

# A short and elementary proof of the main Bahadur-Kiefer theorem

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**A Short and Elementary proof  
of the main Bahadur-Kiefer Theorem**

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The Netherlands

**A SHORT AND ELEMENTARY PROOF  
OF THE MAIN BAHADUR-KIEFER THEOREM**

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A short proof of the lower bound in the strong version of the famous Theorem 1A in Kiefer (1970) on the Bahadur-Kiefer process is presented. The proof is elementary and does in particular not use strong approximations.

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Running head: Proof of Bahadur-Kiefer theorem.

Let  $U_1, U_2, \dots$  be a sequence of independent uniform-(0, 1) random variables and for each  $n \in \mathbb{N}$ , let

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n 1_{[0,t]}(U_i), \quad 0 \leq t \leq 1,$$

be the empirical distribution function at stage  $n$ . The uniform empirical process will be written as

$$\alpha_n(t) = n^{\frac{1}{2}}(F_n(t) - t), \quad 0 \leq t \leq 1; \quad \alpha_n(t) = 0 \text{ for } t < 0 \text{ or } t > 1.$$

Also for each  $n \in \mathbb{N}$ ,

$$Q_n(t) = \inf\{s : F_n(s) \geq t\}, \quad 0 < t \leq 1; \quad Q_n(0) = 0,$$

denotes the empirical quantile function and we write

$$\beta_n(t) = n^{\frac{1}{2}}(Q_n(t) - t), \quad 0 \leq t \leq 1,$$

for the corresponding uniform quantile process. The so-called Bahadur-Kiefer process is defined by

$$R_n(t) = \alpha_n(t) + \beta_n(t), \quad 0 \leq t \leq 1.$$

This process is introduced in Bahadur (1966); in Kiefer (1970, Theorem 1A) the “in-probability-analogue” of the following statement is proved

$$(1) \quad \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{4}}}{(\log n)^{\frac{1}{2}}} \frac{\|R_n\|}{\|\alpha_n\|^{\frac{1}{2}}} = 1 \quad \text{a.s.},$$

where  $\|f\| = \sup_{0 \leq t \leq 1} |f(t)|$  for any real-valued function  $f$  on  $[0, 1]$ . In the latter paper a proof of (1) itself is claimed but not presented. However, it is proved in Shorack (1982) that, indeed, the expression on the left in (1) (with ‘lim’ replaced by ‘limsup’) is not larger than 1, almost surely, (note that  $\|\alpha_n\| = \|\beta_n\|$ ) whereas in a recent paper by Deheuvels and Mason (1990) it is established that the same expression is not smaller than 1, almost surely. The short and elegant proof in Shorack (1982) is based on the Kiefer process strong approximation of  $\alpha_n$ , but in Shorack and Wellner (1986, pp. 590-591) a similar, direct proof of the “upper-bound-part” is given. The ingenious proof of the “lower-bound-part” (which finally led to a complete proof of (1)) in Deheuvels and Mason (1990) is extremely technical, moreover it is again based on the Kiefer process strong approximation of  $\alpha_n$ .

It is the purpose of this note to give a new, short proof of the “lower-bound-part” of (1), i.e. we will prove that

$$(2) \quad \liminf_{n \rightarrow \infty} \frac{n^{\frac{1}{4}}}{(\log n)^{\frac{1}{2}}} \frac{\|R_n\|}{\|\alpha_n\|^{\frac{1}{2}}} \geq 1 \quad \text{a.s.}$$

Our proof is rather easy and not based on strong approximations. It uses as tools the following well-known facts on empirical and quantile processes, although most of them are not required at their full strength.

