

Estimation of convolution tail behaviour

Citation for published version (APA):

Willekens, E. K. E., & Veraverbeke, N. (1989). *Estimation of convolution tail behaviour*. (Memorandum COSOR; Vol. 8904). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/1989

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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- The final published version features the final layout of the paper including the volume, issue and page numbers.

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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

Memorandum COSOR 89-04

ESTIMATION OF CONVOLUTION
TAIL BEHAVIOUR

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Eindhoven, February 1989
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ESTIMATION OF CONVOLUTION TAIL BEHAVIOUR

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ABSTRACT

Several classes of distribution functions are originated by considering distributions whose tailfunctions satisfy special asymptotic relations. A large class sharing this property is provided by the subexponential class S , in which case the asymptotic relation involves tails of convolution powers. In this paper we introduce a statistic which estimates the asymptotic behaviour of convolution tails of a given distribution function and we show that this statistic is asymptotically normal under appropriate conditions. An important tool and result of independent interest is an asymptotic representation in probability for intermediate order statistics.

⁽¹⁾ Research supported by Air Force Office of Scientific Research Grant No F49620-85-C-0144 and by the Belgian National Fund for Scientific Research.

AMS 1980 subject classification. Primary 62E20; secondary 62G05. Key words and phrases. Subexponential distributions, \bar{U} -statistics, intermediate order statistics, asymptotic normality.

1. INTRODUCTION

Throughout we will use distribution functions F with $F(0-) = 0$ and $F(x) < 1$ for all $x \in \mathbb{R}$. We denote the n -fold convolution of F with itself by F^{*n} and use the notation $\bar{F} = 1 - F$ for the tail of F , $\overline{F^{*n}} = 1 - F^{*n}$, etc.

Subexponential distribution functions were introduced independently by Chistyakov (1964) and Chover et al (1973a, 1973b) in the context of branching processes. The subexponential class S is the set of distribution functions F for which

$$(1.1) \quad \lim_{x \rightarrow \infty} \frac{\overline{F^{*2}}(x)}{\bar{F}(x)} = 2$$

or equivalently (see Chistyakov (1964)) for all integers $m \geq 2$,

$$(1.2) \quad \lim_{x \rightarrow \infty} \frac{\overline{F^{*m}}(x)}{\bar{F}(x)} = m .$$

For a probabilistic interpretation of (1.2), let X_1, \dots, X_m be independent random variables with distribution function F . Then (1.2) says that $P(X_1 + \dots + X_m > x) \sim P(\max(X_1, \dots, X_m) > x)$, $x \rightarrow \infty$. This means that the subexponential class S is characterized by the property that the sum and the maximum of a sample are tail equivalent for any sample size.

Chover et al. (1973a) also introduced the class SD of subexponential densities, i.e. probability densities $f > 0$ such that

$$(1.3) \quad \lim_{x \rightarrow \infty} \frac{f(x-y)}{f(x)} = 1, \text{ for every } y \in \mathbb{R} ,$$

and

$$(1.4) \quad \lim_{x \rightarrow \infty} \frac{f^{*2}(x)}{f(x)} = 2 .$$

Here \times denotes density convolution, defined by $(f \times g)(x) = \int_0^x f(x-y) g(y) dy$. It is clear from de l'Hôpital's theorem that if a distribution function F has $f = F' \in SD$, then $F \in S$.

The classes S and SD have been studied extensively by a number of authors; we refer to Teugels (1975), Pitman (1980), Embrechts and Goldie (1980, 1982), Cline (1987), Omeij and Willekens (1986, 1987), Willekens (1988). The property that subexponential distribution functions characterize a certain tail behaviour of compound distributions, has given interesting applications in various domains of stochastic processes, see e.g. Embrechts et al. (1979), Grübel (1984), Willekens (1986) and references therein.

Because of the convolution power in the defining property of S , it is often very hard to check whether a given distribution function F satisfies (1.1) or (1.2). Sufficient conditions for

membership of S only in terms of the tail of F are known, cf. Goldie (1978), Cline (1987), but require an analytical expression for \bar{F} . It is well known that the class S contains every distribution function with regularly varying tail, i.e. $\bar{F}(x) \sim x^{-\alpha}L(x)$, where $\alpha > 0$ and L is slowly varying, meaning that for every $t > 0$: $\lim_{x \rightarrow \infty} L(tx)/L(x) = 1$.

