Infinite hierarchies of t-independent and t-dependent conserved functionals of the Federbush model

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I. INTRODUCTION

In a recent paper, one of the authors constructed four infinite hierarchies of Lie–Bäcklund transformations of the Federbush model. Moreover he computed four creating and annihilating local \((x,t)\)-dependent Lie–Bäcklund transformations to which these hierarchies that lead to these hierarchies. In this paper we show that to these four creating Lie–Bäcklund transformations, we can associate four \(t\)-dependent conserved functionals. By consequence the attempt to construct recursion operators from these creating Lie–Bäcklund transformations failed since they are Hamiltonian vector fields. By recursive action of the Poisson bracket with these functionals we construct infinite hierarchies of conserved functionals associated to the \((x,t)\)-independent Lie–Bäcklund transformations. This will be done in Sec. II. In Sec. III we construct four new \((x,t)\)-dependent Lie–Bäcklund transformations from which we shall prove the existence of four infinite hierarchies of \(t\)-dependent conserved functionals, and consequently hierarchies of \((x,t)\)-dependent Lie–Bäcklund transformations of the Federbush model. A formal proof is given in Appendix B, while a survey of the already known vector fields is given in Appendix A.

We want to stress the fact that all computations have been worked out on a DEC-system 20 computer using REDUCE and a software package to do these calculations.

Lie–Bäcklund transformations are vector fields \(V\) defined on the infinite jet bundle of \(M,N,J = (M,N)\), where \(M\) is the space of independent variables and \(N\) the space of the dependent variables. A Lie–Bäcklund transformation of a differential equation is a vector field \(V\) defined on \(J = (M,N)\) satisfying the condition

\[
\mathcal{L}_V(D + I) \subset D + I, \tag{1.1}
\]

where \(I\) denotes a differential ideal associated to the differential equation at hand, while \(D + I\) denotes its infinite prolongation to \(J = (M,N)\). \(\mathcal{L}_V\) is the Lie derivative with respect to the vector field \(V\) (Ref. 9). Since the vector fields \(V\) are supposed to depend only on a finite number of variables, condition (1.1) reduces to

\[
\mathcal{L}_V \subset I \subset D + I \quad \text{for some } r. \tag{1.2}
\]

Using this method we computed Lie–Bäcklund transformation of the Federbush model. It can be shown that the Lie–Bäcklund transformations in this setting are just symmetries in the works of Magri, Ten Eikelder, and Fuchsteiner and Fokas, where (generators of) symmetries of partial differential equations of evolutionary type are described as transformations on special types of infinite dimensional spaces. Suppose that

\[
\frac{du}{dt} = \Omega^{-1} dH \tag{1.3}
\]

is an infinite dimensional Hamiltonian system, where \(\Omega\) is the symplectic operator, \(H\) the Hamiltonian, \(dH\) is the Fréchet derivative of \(H\). Then to each Hamiltonian symmetry (also called canonical symmetry) \(Y\), there corresponds by definition a Hamiltonian \(F(Y)\) such that

\[
Y = \Omega^{-1} dF(Y), \tag{1.4}
\]

and the Poisson bracket of \(F\) and \(H\) vanishes. Suppose that \(Y_1, Y_2\) are two Hamiltonian symmetries, then \([Y_1, Y_2]\) is a Hamiltonian symmetry and

\[
F([Y_1, Y_2]) = \{F(Y_1), F(Y_2)\}, \tag{1.5}
\]

where \(\{,\}\) is the Poisson bracket defined by

\[
\{F(Y_1), F(Y_2)\} = \langle dF(Y_1), Y_2 \rangle, \tag{1.6}
\]

where \(\langle,\rangle\) denotes the contraction of a one-form and a vector field. These notions shall be used throughout Sec. II and III.

