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P. H. M. Kersten
Department of Applied Mathematics, Twente University of Technology, P. O. Box 217, 7500 AE Enschede, The Netherlands

H. M. M. Ten Eikelder
Department of Mathematics and Computing Science, Eindhoven University of Technology, P. O. Box 513, 5600 MB Eindhoven, The Netherlands

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The construction of four infinite hierarchies of t-independent and t-dependent conserved functionals for the Federbush model is given. A formal proof of the existence of these infinite hierarchies is given in Appendix B.

I. INTRODUCTION

In a recent paper[1] one of the authors constructed four infinite hierarchies of Lie–Bäcklund transformations of the Federbush model.[2,3] Moreover he computed four creating and annihilating local (x,t)-dependent Lie–Bäcklund transformations that lead to these hierarchies. In this paper we show that to these four creating Lie–Bäcklund transformations, we can associate four t-dependent conserved functionals. By consequence the attempt to construct recursion[4,5] operators from these creating Lie–Bäcklund transformations failed since they are Hamiltonian vector fields. By recursive action of the Poisson bracket with these functionals we construct infinite hierarchies of conserved functionals associated to the (x,t)-independent Lie–Bäcklund transformations. This will be done in Sec. II. In Sec. III we construct four new (x,t)-dependent Lie–Bäcklund transformations from which we shall prove the existence of four infinite hierarchies of t-dependent conserved functionals, and consequently hierarchies of (x,t)-dependent Lie–Bäcklund transformations of the Federbush model. A formal proof is given in Appendix B, while a survey of the already known vector fields is given in Appendix A.

We want to stress the fact that all computations have been worked out on a DEC-system 20 computer using REDUCE[6] and a software package[7,8] to do these calculations.

Lie–Bäcklund transformations are vector fields V defined on the infinite jet bundle[9] of M,N, $J^\infty(M,N)$, where M is the space of independent variables and N the space of the dependent variables. A Lie–Bäcklund transformation of a differential equation is a vector field V defined on $J^\infty(M,N)$ satisfying the condition

$$\mathscr{L}_V(D^\infty I)\subset D^\infty I, \tag{1.1}$$

where I denotes a differential ideal associated to the differential equation at hand, while $D^\infty I$ denotes its infinite prolongation to $J^\infty(M,N)$; $\mathscr{L}_V$ is the Lie derivative with respect to the vector field V (Ref. 9). Since the vector fields V are supposed to depend only on a finite number of variables, condition (1.1) reduces to

$$\mathscr{L}_V I\subset D^r I \quad \text{for some } r. \tag{1.2}$$

Using this method we computed Lie–Bäcklund transformation of the Federbush model.[1]

It can be shown that the Lie–Bäcklund transformations in this setting are just symmetries in the works of Magri,[4] Ten Eikelder,[4,5] and Fuchssteiner and Fokas,[10] where (generators of) symmetries of partial differential equations of evolutionary type are described as transformations on special types of infinite dimensional spaces. Suppose that

$$\frac{du}{dt}=\Omega^{-1}\,dH \tag{1.3}$$

is an infinite dimensional Hamiltonian system, where Ω is the symplectic operator, H the Hamiltonian, dH is the Fréchet derivative of H. Then to each Hamiltonian symmetry (also called canonical symmetry) Y, there corresponds by definition a Hamiltonian $F(Y)$ such that

$$Y=\Omega^{-1}\,dF(Y), \tag{1.4}$$

and the Poisson bracket of F and H vanishes.[4,5] Suppose that Y_1, Y_2 are two Hamiltonian symmetries, then $[Y_1,Y_2]$ is a Hamiltonian symmetry and

$$F([Y_2,Y_1])=\{F(Y_1),F(Y_2)\}, \tag{1.5}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined by

$$\{F(Y_1),F(Y_2)\}=\langle dF(Y_1),Y_2\rangle, \tag{1.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the contraction of a one-form and a vector field. These notions shall be used throughout Sec. II and III.

II. CONSERVED FUNCTIONALS FOR THE FEDERBUSH MODEL

We shall discuss conserved functionals for the Federbush model. This model is described by

$$\begin{pmatrix} i(\partial_t+\partial_x) & -m(s) \\ -m(s) & i(\partial_t-\partial_x)\end{pmatrix}\begin{pmatrix}\psi_{s,1}\\ \psi_{s,2}\end{pmatrix} = 4s\pi\lambda\begin{pmatrix}|\psi_{-s,2}|^2 & \psi_{s,1}\\ |\psi_{-s,1}|^2 & \psi_{s,2}\end{pmatrix}\quad (s=\pm 1), \tag{2.1}$$

where $\psi_s(x,t)$ are two-component complex-valued functions.[3] Suppressing the factor 4π ($\lambda'=4\pi\lambda$) and introducing the eight real variables $u_1,v_1,u_2,v_2,u_3,v_3,u_4,v_4$ by

$$\begin{aligned}&\psi_{1,1}=u_1+iv_i,\quad \psi_{-1,1}=u_3+iv_3,\quad m(+1)=m_1,\\ &\psi_{1,2}=u_2+iv_2,\quad \psi_{-1,2}=u_4+iv_4,\quad m(-1)=m_2,\end{aligned} \tag{2.2}$$