If a distribution function F is only known through a finite number of observations, it is impossible with the present theory to decide whether this distribution function is subexponential or not. Such situations may arise in some applied stochastic models such as queueing and risk theory, see Hogg and Klugman (1984).

In this paper we develop a statistical approach to subexponentiality. In Section 2 we therefore define a statistic, based on a sample X_1, \dots, X_n , which gives valuable information to decide whether the underlying distribution function is subexponential or not. This statistic contains in its definition a so called intermediate order statistic of the sample for which we derive an asymptotic representation in probability in Section 3. This result, which is of independent interest is an important tool in Section 4, where we establish asymptotic normality for our basic estimator. Finally, Section 5 contains some related further questions and extensions.

2. DEFINITION OF THE STATISTIC

Let X_1, \dots, X_n be a sequence of independent random variables with common distribution function F and denote by $X_{(1)} \leq \dots \leq X_{(n)}$ the order statistics of this sample. The following statistic which is the sample version of the ratio (1.2) seems to be a plausible choice to describe subexponentiality: for $m \leq n$,

$$H_n(x) = \frac{\overline{F_n^{*m}}(x)}{\bar{F}_n(x)} I(X_{(n)} > x) .$$

Here F_n denotes the empirical distribution function based on X_1, \dots, X_n and the indicator function of the event $\{X_{(n)} > x\}$ ensures that $H_n(x)$ is well defined.

Since $\overline{F_n^{*m}}(x)$ and $\bar{F}_n(x)$ are V -statistics (see Serfling (1980)) and since $I(X_{(n)} > x) \rightarrow 1$ almost surely for fixed x , we get that $\lim_{n \rightarrow \infty} H_n(x) = \overline{F^{*m}}(x)/\bar{F}(x)$ almost surely, for fixed x . In order that $H_n(x)$ can give a meaningful description of subexponential behaviour of F , we have to let x tend to infinity, which gives that $F \in S$ if and only if $\lim_{x \rightarrow \infty} \lim_{n \rightarrow \infty} H_n(x) = m$. This relation shows that H_n contains information about S , but, is in a sense useless because of the two limits. One way to solve this problem is to substitute for x a deterministic sequence $\{x_n\}$ and letting n tend to infinity. In this case however the resulting statistic depends on a parameter which has to be chosen artificially and it turns out that this parameter rather heavily depends on the distribution function F , which is generally unknown (see Willekens (1986)).

We therefore propose to replace the deterministic sequence by a random sequence by taking for each n one of the observations which almost surely tends to infinity as $n \rightarrow \infty$. If we choose for each n the intermediate order statistic $X_{(n-k_n)}$, with $\{k_n\}$ a sequence of integers ($1 \leq k_n < n$) such that $k_n \rightarrow \infty$ and $k_n = o(n)$, $n \rightarrow \infty$, then $H_n(X_{(n-k_n)})$ reduces to

$$H_n(X_{(n-k_n)}) = \frac{1}{k_n n^{m-1}} \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n I(X_{i_1} + \cdots + X_{i_m} > X_{(n-k_n)}) .$$

We now slightly modify $H_n(X_{(n-k_n)})$ by removing the sum over all m -tuples which contain at least two equal integers. This will not affect the asymptotic behaviour because their contribution to the whole sum is of smaller magnitude than the sum of the remaining terms.