II. CONSERVED FUNCTIONALS FOR THE FEDERBUSH MODEL

We shall discuss conserved functionals for the Federbush model. This model is described by

\[
\left( \begin{array}{c} i(\partial_x + \partial_t) - m(s) \\ - m(s) \end{array} \right) (\psi_{s,1}) = \left( \begin{array}{c} \psi_{s,1} \\ \psi_{s,2} \end{array} \right) \tag{2.1}
\]

where \(\psi_{s}(x,t)\) are two-component complex-valued functions. Suppressing the factor \(4\pi (\lambda' = 4\pi \lambda)\) and introducing the eight real variables \(u_1, u_1, u_2, v_2, u_3, v_3, u_4, v_4\) by

\[
\begin{align*}
\psi_{s,1} &= u_1 + iv_1, & \psi_{s,1} &= u_3 + iv_3, & m(\pm 1) = m_1, \\
\psi_{s,2} &= u_2 + iv_2, & \psi_{s,2} &= u_4 + iv_4, & m(-1) = m_2,
\end{align*} \tag{2.2}
\]

where \(m_1, m_2\) are integers.

H. M. Kersten
Department of Applied Mathematics, Twente University of Technology, P. O. Box 217, 7500 AE Enschede, The Netherlands

H. M. M. Ten Eikelder
Department of Mathematics and Computing Science, Eindhoven University of Technology, P. O. Box 513, 5600 MB Eindhoven, The Netherlands

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The construction of four infinite hierarchies of \(t\)-independent and \(t\)-dependent conserved functionals for the Federbush model is given. A formal proof of the existence of these infinite hierarchies is given in Appendix B.
Eq. (2.1) is rewritten as a system of eight nonlinear partial differential equations for the functions $u_1,...,u_4$ i.e.,

\[
\begin{align*}
  u_{1t} + u_{1x} - m_1 v_x &= \lambda R u_{1t}, \\
  - v_{1t} - v_{1x} - m_1 u_x &= \lambda R u_{1t}, \\
  u_{2t} - u_{2x} - m_1 v_x &= \lambda R u_{1t}, \\
  - v_{2t} + v_{2x} - m_1 u_x &= \lambda R u_{1t}, \\
  u_{3t} + u_{3x} - m_2 v_x &= \lambda R u_{1t}, \\
  - v_{3t} - v_{3x} - m_2 u_x &= \lambda R u_{1t}, \\
  u_{4t} - u_{4x} - m_2 v_x &= \lambda R u_{1t}, \\
  - v_{4t} + v_{4x} - m_2 u_x &= \lambda R u_{1t},
\end{align*}
\]

(2.3)

where, in (2.3),

\[
R_1 = u_1^2 + v_1^2, \quad R_2 = u_2^2 + v_2^2, \\
R_3 = u_3^2 + v_3^2, \quad R_4 = u_4^2 + v_4^2.
\]

Equation (2.3) can be written as a Hamiltonian system\(^{4,5}\)

\[
\frac{d\mathbf{u}}{dt} = \Omega^{-1} dH, \tag{2.4}
\]

whereas in (2.4), $\mathbf{u} = (u_1, u_2, u_3, u_4)$,

\[
\Omega = \begin{pmatrix}
  J & 0 \\
  0 & J
\end{pmatrix}, \quad J = \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix}, \tag{2.5a}
\]

and

\[
H = \int_{-\infty}^{\infty} \left\{ u_{1x} v_x - u_1 v_{1x} - u_{2x} v_x + u_2 v_{2x} + u_{3x} v_x - u_3 v_{3x} - u_{4x} v_x + u_4 v_{4x} - m_1 (u_1 u_x + v_1 v_x) - m_2 (u_2 u_x + v_2 v_x) \right\} dx
\]

(2.5b)

(by $f_{-\infty}^{\infty}$ we mean integration of the integrand with respect to $x$). In (2.4), $dH$ is the Fréchet derivative of $H$ defined by

\[
\frac{d}{de} H(x + ey) \bigg|_{e=0} = \langle dH, y \rangle. \tag{2.6}
\]