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Eq. (2.1) is rewritten as a system of eight nonlinear partial differential equations for the functions $u_1,\ldots,v_4$; i.e.,

$$\begin{aligned}
&u_{1t} + u_{1x} - m_1 v_2 = \lambda R_4 v_1,\\
&-v_{1t} - v_{1x} - m_1 u_2 = \lambda R_4 u_1,\\
&u_{2t} - u_{2x} - m_1 v_1 = -\lambda R_3 v_2,\\
&-v_{2t} + v_{2x} - m_1 u_1 = -\lambda R_3 u_2,\\
&u_{3t} + u_{3x} - m_2 v_4 = -\lambda R_2 v_3,\\
&-v_{3t} - v_{3x} - m_2 u_4 = -\lambda R_2 u_3,\\
&u_{4t} - u_{4x} - m_2 v_3 = \lambda R_1 v_4,\\
&-v_{4t} + v_{4x} - m_2 u_3 = \lambda R_1 u_4,
\end{aligned} \tag{2.3}$$

where, in (2.3),

$$\begin{aligned}
&R_1 = u_1^2 + v_1^2, \quad R_2 = u_2^2 + v_2^2,\\
&R_3 = u_3^2 + v_3^2, \quad R_4 = u_4^2 + v_4^3.
\end{aligned}$$

Equation (2.3) can be written as a Hamiltonian system[4,5]

$$\frac{du}{dt} = \Omega^{-1}\, dH, \tag{2.4}$$

whereas in (2.4), $u = (u_1, v_1, \ldots, u_4, v_4)$,

$$\Omega = \begin{pmatrix} J & & & 0 \\ & J & & \\ & & J & \\ 0 & & & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.5a}$$

and

$$\begin{aligned}
H = \int_{-\infty}^{\infty} &\tfrac{1}{2}\{u_{1x}v_1 - u_1 v_{1x} - u_{2x}v_2 + u_2 v_{2x}\\
&+ u_{3x}v_3 - u_3 v_{3x} - u_{4x}v_4 + u_4 v_{4x}\}\\
&- m_1(u_1u_2 + v_1v_2) - m_2(u_3u_4 + v_3v_4)\\
&- (\lambda/2)R_1R_4 + (\lambda/2)R_2R_3
\end{aligned} \tag{2.5b}$$

(by $\int_{-\infty}^{\infty}$ we mean integration of the integrand with respect to x). In (2.4), dH is the Fréchet derivative of H defined by

$$\left.\frac{d}{d\epsilon} H(x + \epsilon y)\right|_{\epsilon = 0} = \langle dH, y\rangle. \tag{2.6}$$

In a previous paper[1] we constructed four first-order Lie–Bäcklund transformations Y_1^+, Y^+_{-1}, Y_1^-, Y^-_{-1} (Appendix A) that are Hamiltonian[4,5] vector fields; the associated Hamiltonian densities are given by

$$\begin{aligned}
\widetilde{F}(Y_1^+) &= -\tfrac{1}{2}(u_{2x}v_2 - u_2v_{2x}) + (\lambda/4)R_{34}R_2\\
&\quad - \tfrac{1}{2}m_1(u_1u_2 + v_1v_2),\\
\widetilde{F}(Y^+_{-1}) &= -\tfrac{1}{2}(u_{1x}v_1 - u_1v_{1x}) + (\lambda/4)R_{34}R_1\\
&\quad + \tfrac{1}{2}m_1(u_1u_2 + v_1v_2),\\
\widetilde{F}(Y_1^-) &= -\tfrac{1}{2}(u_{4x}v_4 - u_4v_{4x}) - (\lambda/4)R_{12}R_4\\
&\quad - \tfrac{1}{2}m_2(u_3u_4 + v_3v_4),\\
\widetilde{F}(Y^-_{-1}) &= -\tfrac{1}{2}(u_{3x}v_3 - u_3v_{3x}) - (\lambda/4)R_{12}R_3\\
&\quad + \tfrac{1}{2}m_2(u_3u_4 + v_3v_4),
\end{aligned} \tag{2.7a}$$

while the Hamiltonian densities associated to the gauge transformations Y_0^+, Y_0^- (see Ref. 1 and Appendix A) are given by

$$\widetilde{F}(Y_0^+) = \tfrac{1}{2}(R_1 + R_2), \quad \widetilde{F}(Y_0^-) = \tfrac{1}{2}(R_3 + R_4). \tag{2.7b}$$