Finally, changing the normalizing factor a little, we end up with the statistic which will be discussed in this paper:

$$(2.1) \quad T_n = \frac{n}{k_n} \binom{n}{m}^{-1} \sum_{C_m} I(X_{i_1} + \cdots + X_{i_m} > X_{(n-k_n)})$$

where $C_m = \{(i_1, \dots, i_m) \mid 1 \leq i_1 < \dots < i_m \leq n\}$. We have that

$$T_n = \frac{n}{k_n} U_n(X_{(n-k_n)})$$

where, for each fixed x , $U_n(x)$ is a U -statistic defined by

$$(2.2) \quad U_n(x) = \binom{n}{m}^{-1} \sum_{C_m} I(X_{i_1} + \cdots + X_{i_m} > x) .$$

The well known asymptotic theory for U -statistics (see Serfling (1980)) is not applicable to $U_n(X_{(n-k_n)})$ because of the presence of $X_{(n-k_n)}$ in the kernel. In fact, $U_n(X_{(n-k_n)})$ is a U -statistic with an estimated parameter as studied by Randles (1982). However, his general result on asymptotic normality does not hold here because it would require $X_{(n-k_n)}$ tending to some constant. In our case $X_{(n-k_n)}$ is, under appropriate conditions, a consistent estimator for x_n , where x_n is the intermediate population quantile defined by the equation

$$\bar{F}(x_n) = \frac{k_n}{n} .$$

Since $k_n = o(n)$, we have that $x_n \rightarrow \infty$ as $n \rightarrow \infty$. In Section 4 we adapt and modify Randles' method to make it work in our case. The basic tool in establishing this is an asymptotic representation in probability for intermediate order statistics. This result, which is of independent interest, is stated and proved in the next section.

3. IN PROBABILITY REPRESENTATION FOR INTERMEDIATE ORDER STATISTICS

The theorem in this section represents the intermediate order statistic $X_{(n-k_n)}$ (with $k_n \rightarrow \infty$ and $k_n = o(n)$, $n \rightarrow \infty$) as a sum of independent random variables plus a remainder term. The almost sure behaviour of the remainder term has been studied by Watts (1980) under much stronger regularity conditions on F . For our purpose however, the in probability approximation suffices and therefore it is important to obtain this under the weakest possible conditions on the underlying distribution function.

THEOREM 1

Let F be a distribution function such that on some interval $(c, +\infty)$, $F'(x) = f(x)$ exists and $f(x) > 0$. Let $\{k_n\}$ be a sequence of integers ($1 \leq k_n < n$) with $k_n \rightarrow \infty$ and $k_n = o(n)$ as $n \rightarrow \infty$.

Let $\bar{F}(x_n) = \frac{k_n}{n}$. Then, for n sufficiently large, we have

$$X_{(n-k_n)} = x_n + \frac{\sum_{i=1}^n I(X_i > x_n) - k_n}{nf(x_n)} + R_n$$

where

$$R_n = o_p\left(\frac{\sqrt{k_n}}{nf(x_n)}\right), n \rightarrow \infty.$$

Proof. Putting $V_n = \frac{nf(x_n)}{\sqrt{k_n}} (X_{(n-k_n)} - x_n)$ and $W_n = \frac{\sum_{i=1}^n I(X_i > x_n) - k_n}{\sqrt{k_n}}$ the proof reduces to showing that $V_n - W_n \rightarrow 0$. For this we will check the conditions of Lemma 1 in Ghosh (1971).

First we have that $W_n = O_p(1)$, since for each $K > 0$, $P(|W_n| > K) \leq K^{-2}$. As to the second condition, first note that for all t ,

$$\{V_n \leq t\} = \{X_{(n-k_n)} \leq t_n^*\} = \{F(x_n) \leq F_n(t_n^*)\} = \{Z_{in} \leq t_n\}$$

where $t_n^* = x_n + t \frac{\sqrt{k_n}}{nf(x_n)}$, $Z_{in} = \frac{n}{\sqrt{k_n}} (F(t_n^*) - F_n(t_n^*))$ and $t_n = \frac{n}{\sqrt{k_n}} (F(t_n^*) - F(x_n))$.

Now using a one term Taylor expansion we obtain that

$$t_n = \frac{n}{\sqrt{k_n}} [F(x_n) + (t_n^* - x_n) f(x_n) + o(|t_n^* - x_n| f(x_n)) - F(x_n)] = t + o(1) \text{ as } n \rightarrow \infty.$$

Since $t_n \rightarrow t$, we have for all t and all $\varepsilon > 0$, $P(V_n \leq t, W_n \geq t + \varepsilon) = P(Z_{in} \leq t_n, W_n > t + \varepsilon) \leq P(|W_n - Z_{in}| \geq \varepsilon/2)$, for n large. To prove that $W_n - Z_{in} \rightarrow 0$, we show that $E[(W_n - Z_{in})^2] \rightarrow 0$.