In a previous paper\(^1\) we constructed four first-order Lie–Bäcklund transformations $Y_0^+, Y_0^-, Y_1^+, Y_1^-$ (Appendix A) that are Hamiltonian\(^{4,5}\) vector fields; the associated Hamiltonian densities are given by

\[
\tilde{F}(Y_0^+) = -\frac{1}{2}(u_{2x} v_x - u_2 v_{2x}) + \frac{1}{2} R_{34} (u_{2x} u_x + v_{2x} v_x) - m_1 (u_1 u_x + v_1 v_x)
\]

and

\[
\tilde{F}(Y_0^-) = -\frac{1}{2}(u_{2x} v_x - u_2 v_{2x}) - \frac{1}{2} R_{34} (u_{2x} u_x + v_{2x} v_x) - m_1 (u_1 u_x + v_1 v_x) - \frac{1}{2} \lambda^2 R_{34} R_2
\]

(2.11a)

while the Hamiltonian densities associated to the gauge transformations $Y_0^+, Y_0^-$ (see Ref. 1 and Appendix A) are given by

\[
\tilde{F}(Y_0^+) = \frac{1}{2} R_1 + R_2, \quad \tilde{F}(Y_0^-) = \frac{1}{2} R_3 + R_4.
\]

(2.7b)

In (2.7a), $R_{12}, R_{34}$ are defined by

\[
R_{12} = R_1 + R_2, \quad R_{34} = R_3 + R_4.
\]

(2.8)

(Note that we use $\tilde{F}$ for the density of the conserved functional $F$, so $F = \int_{-\infty}^{\infty} \tilde{F}$.) The associated Lie–Bäcklund transformations can be derived from (2.7a) by the formula

\[
Y = \Omega^{-1} dF(Y), \quad (2.9)
\]

and for reasons of completeness they are surveyed in Appendix A at the end of this paper. The Hamiltonian densities associated with the second-order Lie–Bäcklund transformations $Y^+_2, Y^+_1, Y^-_2, Y^-_1$ (see Ref. 1 and Appendix A) are computed, yielding

\[
\tilde{F}(Y^+_2) = -\frac{1}{2}(u_{2x} v_x + v_{2x}) + \frac{1}{2} \lambda^2 R_{34} R_2
\]

\[
- \frac{1}{2} m_1 (u_{2x} v_x - u_2 v_{2x}) - \frac{1}{2} \lambda R_{34} R_2
\]

\[
+ \frac{1}{2} \lambda R_{34} R_2 (u_1 u_x + v_1 v_x) - \frac{1}{2} \lambda R_{34} R_2
\]

\[
- \frac{1}{2} m_1 (u_{2x} v_x - u_2 v_{2x}) - \frac{1}{2} \lambda R_{34} R_2
\]

\[
(2.7a)
\]

\[
\tilde{F}(Y^-_2) = -\frac{1}{2}(u_{2x} v_x - v_{2x}) + \frac{1}{2} \lambda^2 R_{34} R_2
\]

\[
- \frac{1}{2} m_1 (u_{2x} v_x - u_2 v_{2x}) - \frac{1}{2} \lambda R_{34} R_2 (u_1 u_x + v_1 v_x)
\]

\[
+ \frac{1}{2} \lambda R_{34} R_2 (u_1 u_x + v_1 v_x) - \frac{1}{2} \lambda R_{34} R_2
\]

\[
- \frac{1}{2} m_1 (u_{2x} v_x - u_2 v_{2x}) - \frac{1}{2} \lambda R_{34} R_2
\]

\[
(2.7b)
\]

\[
\tilde{F}(Y^+_1) = -\frac{1}{2}(u_{2x} v_x - u_2 v_{2x}) + \frac{1}{2} \lambda^2 R_{34} R_2
\]