In (2.7a), R_{12}, R_{34} are defined by

$$R_{12} = R_1 + R_2, \quad R_{34} = R_3 + R_4. \tag{2.8}$$

(Note that we use $\widetilde{F}$ for the density of the conserved functional F, so $F = \int_{-\infty}^{\infty} \widetilde{F}$.) The associated Lie–Bäcklund transformations can be derived from (2.7a) by the formula

$$Y = \Omega^{-1}\, dF(Y), \tag{2.9}$$

and for reasons of completeness they are surveyed in Appendix A at the end of this paper. The Hamiltonian densities associated with the second-order Lie–Bäcklund transformations Y_2^+, Y^+_{-2}, Y_2^-, Y^-_{-2} (see Ref. 1 and Appendix A) are computed, yielding

$$\begin{aligned}
\widetilde{F}(Y_2^+) &= -\tfrac{1}{2}(u_{2x}^2 + v_{2x}^2) + (\lambda/2)R_{34}(u_{2x}v_2 - u_2v_{2x})\\
&\quad - \tfrac{1}{2}m_1(u_{2x}v_1 - u_1v_{2x}) - \tfrac{1}{8}\lambda^2 R_{34}^2 R_2\\
&\quad + \tfrac{1}{4}m_1\lambda R_{34}(u_1u_2 + v_1v_2) - \tfrac{1}{8}m_1^2 R_{12},\\
\widetilde{F}(Y^+_{-2}) &= -\tfrac{1}{2}(u_{1x}^2 + v_{1x}^2) + (\lambda/2)R_{34}(u_{1x}v_1 - u_1v_{1x})\\
&\quad + \tfrac{1}{2}m_1(u_{1x}v_2 - u_2v_{1x}) - \tfrac{1}{8}\lambda^2 R_{34}^2 R_1\\
&\quad - \tfrac{1}{4}m_1\lambda R_{34}(u_1u_2 + v_1v_2) - \tfrac{1}{8}m_1^2 R_{12},\\
\widetilde{F}(Y_2^-) &= -\tfrac{1}{2}(u_{4x}^2 + v_{4x}^2) - (\lambda/2)R_{12}(u_{4x}v_4 - u_4v_{4x})\\
&\quad - \tfrac{1}{2}m_2(u_{4x}v_3 - u_3v_{4x}) - \tfrac{1}{8}\lambda^2 R_{12}^2 R_4\\
&\quad - \tfrac{1}{4}m_2\lambda R_{12}(u_3u_4 + v_3v_4) - \tfrac{1}{8}m_2^2 R_{34},\\
\widetilde{F}(Y^-_{-2}) &= -\tfrac{1}{2}(u_{3x}^2 + v_{3x}^2) - (\lambda/2)R_{12}(u_{3x}v_3 - u_3v_{3x})\\
&\quad + \tfrac{1}{2}m_2(u_{3x}v_4 - u_4v_{3x}) - \tfrac{1}{8}\lambda^2 R_{12}^2 R_3\\
&\quad + \tfrac{1}{4}m_2\lambda R_{12}(u_3u_4 - v_3v_4) - \tfrac{1}{8}m_2^2 R_{34}.
\end{aligned} \tag{2.10}$$

The Hamiltonian densities associated to the vector fields Y_3^+, Y^+_{-3} (see Ref. 1) are computed to be

$$\begin{aligned}
\widetilde{F}(Y_3^+) = &-(u_{2xx}v_{2x} - u_{2x}v_{2xx}) - \lambda R_{34}(u_{2xx}u_2 + v_{2xx}v_2) + (\lambda/2)R_{34}(u_{2x}^2 + v_{2x}^2) - m_1(u_{1x}u_{2x} + v_{1x}v_{2x})\\
&- \tfrac{3}{4}\lambda^2 R_{34}^2(u_{2x}v_2 - u_2v_{2x}) + \tfrac{1}{2}m_1\lambda R_{34}(u_{1x}v_2 - u_1v_{2x} + u_{2x}v_1 - u_2v_{1x}) - \tfrac{1}{4}m_1^2(u_{1x}v_1 - u_1v_{1x})\\
&- \tfrac{1}{2}m_1^2(u_{2x}v_2 - u_2v_{2x}) - \tfrac{1}{4}m_1^3(u_1u_2 + v_1v_2) + \tfrac{1}{8}\lambda^3 R_{34}^3 R_2 - \tfrac{1}{4}m_1\lambda^2 R_{34}^2(u_1u_2 + v_1v_2) + \tfrac{1}{8}m_1^2\lambda R_{34}(R_1 + 2R_2)
\end{aligned} \tag{2.11a}$$

and

$$
\begin{aligned}
\widetilde{F}(Y^{+}_{-3}) = {} & u_{1xx}v_{1x} - u_{1x}v_{1xx} + \lambda R_{34}(u_{1xx}u_1 + v_{1xx}v_1) + (\lambda/2)R_{34}(u_{1x}^2 + v_{1x}^2) - m_1(u_{1x}u_{2x} + v_{1x}v_{2x}) \\
& + \tfrac{3}{4}\lambda^2 R^2_{34}(u_{1x}v_1 - u_1v_{1x}) + \tfrac{1}{2}m_1\lambda R_{34}(u_{1x}v_2 - u_1v_{2x} + u_{2x}v_1 - u_2v_{1x}) + \tfrac{1}{2}m_1^2(u_{1x}v_1 - u_1v_{1x}) \\
& + \tfrac{1}{4}m_1^2(u_{2x}v_2 - u_2v_{2x}) - \tfrac{1}{4}m_1^3(u_1u_2 + v_1v_2) - \tfrac{1}{8}\lambda^3 R^3_{34}R_1 - \tfrac{1}{4}m_1\lambda^2R^2_{34}(u_1u_2 + v_1v_2) - \tfrac{1}{8}m_1^2\lambda R_{34}(2R_1 + R_2).
\end{aligned}
\tag{2.11b}
$$

