tag{4.7a}$$

with

$$\bar{f}_H(\mathbf{x}) = \begin{cases} (A_{11}^H r_l u_h)(\mathbf{x}) - (A_{1\Gamma}^H (u_H)_{|\Gamma^H})(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ f_H(\mathbf{x}) & \mathbf{x} \in \Omega_g^H \backslash \Omega_l^H \end{cases}. \tag{4.7b}$$

The operators $A_{11}^H : l^2(\Omega_l^H) \to l^2(\Omega_l^H)$ and $A_{1\Gamma}^H : l^2(\Gamma^H) \to l^2(\Omega_l^H)$ are coarse grid analogues of A_{11}^h and $A_{1\Gamma}^h$ in (4.6).
Using (4.5) we can rewrite (4.7a), (4.7b) as follows

$$A_H \bar{u}_H = f_H + \chi(A_{11}^H r_l u_h - A_{1\Gamma}^H (u_H)_{|\Gamma^H} - f_H). \tag{4.8}$$

So the right hand side of the global coarse grid problem is corrected by the defect of a local fine grid approximation. Once we have solved (4.8) we can update the local fine grid problem:

$$A_{11}^h \bar{u}_h = f_h + A_{1\Gamma}^h p_\Gamma \bar{u}_H. \tag{4.9}$$

The approximations $\bar{u}_H$ and $\bar{u}_h$ of U can be used to define an approximation of U on the composite grid:

$$\bar{u}_c(\mathbf{x}) := \begin{cases} \bar{u}_h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ \bar{u}_H(\mathbf{x}) & \mathbf{x} \in \Omega_c^H = \Omega_c^{h,H} \backslash \Omega_l^h \end{cases}. \tag{4.10}$$

In the LDC iteration global problems like (4.8) and local problems like (4.9) are combined in the way described above.

LDC

Start: solve the global problem

$$A_H u_{H,0} = f_H \text{ on } \Omega_g^H$$

solve the local problem

$$A_{11}^h u_{h,0} = f_h + A_{1\Gamma}^h p_\Gamma u_{H,0} \text{ on } \Omega_l^h$$

$i = 1, 2, \dots$:

a) compute the right hand side of the global problem

$$\bar{f}_H := (1-\chi) f_H + \chi (A_{11}^H r_l u_{h,i-1} - A_{1\Gamma}^H (u_{H,i-1})_{|\Gamma^H}) \tag{4.11a}$$

b) solve the global problem

$$A_H u_{H,i} = \bar{f}_H \text{ on } \Omega_g^H \tag{4.11b}$$

c) solve the local problem

$$A_{11}^h u_{h,i} = f_h + A_{1\Gamma}^h p_\Gamma u_{H,i} \text{ on } \Omega_l^h \tag{4.11c}$$

Corresponding to $u_{H,i}$ and $u_{h,i}$ one can define a composite grid approximation $u_{c,i}$ as in (4.10).

In practice the systems in (4.11b), (4.11c) will be solved approximately by a fast iterative method. Then one can take advantage of the fact that one has to solve (standard) problems on uniform grids.

Any fixed point $(\hat{u}_H, \hat{u}_h)$ of the iterative process (4.11) is characterized by the system (see [9])

$$\begin{aligned} &A_H \hat{u}_H + \chi (A_{1\Gamma}^H (\hat{u}_H)_{|\Gamma^H} - A_{11}^H r_l \hat{u}_h) = (1-\chi) f_H && \text{on } \Omega_g^H, \\ &A_{11}^h \hat{u}_h = f_h + A_{1\Gamma}^h p_\Gamma \hat{u}_H && \text{on } \Omega_l^h. \end{aligned}$$

Corresponding to $\hat{u}_H$ and $\hat{u}_h$ one can define a composite grid approximation $\hat{u}_c$ as in (4.10). We now discuss two main results from [7]. Firstly, it is proved in [7] that $\hat{u}_c$ *is the solution of the composite grid problem that is analyzed in Section 3* (cf. (3.4)). Secondly, it is shown in [7] that the *LDC iterates are equal to the iterates resulting from a Fast Adaptive Composite grid method* (FAC, cf. [12]) applied to this composite grid problem. Using these results we make the following observations:

- The LDC method seems a natural approach for computing discrete approximations on a composite grid. The close connection between LDC and the composite grid discretization of Section 3 (where with respect to discretization an interface point is treated as a coarse grid point) yields a further justification of this discretization method.
- The result of Theorem 3.7 yields a discretization error bound for the limit ($\hat{u}_c$) of the LDC iteration.
- The LDC method can be used for solving the composite grid system of Section 3. Note that in the LDC solution process we do *not* need the composite grid operator $A_{h,H}$. We only use the discretizations on the local fine grid (A_{11}^h) and on the global coarse grid (A_H).
- Due to the equivalence of LDC and FAC we expect fast convergence of the LDC iteration and a convergence rate independent of H, h and σ. This convergence behaviour is also observed in numerical experiments (cf. [7]). Thus we expect the LDC method to be an efficient solver for the composite grid system of Section 3.