\[
- \frac{1}{2} m_1 (u_{2x} v_x - u_2 v_{2x}) - \frac{1}{2} \lambda R_{34} R_2 (u_1 u_x + v_1 v_x) - \frac{1}{2} \lambda R_{34} R_2
\]

\[
+ \frac{1}{2} \lambda R_{34} R_2 (u_1 u_x + v_1 v_x) + \frac{1}{2} \lambda R_{34} R_2
\]

\[
\tilde{F}(Y^-_1) = -\frac{1}{2}(u_{2x} v_x - u_2 v_{2x}) - \frac{1}{2} \lambda^2 R_{34} R_2
\]

\[
- \frac{1}{2} m_1 (u_{2x} v_x - u_2 v_{2x}) - \frac{1}{2} \lambda R_{34} R_2 (u_1 u_x + v_1 v_x)
\]

\[
+ \frac{1}{2} \lambda R_{34} R_2 (u_1 u_x + v_1 v_x) + \frac{1}{2} \lambda R_{34} R_2
\]

\[
(2.7a)
\]
Similar results are obtained for the Hamiltonians associated to the Lie–Bäcklund transformations $Y^{-1}$, $Y^{-3}$. The vector fields $Z_0^+$, $Z_0^-$ (see Ref. 1 and Appendix A) are Hamiltonian vector fields also, and the associated Hamiltonian densities are

$$\tilde{F}(Y_{+}^{-}) = u_{1xx}v_{1x} - u_{1x}v_{1xx} + \lambda R_{34}(u_{1xx}u_1 + v_{1xx}v_1) + (\lambda/2) R_{34}(u_{1xx}^2 + v_{1xx}^2) - m_1(u_{1xx^2} + v_{1xx^2})$$
$$+ \frac{1}{2}\lambda^2 R_{34}(u_{1xx}v_1 - u_{1x}v_{1xx}) + m_1\lambda R_{34}(u_{1x}v_2 - u_{2x}v_1 + u_{2x}v_1) + \frac{1}{2}\lambda^2 R_{34}(u_{1xx}v_1 - u_{1x}v_{1xx})$$
$$+ \frac{1}{2}m_1^2(u_{2xx^2} - u_{2x^2x}) - \frac{1}{2}m_1^2(u_{1xx^2} + v_{1xx^2}) - \frac{1}{2}\lambda^2 R_{34}(u_{1xx^2} + v_{1xx^2}) - \frac{1}{2}\lambda^2 R_{34}(u_{1xx^2} + v_{1xx^2})$$

(2.11b)

III. INFINITE HIERARCHIES OF $(x,t)$-DEPENDENT LIE–BÄCKLUND TRANSFORMATIONS AND THEIR ASSOCIATED HAMILTONIANS

In this section we shall prove by construction the existence of infinite hierarchies of $(x,t)$-dependent Lie–Bäcklund transformations

$$Z_0^+, Z_1^+, Z_2^+, Z_3^+, ... = [Z_1^+, Z_2^+]$$
$$Z_k^+ = [Z_{k-1}^+, Z_1^+]$$
$$Z_0^+, Z_1^+, Z_2^+, Z_3^+ = [Z_1^+, Z_2^+]$$
$$Z_{-k}^- = [Z_{-1}^-, Z_{-k+1}^-]$$

(3.1)

Since the Lie algebra of Lie–Bäcklund transformations is a direct sum of two Lie algebras, we shall restrict our considerations from now on to the “+” part. First of all we construct the vector fields $Z_1^+$, $Z_3^+$ (cf. Table I). Second, we prove that $[Z_1^+, Z_2^+]$ is independent of $Z_0^+$, $Z_1^+$, $Z_3^+$, and by an induction argument we obtain an infinite hierarchy. The same arguments apply to the other hierarchies. Moreover we shall prove that the vector fields $Z_{+k}$ are Hamiltonian vector fields, and the associated Hamiltonian densities are given.