Similar results are obtained for the Hamiltonians associated to the Lie–Bäcklund transformations Y_3^-, Y^-_{-3}. The vector fields Z_0^+, Z_0^- (see Ref. 1 and Appendix A) are Hamiltonian vector fields also, and the associated Hamiltonian densities are

$$
\begin{aligned}
\widetilde{F}(Z_0^+) &= x(\widetilde{F}(Y_1^+) - \widetilde{F}(Y^+_{-1})) \\
&\quad + t(\widetilde{F}(Y_1^+) + \widetilde{F}(Y^+_{-1})), \\
\widetilde{F}(Z_0^-) &= x(\widetilde{F}(Y_1^-) - \widetilde{F}(Y^-_{-1})) \\
&\quad + t(\widetilde{F}(Y^-_{-1}) + \widetilde{F}(Y^-_{-1})).
\end{aligned}
\tag{2.12}
$$

Now we arrive at the remarkable fact that the creating and annihilating Lie–Bäcklund transformations Z_1^+, Z^+_{-1}, Z_1^-, Z^-_{-1}, (see Ref. 1 and Appendix A) turn out to be Hamiltonian vector fields. The corresponding Hamiltonian densities are

$$
\begin{aligned}
\widetilde{F}(Z_1^+) &= x\{\widetilde{F}(Y_2^+) - \tfrac{1}{4}m_1^2\widetilde{F}(Y_0^+)\} \\
&\quad + t\{\widetilde{F}(Y_2^+) + \tfrac{1}{4}m_1^2\widetilde{F}(Y_0^+)\}, \\
\widetilde{F}(Z^+_{-1}) &= x\{-\widetilde{F}(Y^+_{-2}) + \tfrac{1}{4}m_1^2\widetilde{F}(Y_0^+)\} \\
&\quad + t\{\widetilde{F}(Y^+_{-2}) + \tfrac{1}{4}m_1^2\widetilde{F}(Y_0^+)\}, \\
\widetilde{F}(Z_1^-) &= x\{\widetilde{F}(Y_1^-) - \tfrac{1}{4}m_2^2\widetilde{F}(Y_0^-)\} \\
&\quad + t\{\widetilde{F}(Y_2^-) + \tfrac{1}{4}m_2^2\widetilde{F}(Y_0^-)\}, \\
\widetilde{F}(Z^-_{-1}) &= x\{-\widetilde{F}(Y^-_{-2}) + \tfrac{1}{4}m_2^2\widetilde{F}(Y_0^-)\} \\
&\quad + t\{\widetilde{F}(Y^-_{-2}) + \tfrac{1}{4}m_2^2\widetilde{F}(Y_0^-)\}.
\end{aligned}
\tag{2.13}
$$

The Hamiltonians $F(Z_1^+),..., F(Z^-_{-1})$ act as creating and annihilating operators on the t-independent Hamiltonians $F(Y^+_{-3}),..., F(Y_3^+), F(Y^-_{-3}),..., F(Y_3^-)$ by the action of the Poisson bracket (1.6), for example,

$$
\begin{aligned}
&\{F(Z_1^+), F(Y_0^+)\} = 0, \\
&\{F(Z_1^+), F(Y^+_{-1})\} = \tfrac{1}{4}m_1^2\{\tfrac{1}{2}R_1 + \tfrac{1}{2}R_2\} = \tfrac{1}{4}m_1^2\, F(Y_0^+),
\end{aligned}
\tag{2.14}
$$

$$
\{F(Z_1^+), F(Y_1^+)\} = -F(Y_2^+),
$$

and similar results for $F(Z^+_{-1})$, $F(Z_1^-)$, $F(Z^-_{-1})$. So the Hamiltonians $F(Z_1^+),...,F(Z^-_{-1})$ generate four hierarchies of (probably commuting t-independent) Hamiltonians

$$
F(Y^{\pm}_{\pm i}) \quad (i = 0,1,...).
\tag{2.15}
$$

Note that due to results described in Sec. III, we are more likely to consider

$$
...,F(Y^+_{-3}),...,F(Y_0^+),...,F(Y_3^+),...
\tag{2.16a}
$$

and

$$
...,F(Y^-_{-3}),...,F(Y_0^-),...,F(Y_3^-),...
\tag{2.16b}
$$

as two hierarchies instead of four.

III. INFINITE HIERARCHIES OF (*x,t*)-DEPENDENT LIE–BÄCKLUND TRANSFORMATIONS AND THEIR ASSOCIATED HAMILTONIANS

In this section we shall prove by construction the existence of infinite hierarchies of (x,t)-dependent Lie–Bäcklund transformations

$$
\begin{aligned}
&Z_0^+, Z_1^+, Z_2^+, Z_3^+ = [Z_1^+, Z_2^+],..., \\
&\qquad\qquad Z_k^+ = [Z_1^+, Z^+_{k-1}],..., \\
&Z_0^+, Z^+_{-1}, Z^+_{-2}, Z^+_{-3} = [Z^+_{-1}, Z^+_{-2}],..., \\
&\qquad\qquad Z^+_{-k} = [Z^+_{-1}, Z^+_{-k+1}],... .
\end{aligned}
\tag{3.1}
$$

Since the Lie algebra of Lie–Bäcklund transformations is a direct sum of two Lie algebras,[1] we shall restrict our considerations from now on to the " + " part. First of all we construct the vector fields Z_2^+, Z^+_{-2} (cf. Table I). Second, we prove that $[Z_1^+, Z_2^+]$ is independent of Z_0^+, Z_1^+, Z_2^+, and by an induction argument we obtain an infinite hierarchy. The same arguments apply to the other hierarchies. Moreover we shall prove that the vector fields $Z^+_{\pm i}$ are Hamiltonian vector fields, and the associated Hamiltonian densities are given.