FACT 1 (Mogul'skii (1979)).

$$(3) \quad \liminf_{n \rightarrow \infty} (\log \log n)^{\frac{1}{2}} \|\alpha_n\| = \pi/8^{\frac{1}{2}} \quad \text{a.s.}$$

FACT 2 (easy).

$$(4) \quad \|\beta_n + \alpha_n \circ Q_n\| = n^{-\frac{1}{2}} \quad \text{a.s.}$$

FACT 3 (Kiefer (1970)).

$$(5) \quad \limsup_{n \rightarrow \infty} n^{\frac{1}{4}} (\log n)^{-\frac{1}{2}} (\log \log n)^{-\frac{1}{4}} \|R_n\| = 2^{-\frac{1}{4}} \quad \text{a.s.}$$

Define the oscillation modulus of  $\alpha_n$  by

$$\omega_n(a) = \sup_{\substack{|t-s| \leq a \\ 0 \leq s \leq t \leq 1}} |\alpha_n(t) - \alpha_n(s)|, \quad 0 < a \leq 1;$$

let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of positive numbers with  $a_n \downarrow 0$  and  $n a_n \uparrow$ .

FACT 4 (Mason, Shorack and Wellner (1983)). If  $\log(1/a_n)/\log \log n \rightarrow c \in [0, \infty)$ , then

$$(6) \quad \limsup_{n \rightarrow \infty} \frac{\omega_n(a_n)}{(a_n \log \log n)^{\frac{1}{2}}} = (2(1+c))^{\frac{1}{2}} \quad \text{a.s.}$$

FACT 5 (Stute (1982)). If  $\log(1/a_n)/\log \log n \rightarrow \infty$  and  $n a_n/\log n \rightarrow \infty$ , then

$$(7) \quad \lim_{n \rightarrow \infty} \frac{\omega_n(a_n)}{(a_n \log(1/a_n))^{\frac{1}{2}}} = 2^{\frac{1}{2}} \quad \text{a.s.}$$

FACT 6 (Mallows (1968)). If  $(N_1, \dots, N_k), k \in \mathbb{N}$ , has a multinomial distribution with parameters  $m$  and  $p_1, \dots, p_k$ , where  $m \in \mathbb{N}$  and  $p_1, \dots, p_k$  are non-negative with  $\sum_{i=1}^k p_i = 1$ , then for all  $\lambda_1, \dots, \lambda_k$

$$P(N_1 \leq \lambda_1, \dots, N_k \leq \lambda_k) \leq \prod_{i=1}^k P(N_i \leq \lambda_i).$$

FACT 7 (Kolmogorov (1929)). Let  $m \in \mathbb{N}$  and  $t \in (0, \frac{1}{2})$ . Then for every  $\delta > 0$  there exist  $K_1, K_2 \in (0, \infty)$  such that for  $K_1 t^{\frac{1}{2}} \leq \lambda \leq K_2 m^{\frac{1}{2}} t$

$$P(\alpha_m(t) > \lambda) \geq \exp(-(1 + \delta)\lambda^2/(2t(1 - t))).$$

FACT 8 (Dvoretzky, Kiefer and Wolfowitz (1956), Massart (1990)). Let  $n \in \mathbb{N}$ . Then for all  $\lambda \geq 0$

$$(8) \quad P(\|\alpha_n\| \geq \lambda) \leq 2 \exp(-2\lambda^2).$$

PROOF OF (2). Let  $I$  denote the identity function on  $[0, 1]$ . From (4), (3) and  $Q_n = I + \beta_n/n^{\frac{1}{2}}$  we see that it suffices to show that

$$(9) \quad \liminf_{n \rightarrow \infty} \frac{n^{\frac{1}{4}}}{(\log n)^{\frac{1}{2}}} \frac{\|\alpha_n(I + \beta_n/n^{\frac{1}{2}}) - \alpha_n\|}{\|\alpha_n\|^{\frac{1}{2}}} \geq 1 \quad \text{a.s.}$$

Using (5), (7) and (3), (9) can in turn be replaced by

$$(10) \quad \liminf_{n \rightarrow \infty} \frac{n^{\frac{1}{4}}}{(\log n)^{\frac{1}{2}}} \frac{\|\alpha_n(I - \alpha_n/n^{\frac{1}{2}}) - \alpha_n\|}{\|\alpha_n\|^{\frac{1}{2}}} \geq 1 \quad \text{a.s.}$$