Motivated by the result of $Z_2^+ + Z_3^-$ (Ref. 1) we search for a local $(x,t)$-dependent Lie–Bäcklund transformation, linear in $x,t$ and of degree 4. The structure of such a Lie–Bäcklund transformation has to be

$$x\left(\sum_{i=-3}^{3} \alpha_i m_1^{-i} Y_1^+ \right) + t\left(\sum_{i=-3}^{3} \beta_i m_1^{-i} Y_1^+ \right) + C$$

(3.2)

where, in (3.2), $\alpha_i, \beta_i (i = -3, ..., 3)$ are constants and $C$ is $(x,t)$ independent of degree 4. Eventually, after a huge computation, we obtained two Lie–Bäcklund transformations

$$Z_1^+ = x(Y_1^+ + m_1^{-2} Y_1^+) + t(Y_1^+ - m_1^{-2} Y_1^+) + C_1^+$$
$$Z_{-2}^- = x(Y_{-2}^+ + m_1^{-2} Y_{-2}^+) + t(Y_{-2}^+ - m_1^{-2} Y_{-2}^+) + C_{-2}^-$$

(3.3)

where, in (3.3),

<table>
<thead>
<tr>
<th>TABLE I. The Lie-algebraic picture of the Federbush model.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_0^+$</td>
</tr>
<tr>
<td>$Z_1^+$</td>
</tr>
<tr>
<td>$Z_2^+$</td>
</tr>
<tr>
<td>$Z_3^+$</td>
</tr>
<tr>
<td>$Z_{-1}^-$</td>
</tr>
<tr>
<td>$Z_{-2}^-</td>
</tr>
</tbody>
</table>

P. H. M. Kersten and H. M. M. Ten Eikelder 2142
Remarkably, the vector fields are computed to be

\[ C_{x^+} = \pmatrix{ -2v_2 & -\lambda R_{34}v_2 + m_1u_1, \\
 -2u_2 & -\lambda R_{34}v_2 + m_1u_1 }, \]

and

\[ C_{\pm} = \pmatrix{ -4u_{1x} + 4\lambda R_{34}v_1 + 2\lambda (R_{34})_x v_2 - 2m_1v_1x \\
 -\lambda^2 R_{34}v_2 - m_1 R_{34}u_1 + m_1^2u_2 }, \]  

(3.4a)

\[ C_{2x} = \pmatrix{ -4u_{2x} - 4\lambda R_{34}u_2 - 2\lambda (R_{34})_x u_1 + 2m_1u_2x \\
 -\lambda^2 R_{34}v_2 - m_1 R_{34}u_1 + m_1^2u_2 }, \]

(3.4b)

\[ C_{2x} = \pmatrix{ -4u_{1x} + 4\lambda R_{34}v_1 + 2\lambda (R_{34})_x v_2 - 2m_1v_1x \\
 -\lambda^2 R_{34}v_2 - m_1 R_{34}u_1 + m_1^2u_2 }, \]

while

\[ L_{\mp} = 2u_2x, u_2 + 2v_2x, v_2 - m_1(u_1v_2 - u_2v_1), \]

(3.4c)

Remarkably, the vector fields \( Z_1^+ \) and \( Z_2^+ \) are again Hamiltonian vector fields, and the associated Hamiltonian densities are computed to be

\[ F(Z_1^+) = x(\bar{F}(Y_1^+) + \pmatrix{ 0 \cr 0 } (\bar{F}(Y_1^+)) + (\bar{F}(Y_1^+)) \]

\[ + \pmatrix{ -\lambda /4m_1 R_{34}(u_1v_2 - u_2v_1) - \lambda m_1(u_1v_2 + v_2u_1) }, \]