Motivated by the result of Z_0^+, Z_1^+, Z^+_{-1} (Ref. 1) we search for a local (x,t)-dependent Lie–Bäcklund transformation, linear in x,t and of degree 4. The structure of such a Lie–Bäcklund transformation has to be

$$
x\left(\sum_{i=-3}^{3} \alpha_i m_1^{3-|i|} Y_i^+\right) + t\left(\sum_{i=-3}^{3} \beta_i m_1^{3-|i|} Y_i^+\right) + C,
\tag{3.2}
$$

where, in (3.2), α_i, β_i $(i = -3,...,3)$ are constants and C is (x,t) independent of degree 4. Eventually, after a huge computation, we obtained two Lie–Bäcklund transformations

$$
\begin{aligned}
Z_2^+ &= x(Y_3^+ + \tfrac{1}{2}m_1^2Y_1^+) + t(Y_3^+ - \tfrac{1}{2}m_1^2Y_1^+) + C_2^+, \\
Z^+_{-2} &= x(-Y^+_{-3} + \tfrac{1}{2}m_1^2Y^+_{-1}) \\
&\quad + t(Y^+_{-3} + \tfrac{1}{2}m_1^2Y^+_{-1}) + C^+_{-2},
\end{aligned}
\tag{3.3}
$$

where, in (3.3),

TABLE I. The Lie-algebraic picture of the Federbush model.

	Y_3^+	Y_3^-		deg = 6
Z_2^+	Y_2^+	Y_2^-	Z_2^-	deg = 4
Z_1^+	Y_1^+	Y_1^-	Z_1^-	deg = 2
Z_0^+	Y_0^+	Y_0^-	Z_0^-	deg = 0
Z^+_{-1}	Y^+_{-1}	Y^-_{-1}	Z^-_{-1}	deg = 2
Z^+_{-2}	Y^+_{-2}	Y^-_{-2}	Z^-_{-2}	deg = 4
	Y^+_{-3}	Y^-_{-3}		deg = 6

$$C_2^{+,u_1} = \tfrac14 m_1(-2v_{2x} - \lambda R_{34}u_2 + m_1u_1),$$
$$C_2^{+,v_1} = \tfrac14 m_1(-2u_{2x} - \lambda R_{34}v_2 + m_1v_1),$$
$$C_2^{+,u_2} = \tfrac34(-4u_{2xx} + 4\lambda R_{34}v_{2x} + 2\lambda(R_{34})_x v_2 - 2m_1v_{1x} + \lambda^2R_{34}^2u_2 - m_1\lambda R_{34}u_1 + m_1^2u_2), \quad (3.4a)$$
$$C_2^{+,v_2} = \tfrac34(-4v_{2xx} - 4\lambda R_{34}u_{2x} - 2\lambda(R_{34})_x u_2 + 2m_1u_{1x} + \lambda^2R_{34}^2v_2 - m_1\lambda R_{34}v_1 + m_1^2v_2),$$
$$C_2^{+,u_3} = (\lambda/3)v_3L_2^+, \quad C_2^{+,v_3} = -(\lambda/2)u_3L_2^+,$$
$$C_2^{+,u_4} = (\lambda/2)v_4L_2^+, \quad C_2^{+,v_4} = -(\lambda/2)u_4L_2^+,$$
and
$$C_{-2}^{+,u_1} = \tfrac34(-4u_{1xx} + 4\lambda R_{34}v_{1x} + 2\lambda(R_{34})_x v_1 + 2m_1v_{2x} + \lambda^2R_{34}^2u_1 + m_1\lambda R_{34}u_2 + m_1^2u_1),$$
$$C_{-2}^{+,v_1} = \tfrac34(-4v_{1xx} - 4\lambda R_{34}u_{1x} - 2\lambda(R_{34})_x u_1 - 2m_1u_{2x} + \lambda^2R_{34}^2v_1 + m_1\lambda R_{34}v_2 + m_1^2v_1), \quad (3.4b)$$
$$C_{-2}^{+,u_2} = \tfrac14 m_1(2v_{1x} + \lambda R_{34}u_1 + m_1u_2),$$
$$C_{-2}^{+,v_2} = \tfrac14 m_1(-2u_{1x} + \lambda R_{34}v_1 + m_1v_2),$$
$$C_{-2}^{+,u_3} = (\lambda/2)v_3L_{-2}^+, \quad C_{-2}^{+,v_3} = -\frac{\lambda}{2}u_3L_{-2}^+,$$
$$C_{-2}^{+,u_4} = (\lambda/2)v_4L_{-2}^+, \quad C_{-2}^{+,v_4} = -(\lambda/2)u_4L_{-2}^+,$$
while
$$L_2^+ = 2u_{2x}u_2 + 2v_{2x}v_2 - m_1(u_1v_2 - u_2v_1),$$
$$L_{-2}^+ = 2u_{1x}u_1 + 2v_{1x}v_1 - m_1(u_1v_2 - u_2v_1). \quad (3.4c)$$
Remarkably, the vector fields Z_2^+, Z_{-2}^+ are again Hamiltonian vector fields, and the associated Hamiltonian densities are computed to be
$$F(Z_2^+) = x(\tilde F(Y_3^+) + \tfrac12 m_1^2\tilde F(Y_1^+)) + t(\tilde F(Y_3^+) - \tfrac12 m_1^2\tilde F(Y_1^+)) - (\lambda/2)R_{34}(u_2u_{2x} + v_2v_{2x}) + (\lambda/4)m_1R_{34}(u_1v_2 - u_2v_1) - \tfrac12 m_1(u_1u_{2x} + v_1v_{2x}) \quad (3.5a)$$
and
$$\tilde F(Z_{-2}^+) = x(-\tilde F(Y_{-3}^+) + \tfrac12 m_1^2\tilde F(Y_{-1}^+)) + t(\tilde F(Y_{-3}^+) + \tfrac12 m_1^2\tilde F(Y_{-1}^+)) - (\lambda/2)R_{34}(u_1u_{1x} + v_1v_{1x}) + (\lambda/4)m_1R_{34}(u_1v_2 - u_2v_1) - \tfrac12 m_1(u_1u_{2x} + v_1v_{2x}). \quad (3.5b)$$