5 Numerical Experiments

In this section we will show results of a few numerical experiments. First, we present results related to the global discretization error bound proved in Theorem 3.7. In the second part of this section we discuss a two dimensional nonuniform discretization method which can be seen as a generalization of the one dimensional method with stiffness matrix $\tilde{A}_{h,H}$ of Section 2 (cf. (2.5b)).

Below we will illustrate certain phenomena using numerical results for the following model problem:

$$\begin{aligned} -\Delta U &= f \quad \text{in } \Omega = (0,1) \times (0,1) \\ U &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

We consider two cases:
Case 1: f, g such that the solution U is given by

$$U(x,y) = x^2 + y^2. \tag{5.2}$$

Case 2: f, g such that the solution U is given by

$$U(x,y) = \frac{1}{2}\{\tanh(25(x+y-\frac{1}{8}))+1\}. \tag{5.3}$$

Clearly in Case 1 we have a very smooth solution and we do not need a composite grid. This example is used below for theoretical considerations. The solution U in Case 2 is shown in Figure 6. The solution varies very rapidly in a small part of the domain and is relatively

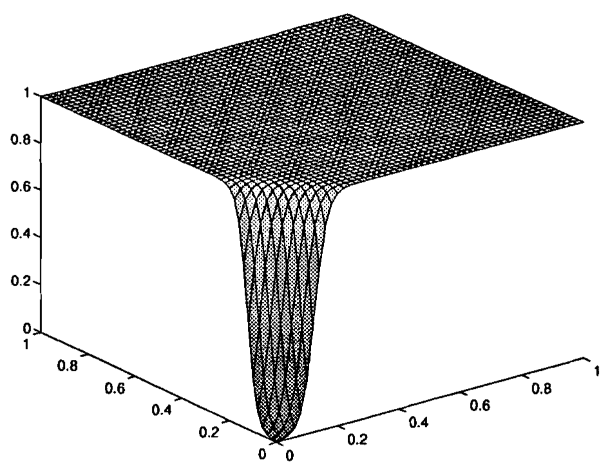

Figure 6: The solution U from (5.3).

smooth in the remaining part of the domain.
In both cases for Ω_l we take

$$\Omega_l = \{(x,y) \in \Omega \mid x \le \frac{1}{4} \wedge y \le \frac{1}{4}\}.$$

Experiment 1. In the upper bound for the global discretization error as proved in Theorem 3.7 we have a term $C_3 H$ if we use linear interpolation on the interface ($j = 1$). In this experiment

$\sigma = 2$			
$H = 1/16$	$H = 1/32$	$H = 1/64$	$H = 1/128$
$1.08e-03$	$4.47e-04$	$2.01e-04$	$9.60e-05$
$H = 1/16$			
$\sigma = 2$	$\sigma = 4$	$\sigma = 8$	$\sigma = 16$
$1.08e-03$	$1.26e-03$	$1.35e-03$	$1.42e-03$

Table 1: Global discretization errors; Case 1; linear interpolation.

we show that the bound is sharp with respect to this C_3H term. We consider Case 1. Then for C_1, C_2 in (3.27) we have $C_1 = C_2 = 0$. In Table 1 we show values of the global discretization error $\|u_{h,H} - U_{|\Omega_c^{h,H}}\|_\infty$ for several values of H and $\sigma = H/h$. We clearly observe the linear dependence on H.

Experiment 2. We consider Case 2 and use quadratic interpolation on the interface. For this (model) composite grid problem Theorem 3.7 yields a discretization error bound of the form $D_1h^2 + D_2H^2$ with $D_1 \gg D_2$. Based on this bound we expect the following. If we take H fixed then decreasing h (i.e. increasing σ) should result in h^2 convergence until a certain threshold value σ_{max} is reached. This convergence behaviour can be observed in the rows of Table 2. Furthermore, for $H = 1/8$ we see a threshold value $\sigma_{max} \approx 16$. Also note that in Table 2 there is only little variation in the values if we take h fixed and vary σ. For example, along the diagonal from $(H, \sigma) = (1/64, 1)$ to $(H, \sigma) = (1/8, 8)$ (i.e. $h = 1/64$) all values are about $5.5e-3$. This means that the global discretization error corresponding to the composite grid problem with $H = 1/8$, $h = 1/64$ is approximately of the same size as the global discretization error corresponding to the standard discrete problem on the global uniform grid with $h = 1/64$. So in this sense the quality of the discrete solutions of these two problems is the same. However, in the composite grid problem the discrete solution can be computed with significantly lower arithmetic costs.