(3.5a)

\[ F(Z_2^+) = x(\bar{F}(Y_2^+) + \pmatrix{ 0 \cr 0 } (\bar{F}(Y_2^+)) + (\bar{F}(Y_2^+)) \]

\[ + \pmatrix{ -\lambda /4m_1 R_{34}(u_1v_2 - u_2v_1) - \lambda m_1(u_1v_2 + v_2u_1) }, \]

(3.5b)

Obviously, similar results will hold for vector fields \( Z_2^-, Z_{\pm} \) and their associated Hamiltonian densities. A formal proof of the existence of infinite hierarchies of \( t \)-dependent Hamiltonians and corresponding Lie-Bäcklund transformations is given in Appendix B by application of Lemma 1.

Finally we computed the action of the vector fields \( Z^+ \) on the hierarchy \( (Y_1^+)_{\infty} \) by calculation of the Poisson bracket of the associated Hamiltonians, which resulted in

\[ \{ F(Z_1^+), F(Y_0^+) \} = -H m_1 F(Y_0^+), \]

\[ \{ F(Z_2^+) \}, F(Y_0^+) \} = -H m_1 F(Y_0^+) \]

(3.6)

\[ \{ F(Z_2^+), F(Y_0^+) \} = 0, \quad \{ F(Z_2^+), F(Y_0^+) \} = 0, \]

while the action on the \( F(Z_1^+) \) \( \infty \) hierarchy is

\[ \{ F(Z_1^+), F(Z_{-1}) \} = -H m_1 F(Z_{-1}), \]

(3.7)

\[ \{ F(Z_1^+), F(Z_{-1}) \} = -H m_1 F(Z_{-1}), \]

a result which is twice the action of \( Z_{\pm} \), being similar to the result obtained by Ten Eikelder\(^{11} \) for the massive Thirring model.

**IV. CONCLUSION**

We obtained four infinite hierarchies of \( (x,t) \)-independent Lie-Bäcklund transformations and four infinite hierarchies of \( (x,t) \)-dependent Lie-Bäcklund transformations, which are all Hamiltonian vector fields. The corresponding densities are given.

**ACKNOWLEDGMENTS**

The authors wish to thank Professor R. Martini and Professor J. de Graaff for stimulating this joint research.

**APPENDIX A: LIE-BÄCKLUND TRANSFORMATIONS OF THE FEDERBUSCH MODEL**

We summarize the Lie-Bäcklund transformations obtained in Ref. 1, only giving the " + " part, \( Y_0^+, Y_1^+, Y_2^+, Z_0^+, Z_{\pm}^+, \) i.e.,

\[ Y_0^+ = -v_1 \partial_{x_1} + u_1 \partial_{v_1} - v_2 \partial_{u_2} + u_2 \partial_{v_2}, \]

\[ Y_1^+ = \pmatrix{ m_1 \partial_{u_1} - m_1 \partial_{v_1} + (2v_2 - v_2) \partial_{u_2} + (2v_2 - v_2) \partial_{v_2}, \\
 m_1 \partial_{u_2} - m_1 \partial_{v_2} + (2v_2 - v_2) \partial_{u_2} + (2v_2 - v_2) \partial_{v_2} }, \]

\[ Y_{\pm} = \pmatrix{ 2(m_1 \partial_{u_2} - m_1 \partial_{v_2} + \lambda v_2(R_{34})_x \partial_{u_2} + \lambda v_2(R_{34})_x \partial_{v_2}, \\
 2(m_1 \partial_{u_2} - m_1 \partial_{v_2} + \lambda v_2(R_{34})_x \partial_{u_2} + \lambda v_2(R_{34})_x \partial_{v_2} ), \}

\[ Y_{\pm}^{2+} \partial_{x_1} = \pmatrix{ \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2}, \\
 \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2} }, \]

\[ Y_{\pm}^{2\pm} = \pmatrix{ \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2}, \\
 \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2} }, \]

where

\[ Y_{\pm}^{2+} = \pmatrix{ \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2}, \\
 \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2} }, \]