Obviously, similar results will hold for vector fields Z_2^-, Z_{-2}^- and their associated Hamiltonian densities. A formal proof of the existence of infinite hierarchies of t-dependent Hamiltonians and corresponding Lie–Bäcklund transformations is given in Appendix B by application of Lemma 1.

Finally we computed the action of the vector fields Z_2^+ on the hierarchy $(Y_i^+)_{i\in\mathbb{Z}}$ by a calculation of the Poisson bracket of the associated Hamiltonians, which resulted in
$$\{F(Z_2^+), F(Y_{-2}^+)\} = -\tfrac14 m_1^4F(Y_0^+),$$
$$\{F(Z_{-2}^+), F(Y_2^+)\} = -\tfrac14 m_1^4F(Y_0^+),$$
$$\{F(Z_2^+), F(Y_{-1}^+)\} = -\tfrac12 m_1^2F(Y_1^+), \quad (3.6)$$
$$\{F(Z_{-2}^+), F(Y_1^+)\} = -\tfrac12 m_1^2F(Y_{-1}^+),$$
$$\{F(Z_2^+), F(Y_0^+)\} = 0, \quad \{F(Z_{-2}^+), F(Y_0^+)\} = 0,$$
while the action on the $F(Z_i^+)_{i\in\mathbb{Z}}$ hierarchy is
$$\{F(Z_2^+), F(Z_{-1}^+)\} = -\tfrac32 m_1^2F(Z_1^+),$$
$$\{F(Z_{-2}^+), F(Z_{+1}^+)\} = -\tfrac32 m_1^2F(Z_{-1}^+), \quad (3.7)$$
$$\{F(Z_2^+), F(Z_{-2}^+)\} = -m_1^4F(Z_0^+),$$
a result which is twice the action of $Z_{\pm1}^+$, being similar to the result obtained by Ten Eikelder[11] for the massive Thirring model.

IV. CONCLUSION

We obtained four infinite hierarchies of (x,t)-independent Lie–Bäcklund transformations and four infinite hierarchies of (x,t)-dependent Lie–Bäcklund transformations, which are all Hamiltonian vector fields. The corresponding densities are given.

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APPENDIX A: LIE–BÄCKLUND TRANSFORMATIONS OF THE FEDERBUSH MODEL

We summarize the Lie–Bäcklund transformations obtained in Ref. 1, only giving the "+" part, Y_0^+, Y_1^+, $Y_{\pm2}^+$, Z_0^+, $Z_{\pm+}^+$, i.e.,