Set, for  $0 \leq t \leq 1$ ,

$$\bar{\alpha}_n(t) = \begin{cases} \alpha_n(t), & \text{if } |\alpha_n(t)| > 1/\log n, \\ 1/\log n, & \text{if } |\alpha_n(t)| \leq 1/\log n. \end{cases}$$

Define the following grid on  $[0, 1]$ :  $t_{i,n} = i/[\log n]$ ,  $i = 0, 1, \dots, [\log n]$ , where  $[x]$  denotes the integer part of  $x \in \mathbb{R}$ . Now using (6) and (7), and again (3), it follows that instead of proving (10), it suffices to prove that

$$(11) \quad \liminf_{n \rightarrow \infty} \frac{n^{\frac{1}{4}}}{(\log n)^{\frac{1}{2}}} \frac{\max_{0 \leq i \leq [\log n] - 1} \sup_{t_{i,n} \leq t \leq t_{i+1,n}} |\alpha_n(t - \bar{\alpha}_n(t_{i,n})/n^{\frac{1}{2}}) - \alpha_n(t)|}{\max_{0 \leq i \leq [\log n]} |\bar{\alpha}_n(t_{i,n})|^{\frac{1}{2}}} \geq 1 \quad \text{a.s.}$$

Using the Borel-Cantelli lemma a proof of (11) is established if we show that for all  $\varepsilon \in (0, 1)$ ,  $\sum_{n=3}^{\infty} P A_n < \infty$ , where

$$A_n = \left\{ n^{\frac{1}{4}} \max_{0 \leq i \leq [\log n] - 1} \sup_{t_{i,n} \leq t \leq t_{i+1,n}} |\alpha_n(t - \bar{\alpha}_n(t_{i,n})/n^{\frac{1}{2}}) - \alpha_n(t)| \right.$$

$$\leq \left( (1 - \varepsilon) \max_{0 \leq i \leq [\log n]} |\bar{\alpha}_n(t_{i,n})| \log n \right)^{\frac{1}{2}} \Big\}.$$

Write  $C_n = C_n(c_{1,n}, c_{2,n}, \dots, c_{[\log n]-1,n}) = \{\alpha_n(t_{i,n}) = c_{i,n}, 1 \leq i \leq [\log n] - 1\}$ ,  $c_{i,n} \in [-\log n, \log n]$  and  $c_{i,n}$  such that  $n t_{i,n} + n^{\frac{1}{2}} c_{i,n} \in \{0, 1, \dots, n\}$  and such that  $n t_{i,n} + n^{\frac{1}{2}} c_{i,n}$  is non-decreasing in  $i$ . Observe that  $PC_n > 0$ . Set  $\bar{c}_n = \left( \max_{1 \leq i \leq [\log n]-1} |c_{i,n}| \right) \vee (1/\log n)$  and, on  $C_n$ , let  $t_n$  be the smallest  $t_{i,n}, 0 \leq i \leq [\log n]$ , such that  $|\bar{\alpha}_n(t_{i,n})| = \bar{c}_n$ , write  $d_n = \alpha_n(t_n)$  and  $\bar{d}_n = \bar{\alpha}_n(t_n)$ ; set  $t'_n = t_n + 1/[\log n]$  and  $d'_n = \alpha_n(t'_n)$ . Now we have

$$(12) \quad P(A_n | C_n) \leq P\left(n^{\frac{1}{4}} \sup_{t_n + \frac{\varepsilon_n}{n^{\frac{1}{2}}} \leq t \leq t'_n - \frac{\varepsilon_n}{n^{\frac{1}{2}}}} |\alpha_n(t - \bar{d}_n/n^{\frac{1}{2}}) - \alpha_n(t)| \leq ((1 - \varepsilon)\bar{c}_n \log n)^{\frac{1}{2}} | C_n\right).$$