\[ Y_{\pm}^{2\pm} = \pmatrix{ \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2}, \\
 \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2} }, \]

\[ Y_{\pm}^{2+} = \pmatrix{ \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2}, \\
 \lambda \partial_{u_1} \partial_{u_2} + \lambda \partial_{v_1} \partial_{v_2} }, \]
\[ K_{\pm} = -2u_{\pm}v_{\pm} + 2u_{\mp}v_{\mp} + m_{\pm}(u_{\pm}v_{\pm} + v_{\mp}u_{\mp}) + \lambda R_{\mp}R_{\pm}, \]

\[ Y_{\pm}^\pm = \frac{1}{2} \{ -4v_{\pm}x - 2\lambda u_{\pm}(R_{\mp}x) - 4\lambda u_{\pm}R_{\mp} - 2m_{\pm}x + \lambda m_{\pm}v_{\pm}R_{\mp} + \lambda v_{\mp}R_{\pm}^2 + m_{\pm}^2v_{\pm} \}, \]

\[ Y_{\pm}^\mp = \frac{1}{2} \{ -4u_{\mp}x - 2\lambda v_{\mp}(R_{\pm}x) - 4\lambda v_{\mp}R_{\pm} - 2m_{\mp}x - \lambda m_{\pm}u_{\pm}R_{\pm} - \lambda v_{\mp}R_{\pm}^2 - m_{\pm}^2u_{\pm} \}, \]

\[ Y_{\pm}^\pm = (\lambda/2)v_{\pm}K_{\pm}^\pm, \quad Y_{\pm}^\mp = (\lambda/2)u_{\pm}K_{\pm}^\mp, \quad Y_{\pm} = (\lambda/2)u_{\pm}K_{\pm}, \quad Y_{\pm}^\pm = -(\lambda/2)u_{\pm}K_{\pm}^\mp, \]

where

\[ K_{\pm} = -2u_{\pm}v_{\pm} + 2u_{\mp}v_{\mp} + m_{\pm}(u_{\pm}v_{\pm} + v_{\mp}u_{\mp}) + \lambda R_{\mp}R_{\pm}, \]

while the \((x,t)\)-dependent Lie-Backlund transformations are given by

\[ Z_0^+ = x(Y_t^+ - Y_{\pm}^1) + t(Y_t^+ + Y_{\pm}^1) + \left( -u_{\pm} \frac{\partial}{\partial u_{\pm}} - v_{\mp} \frac{\partial}{\partial v_{\mp}} + u_{\mp} \frac{\partial}{\partial u_{\mp}} + v_{\pm} \frac{\partial}{\partial v_{\pm}} \right), \]

\[ Z_1^+ = x(Y_{\pm}^+ - 4u_{\pm}Y_{\pm}^0) + t(Y_{\pm}^+ + 4u_{\pm}Y_{\pm}^0) + \left( -u_{\pm} \frac{\partial}{\partial u_{\pm}} - v_{\mp} \frac{\partial}{\partial v_{\mp}} + u_{\mp} \frac{\partial}{\partial u_{\mp}} + v_{\pm} \frac{\partial}{\partial v_{\pm}} \right), \]

\[ Z_2^+ = x(-Y_{\pm}^+ - 4u_{\pm}Y_{\pm}^0) + t(-Y_{\pm}^+ + 4u_{\pm}Y_{\pm}^0) + \left( -u_{\pm} \frac{\partial}{\partial u_{\pm}} - v_{\mp} \frac{\partial}{\partial v_{\mp}} + u_{\mp} \frac{\partial}{\partial u_{\mp}} + v_{\pm} \frac{\partial}{\partial v_{\pm}} \right), \]

\[ Z_3^+ = x(Y_{\pm}^+ - 4u_{\pm}Y_{\pm}^0) + t(-Y_{\pm}^+ + 4u_{\pm}Y_{\pm}^0) + \left( -u_{\pm} \frac{\partial}{\partial u_{\pm}} - v_{\mp} \frac{\partial}{\partial v_{\mp}} + u_{\mp} \frac{\partial}{\partial u_{\mp}} + v_{\pm} \frac{\partial}{\partial v_{\pm}} \right). \]

Similar results have been obtained for the “−” part.