$$Y_0^+ = -v_1\,\partial_{u_1} + u_1\,\partial_{v_1} - v_2\,\partial_{u_2} + u_2\,\partial_{v_2},$$
$$Y_1^+ = \tfrac12 m_1v_2\,\partial_{u_1} - \tfrac12 m_1u_2\,\partial_{v_1} + \tfrac12(2u_{2x} + m_1v_1 - \lambda v_2(R_{34}))\partial_{u_2} + \tfrac12(2v_{2x} - m_1u_1 + \lambda u_2(R_{34}))\partial_{v_2} - (\lambda/2)v_3R_2\,\partial_{u_3} + (\lambda/2)u_3R_2\,\partial_{v_3} - (\lambda/2)v_4R_2\,\partial_{u_4} + (\lambda/2)u_4R_2\,\partial_{v_4},$$
$$Y_{-1}^+ = \tfrac12(2u_{1x} - m_1v_2 - \lambda v_1(R_{34}))\partial_{u_1} + \tfrac12(2v_{1x} + m_1u_2 + \lambda u_1(R_{34}))\partial_{v_1} - \tfrac12 m_1v_1\,\partial_{u_2} + \tfrac12 m_1u_1\,\partial_{v_2} - (\lambda/2)v_3R_1\,\partial_{u_3} + (\lambda/2)u_3R_1\,\partial_{v_3} - (\lambda/2)v_4R_1\,\partial_{u_4} + (\lambda/2)u_4R_1\,\partial_{v_4},$$
$$Y_2^{+,u_1} = \tfrac14 m_1\{+2u_{2x} - \lambda v_2R_{34} + m_1v_1\}, \quad Y_2^{+,v_1} = \tfrac14 m_1\{+2v_{2x} + \lambda u_2R_{34} - m_1u_1\},$$
$$Y_2^{+,u_2} = \tfrac14\{-4v_{2xx} - 2\lambda u_2(R_{34})_x - 4\lambda u_{2x}R_{34} + 2m_1u_{1x} - \lambda m_1v_1R_{34} + \lambda^2v_2R_{34}^2 + m_1^2v_2\},$$
$$Y_2^{+,v_2} = \tfrac14\{+4u_{2xx} - 2\lambda v_2(R_{34})_x - 4\lambda v_{2x}R_{34} + 2m_1v_{1x} + \lambda m_1u_1R_{34} - \lambda^2u_2R_{34}^2 + m_1^2u_2\},$$
$$Y_2^{+,u_3} = (\lambda/2)v_3K_2^+, \quad Y_2^{+,v_3} = -(\lambda/2)u_3K_2^+, \quad Y_2^{+,u_4} = (\lambda/2)v_4K_2^+, \quad Y_2^{+,v_4} = -(\lambda/2)u_1K_2^+,$$
where

$$K_2^+ = -2u_{1x}v_1 + 2u_1v_{1x} + m_1(u_1u_2 + v_1v_2) + \lambda R_1R_{34},$$
$$Y_{-2}^{+;u_1} = \tfrac{1}{4}\{-4v_{1xx} - 2\lambda u_1(R_{34})_x - 4\lambda u_{1x}R_{34} - 2m_1u_{2x} + \lambda m_1v_2R_{34} + \lambda^2 v_1R_{34}^2 + m_1^2v_1\},$$
$$Y_{-2}^{+;v_1} = \tfrac{1}{4}\{+4u_{1xx} - 2\lambda v_1(R_{34})_x - 4\lambda v_{1x}R_{34} - 2m_1v_{2x} - \lambda m_1u_2R_{34} - \lambda^2 u_1R_{34}^2 - m_1^2u_1\},$$
$$Y_{-2}^{+;u_2} = \tfrac{1}{4}m_1\{-2u_{1x} + \lambda v_1R_{34} + m_1v_2\}, \quad Y_{-2}^{+;v_2} = \tfrac{1}{4}m_1\{-2v_{1x} - \lambda u_1R_{34} - m_1u_2\},$$
$$Y_{-2}^{+;u_3} = (\lambda/2)v_3K_{-2}^+, \quad Y_{-2}^{+;v_3} = -(\lambda/2)u_3K_{-2}^+, \quad Y_{-2}^{+;u_4} = (\lambda/2)v_4K_{-2}^+, \quad Y_{-2}^{+;v_4} = -(\lambda/2)u_4K_{-2}^+,$$
where
$$K_{-2}^+ = -2u_{1x}v_1 + 2u_1v_{1x} + m_1(u_1u_2 + v_1v_2) + \lambda R_1R_{34},$$
while the (x,t)-dependent Lie–Bäcklund transformations are given by
$$Z_0^+ = x(Y_1^+ - Y_{-1}^+) + t(Y_1^+ + Y_{-1}^+) + \tfrac{1}{2}(-u_1\partial_{u_1} - v_1\partial_{v_1} + u_2\partial_{u_2} + v_2\partial_{v_2}),$$
$$Z_1^+ = x(+Y_2^+ - \tfrac{1}{4}m_1^2Y_0^+) + t(+Y_2^+ + \tfrac{1}{4}m_1^2Y_0^+) + \tfrac{1}{2}(-2v_{2x} + m_1u_1 - \lambda u_2R_{34})\partial_{u_2}$$
$$+ \tfrac{1}{2}(+2u_{2x} + m_1v_1 - \lambda v_2R_{34})\partial_{v_2},$$
$$Z_{-1}^+ = x(-Y_{-2}^+ - \tfrac{1}{4}m_1^2Y_0^+) + t(+Y_{-2}^+ + \tfrac{1}{4}m_1^2Y_0^+) + \tfrac{1}{2}(+2v_{1x} + m_1u_2 + \lambda u_1R_{34})\partial_{u_1}$$
$$+ \tfrac{1}{2}(-2u_{1x} + m_1v_2 - \lambda v_1R_{34})\partial_{v_1},$$
$$Y_3^+ = [Z_1^+, Y_2^+], \quad Y_{-3}^+ = [Z_{-1}^+, Y_{-2}^+].$$
Similar results have been obtained for the " $-$ " part.[1]