H	1	2	4	8	16	32	σ
1/8	$2.55e-1$	$6.02e-2$	$2.29e-2$	$5.39e-3$	$1.49e-3$	$1.54e-3$	
1/16	$6.08e-2$	$2.29e-2$	$5.54e-3$	$1.35e-3$	$8.03e-4$		
1/32	$2.30e-2$	$5.61e-3$	$1.41e-3$	$3.33e-4$			
1/64	$5.63e-3$	$1.43e-3$	$3.51e-4$				

Table 2: Global discretization errors; Case 2; quadratic interpolation.

We now discuss an obvious two dimensional generalization of the one dimensional approach in (2.5b). We use the same discretization stencils as in Section 3 at all grid points of $\Omega_c^{h,H} \backslash \Gamma^H$. Again, we use linear ($j = 1$) or quadratic ($j = 2$) interpolation. On Γ^H we do not use a coarse grid stencil as in Section 3, but a nonsymmetric stencil of the same type as in (2.5b). For example, in $M \in \Gamma_{vert}^H$ we use ($u \in l^2(\Omega_c^{h,H})$):

$$
\begin{aligned}
[\tilde{A}_{h,H}]_M u &= H^{-2}(-\frac{2\sigma^2}{\sigma+1}u(M-(h,0)) + 2\sigma u(M) - \frac{2\sigma}{\sigma+1}u(M+(H,0))) + \\
&\quad H^{-2}(-u(M-(0,H)) + 2u(M) - u(M+(0,H))).
\end{aligned} \tag{5.4}
$$

This results in a discretization with stiffness matrix denoted by $\tilde{A}_{h,H}$ and with local discretization errors as in (3.25) but now with an $\mathcal{O}(H)$ error at points $M \in \Gamma^H$. In Section 2 we noticed that in the one dimensional case the local discretization error on $\Gamma_h^\star$ is reduced only by a factor h (cf. (2.14)). Numerical experiments show that in the two dimensional case we also have $\|\tilde{A}_{h,H}^{-1}\mathbb{1}_{\Gamma_h^\star}\|_\infty \approx ch$. So then for the local discretization errors on $\Gamma_h^\star$ of size $C_3\sigma^2 H^{j-1} + C_1 h^2$ (cf. (3.25c)) we only have a damping factor $ch = cH/\sigma$, instead of the damping factor cH/σ^2 as in Theorem 3.4. This then implies a global discretization error estimate of the form

$$
\|\tilde{u}_{h,H} - U_{|\Omega_c^{h,H}}\|_\infty \lesssim \frac{1}{8}C_1 h^2 + \frac{1}{8}C_2 H^2 + C_3 c\sigma H^j, \tag{5.5}
$$

with C_i as in (3.27). Clearly, due to the factor σ the bound in (5.5) is less favourable than the result in Theorem 3.7. We also note that for solving the resulting discrete problem an FAC type of method can be used. Then we need the composite grid operator $\tilde{A}_{h,H}$ in the solution method, whereas in the LDC approach (cf. Section 4) the composite grid operator $A_{h,H}$ is not needed. So the composite grid discretization with stiffness matrix $\tilde{A}_{h,H}$ has disadvantages when compared with the composite grid discretization of Section 3.

Experiment 3. This experiment is similar to Experiment 1 but now with the stiffness matrix $\tilde{A}_{h,H}$ instead of the stiffness matrix $A_{h,H}$. We use linear interpolation on the interface and we consider Case 1. Then the bound in (5.5) is of the form $C_3 c\sigma H$, so we expect a growing discretization error if σ is increased. A dependence of the global discretization error on σ is observed in Table 3, too. Apparently this dependence is not linear in σ. Probably this is due to the fact that the local discretization errors on $\Gamma_h^\star$, i.e. $d_{h,H}(\mathbf{y})$ with $\mathbf{y} \in \Gamma_h^\star$, show an oscillating behaviour and approximating $d_{h,H}(\mathbf{y})_{|\mathbf{y}\in\Gamma_h^\star}$ by the constant vector $\|d_{h,H}\|_{\infty,\Gamma_h^\star}\mathbb{1}_{\Gamma_h^\star}$ (as is done in the proof of (5.5)) is rather crude.

$\sigma = 2$			
$H = 1/16$	$H = 1/32$	$H = 1/64$	$H = 1/128$
$1.48e-03$	$6.82e-04$	$3.25e-04$	$1.60e-04$
$H = 1/16$			
$\sigma = 2$	$\sigma = 4$	$\sigma = 8$	$\sigma = 16$
$1.48e-03$	$2.54e-03$	$3.84e-03$	$5.30e-03$

Table 3: Global discretization errors; Case 1;linear interpolation; stiffness matrix $\tilde{A}_{h,H}$.

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