**APPENDIX B: THE INFINITY OF THE HIERARCHIES**

We shall prove a lemma from which the existence of infinite hierarchies of Hamiltonians

\[ F(Y_{\pm}^+), F(Y_{\pm}^0), F(Y_{\pm}^1), \ldots, \]

\[ F(Z_{0+}), F(Z_{1+}), F(Z_{2+}), \ldots, \]

and their associated Lie–Bäcklund transformations

\[ Y_{0+}, Y_{1+}, Y_{2+}, \ldots, \quad Z_{0+}, Z_{1+}, Z_{2+}, \ldots, \]

immediately follow. In this lemma the lower indices of \( u, v \) refer to partial derivatives with respect to \( x \) (i.e., \( U_{\pm} = u_x, \quad U_{2+} = u_{xx} \),...).

**Lemma:** Let \( H_n(u,v), K_n(u,v), \bar{H}_n(u,v), \) and \( \bar{K}_n(u,v) \) be defined by

\[ H_n(u,v) = \int_{-\infty}^{\infty} \left( u_x^2 + v_x^2 \right), \]

\[ K_n(u,v) = \int_{-\infty}^{\infty} \left( u_{n+1}v_n - u_{n+1}u_n \right), \]

\[ \bar{H}_n(u,v) = \int_{-\infty}^{\infty} x(u_x^2 + v_x^2), \]

\[ \bar{K}(u,v) = \int_{-\infty}^{\infty} x(u_{n+1}v_n - u_{n+1}u_n), \]

and define the Poisson bracket of \( F \) and \( L \) \( \{F,L\} \) by

\[ \{F,L\} = \int_{-\infty}^{\infty} \left( \frac{\delta F}{\delta u} \frac{\delta L}{\delta v} - \frac{\delta F}{\delta v} \frac{\delta L}{\delta u} \right), \]

then the following results hold

\[ \{\bar{H}_1,H_1\} = + 4nK_n, \]

\[ \{\bar{H}_1,K_n\} = + 2(2n+1)H_{n+1}, \]

\[ \{\bar{H}_1,\bar{H}_n\} = + 4(n-1)\bar{K}_n, \]

where

\[ \{\bar{H}_1,H_n\} = + 2(2n-1)\bar{H}_{n+1} + 1, \]

\[ \{\bar{H}_1,K_n\} = + 2(2n+1)H_{n+1}, \]

\[ \{\bar{H}_1,\bar{H}_n\} = + 4(n-1)\bar{K}_n, \]

which proves relation (B5a). Substitution of (B6b) into (2.4) yields

\[ \{\bar{H}_1,\bar{H}_n\} = + 4(n-1)\bar{K}_n, \]

and this existence of infinite hierarchies \( H(Y_{\pm}^+) \) now follows from the explicit structure of \( H(Z_{\pm}^+ \) [Eq. (2.12)] and \( H(Y_{\pm}^+) \) \[ Eq. (2.6) \] by consid-
ering the $(\lambda,m_1,m_2)$-independent parts and application of part a and b of this Lemma. The existence of the infinite hierarchies $H(Z^{n})$ follows from a similar argument using $H_{m}(u,v), K_{n}(u,v)$.

5H. M. Ten Eikelder, “Symmetries for dynamical and Hamiltonian syst-

2145 J. Math. Phys., Vol. 27, No. 8, August 1986


2145

8P. H. M. Kersten, Ph.D. thesis, Twente University of Technology, Ens-

formulations of Bäcklund transformations,” in Mathematical Physics Stud-