APPENDIX B: THE INFINITY OF THE HIERARCHIES

We shall prove a lemma from which the existence of infinite hierarchies of Hamiltonians

$$\begin{aligned} &F(Y_0^+), F(Y_1^+), F(Y_2^+), \ldots, \\ &F(Y_0^+), F(Y_{-1}^+), F(Y_{-2}^+), \ldots, \\ &F(Z_0^+), F(Z_1^+), F(Z_2^+), \ldots, \\ &F(Z_0^+), F(Z_{-1}^+), F(Z_{-2}^+), \ldots, \end{aligned} \tag{B1}$$

and their associated Lie–Bäcklund transformations

$$Y_0^+, Y_1^+, Y_{\pm 2}^+, \ldots, \quad Z_0^+, Z_{\pm 1}^+, Z_2^+, \ldots, \tag{B2}$$

immediately follow. In this lemma the lower indices of u, v refer to partial derivatives with respect to x (i.e., $u_1 = u_x$, $u_2 = u_{xx}, \ldots$).

Lemma: Let $H_n(u,v)$, $K_n(u,v)$, $\overline{H}_n(u,v)$, and $\overline{K}_n(u,v)$ be defined by

$$\begin{aligned} H_n(u,v) &= \int_{-\infty}^{\infty} (u_n^2 + v_n^2), \\ K_n(u,v) &= \int_{-\infty}^{\infty} (u_{n+1}v_n - v_{n+1}u_n), \\ \overline{H}_n(u,v) &= \int_{-\infty}^{\infty} x(u_n^2 + v_n^2), \\ \overline{K}(u,v) &= \int_{-\infty}^{\infty} x(u_{n+1}v_n - v_{n+1}u_n), \end{aligned} \tag{B3}$$

and define the Poisson bracket of F and L $\{F,L\}$ by

$$\{F,L\} = \int_{-\infty}^{\infty} \left(+ \frac{\delta F}{\delta v}\frac{\delta L}{\delta u} - \frac{\delta F}{\delta u}\frac{\delta L}{\delta v} \right), \tag{B4}$$

then the following results hold

$$\{\overline{H}_1, H_n\} = +4nK_n, \tag{B5a}$$
$$\{\overline{H}_1, K_n\} = +2(2n+1)H_{n+1}, \tag{B5b}$$
$$\{\overline{H}_1, \overline{H}_n\} = +4(n-1)\overline{K}_n, \tag{B5c}$$
$$\{\overline{H}_1, \overline{K}_n\} = +2(2n-1)\overline{H}_{n+1}. \tag{B5d}$$

Proof: We shall prove relations (B5a) and (B5c) (the other proofs run along the same lines):

$$\frac{\delta H_n}{\delta u} = (-1)^n 2u_{2n}, \quad \frac{\delta H_n}{\delta v} = (-1)^n 2v_{2n}, \tag{B6a}$$
$$\frac{\delta \overline{H}_n}{\delta u} = (-1)^n 2(xu_n)^{(n)}, \quad \frac{\delta \overline{H}_n}{\delta v} = (-1)^{(n)} 2(xv_n)^{(n)}. \tag{B6b}$$

Substitution of (B6a) and (B6b) into (B4) yields

$$\begin{aligned} \{\overline{H}_1, H_n\} &= -\int_{-\infty}^{\infty} 4(-1)^n u_{2n}(xv_1)^{(1)} - 4(-1)^n v_{2n}(xu_1)^{(1)} \\ &= 4(-1)^{2n}\int_{-\infty}^{\infty} (xv_1)^{(n)}u_{n+1} - (xu_1)^{(n)}v_{n+1} \\ &= +4n\int_{-\infty}^{\infty} v_nu_{n+1} - u_nv_{n+1} = +4nK_n, \end{aligned}$$

which proves relation (B5a). Substitution of (B6b) into (2.4) yields

$$\begin{aligned} \{\overline{H}_1, \overline{H}_n\} &= -\int_{-\infty}^{\infty} 4(-1)^n(xv_1)^{(1)}(xu_n)^{(n)} \\ &\quad - 4(-1)^n(xu_1)^{(1)}(xv_n)^{(n)} \\ &= 4(-1)^n(-1)^n\int_{-\infty}^{\infty} (xv_1)^{(n)}(xu_n)^{(1)} \\ &\quad - (xu_1)^{(n)}(xv_n)^{(1)} \\ &= +4(n-1)\int_{-\infty}^{\infty} x(u_{n+1}v_n - u_nv_{n+1}) \\ &= +4(n-1)\overline{K}_n, \end{aligned}$$

which proves relation (B5c). This existence of infinite hierarchies $H(Y_{\pm i}^+)$ now follows from the explicit structure of $H(Z_{\pm 1}^+)$ [Eq. (2.12)] and $H(Y_{\pm 1}^+)$ [Eq. (2.6)] by considering the (λ, m_1, m_2)-independent parts and application of part a and b of this Lemma. The existence of the infinite hierarchies $H(Z^{\pm}_{\pm n})$ follows from a similar argument using $\overline{H}_m(u,v), \overline{K}_n(u,v)$.

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