Write  $m_n = n/[\log n] + n^{\frac{1}{2}}(d'_n - d_n)$ . Note that, on  $C_n$ ,  $m_n$  is the number of observations falling in the interval  $(t_n, t'_n]$ . Now it is not hard to see that on  $C_n$ , the process  $\tilde{\alpha}_{m_n}$  defined by

$$\tilde{\alpha}_{m_n}(s) = (n/m_n)^{\frac{1}{2}} \{ \alpha_n(t_n + s/[\log n]) - (d_n(1-s) + d'_n s) \}, \quad 0 \leq s \leq 1,$$

is a uniform empirical process based on  $m_n$  observations. Hence the right hand side of (12) can be written as

$$(13) \quad P\left(n^{\frac{1}{4}} \sup_{\frac{\bar{c}_n[\log n]}{n^{\frac{1}{2}}} \leq s \leq 1 - \frac{\bar{c}_n[\log n]}{n^{\frac{1}{2}}}} \left| \left(\frac{m_n}{n}\right)^{\frac{1}{2}} \{ \tilde{\alpha}_{m_n}\left(s - \frac{\bar{d}_n[\log n]}{n^{\frac{1}{2}}}\right) - \tilde{\alpha}_{m_n}(s) \} \right. \right. \\ \left. \left. + \bar{d}_n[\log n](d_n - d'_n)n^{-\frac{1}{2}} \right| \leq ((1 - \varepsilon)\bar{c}_n \log n)^{\frac{1}{2}}\right).$$

Now observe that

$$\begin{aligned} |n^{\frac{1}{4}} \bar{d}_n[\log n](d_n - d'_n)n^{-\frac{1}{2}}| / (\bar{c}_n \log n)^{\frac{1}{2}} &\leq 2\bar{c}_n^{\frac{1}{2}} (\log n)^{\frac{1}{2}} n^{-\frac{1}{4}} \\ &\leq 2(\log n)^2 n^{-\frac{1}{4}} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, for large  $n$ , the expression in (13) is bounded from above by

$$(14) \quad P\left(n^{\frac{1}{4}} \left(\frac{m_n}{n}\right)^{\frac{1}{2}} \sup_{\frac{\bar{c}_n[\log n]}{n^{\frac{1}{2}}} \leq s \leq 1 - \frac{\bar{c}_n[\log n]}{n^{\frac{1}{2}}}} \left| \tilde{\alpha}_{m_n}\left(s - \frac{\bar{d}_n[\log n]}{n^{\frac{1}{2}}}\right) - \tilde{\alpha}_{m_n}(s) \right| \right. \\ \left. \leq ((1 - \frac{1}{2}\varepsilon)\bar{c}_n \log n)^{\frac{1}{2}}\right),$$

which by Fact 6 is less than or equal to

$$(15) \quad \{P(n^{\frac{1}{4}}(\frac{m_n}{n})^{\frac{1}{2}}\tilde{c}_{m_n}\left(\frac{\bar{c}_n\lfloor\log n\rfloor}{n^{\frac{1}{2}}}\right)\} \leq ((1 - \frac{1}{2}\varepsilon)\bar{c}_n \log n)^{\frac{1}{2}}\}^{n^{\frac{1}{2}}/(2(\log n)^2)}.$$

It is easy to check that, for large  $n$ , Fact 7 applies to the probability in (15). This yields, with  $\delta = \varepsilon/4$ , the following upper bound for the expression in (15)

$$(16) \quad (1 - n^{-\frac{1}{2}(1-\varepsilon/4)})^{n^{\frac{1}{2}}/(2(\log n)^2)} \leq \exp(-n^{\varepsilon/8}/(2(\log n)^2)) \leq 1/n^2.$$

Now we are ready to complete the proof. Combining (12)-(16) we have  $P(A_n|C_n) \leq 1/n^2$  ( $n$  large). Set  $D_n = \{\|\alpha_n\| > \log n\}$  and note that (8) implies that  $PD_n \leq 1/n^2$  ( $n \geq 4$ ). Hence for large  $n$

$$(17) \quad PA_n \leq P(A_n \cap D_n^c) + PD_n \leq (\sup^* P(A_n|C_n)) + PD_n \leq 1/n^2 + 1/n^2 = 2/n^2,$$

where  $\sup^*$  denotes the supremum over all  $C_n$  as defined before. Now, of course,  $\sum_{n=3}^{\infty} PA_n < \infty$  because of (17). This proves (11) and hence (2).  $\square$

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