Since $W_n - Z_{in} = \frac{n}{\sqrt{k_n}} (F_n(t_n^*) - F(t_n^*) - F_n(x_n) + F(x_n))$ we have

$$\begin{aligned} E[(W_n - Z_{in})^2] &= \frac{n^2}{k_n} [\text{Var } F_n(t_n^*) + \text{Var } F_n(x_n) - 2 \text{Cov}(F_n(t_n^*), F_n(x_n))] \\ &= \frac{n}{k_n} [F(t_n^*) - F(x_n)] [\pm 1 - F(t_n^*) + F(x_n)] \end{aligned}$$

with the upper sign if $t > 0$ and the lower if $t \leq 0$. Now, by one term Taylor expansion we see that $F(t_n^*) = F(x_n) + t \frac{\sqrt{k_n}}{n} + o(\frac{\sqrt{k_n}}{n})$, $n \rightarrow \infty$. Hence, $E[(W_n - Z_{in})^2] = O(\frac{1}{\sqrt{k_n}}) \rightarrow 0$, $n \rightarrow \infty$.

An immediate consequence of the representation in the foregoing theorem is the asymptotic normality for intermediate order statistics.

COROLLARY 1

Under the conditions of Theorem 1,

$$\frac{nf(x_n)}{\sqrt{k_n}} (X_{(n-k_n)} - x_n) \xrightarrow{d} N(0; 1), \quad n \rightarrow \infty.$$

Proof. From Theorem 1, the limiting distribution of $\frac{nf(x_n)}{\sqrt{k_n}} (X_{(n-k_n)} - x_n)$ coincides with that of $\sum_{i=1}^n Y_{in}$, where

$$Y_{in} = \frac{I(X_i > x_n) - k_n/n}{\sqrt{k_n}}.$$

Since $E(Y_{in}) = 0$, $E(Y_{in}^2) = \frac{1}{n}(1 - \frac{k_n}{n})$, $E[|Y_{in}|^3] \leq \frac{1}{n\sqrt{k_n}}$, the stated result follows from Liapunov's central limit theorem for triangular arrays.

4. ASYMPTOTIC NORMALITY OF THE PROPOSED STATISTIC

We now turn back to the statistic T_n in (2.1). For proving its asymptotic normality we will have to assume that F is such that $F' = f$ is in a class SD' which is slightly more restrictive than the class SD of subexponential densities. The class SD' consists of probability densities $f > 0$ such that (1.4) holds, together with the following smoothness condition (which replaces (1.3)): there are constants $M > 0$, $0 < p \leq 1$, and a slowly varying function L , such that

$$(4.1) \quad \frac{\bar{F}(x)}{x^p L(x) f(x)} < M, \text{ for large } x,$$

and

$$(4.2) \quad \lim_{x \rightarrow \infty} \frac{f(x + o(x^p L(x)))}{f(x)} = 1.$$

Some archetypes of distribution functions with density in SD' are:

- (i) $\bar{F}(x) = \exp(-x^\alpha)$, $x > 0$, $0 < \alpha < 1$ (take $p = 1 - \alpha$ and $L(x) = 1$)
- (ii) $\bar{F}(x) = x^{-\alpha}$, $x > 1$, $\alpha > 0$ (take $p = 1$ and $L(x) = 1$)
- (iii) lognormal (take $p = 1$ and $L(x) = 1/\log x$).

THEOREM 2

Let F be a distribution function such that $F' = f \in SD'$. Let $\{k_n\}$ be a sequence of integers ($1 \leq k_n < n$) with $k_n \rightarrow \infty$ and $k_n = o(n)$ as $n \rightarrow \infty$. Let $\bar{F}(x_n) = \frac{k_n}{n}$. Then

$$\sqrt{k_n} \left(T_n - \frac{\bar{F}^{*m}(x_n)}{\bar{F}(x_n)} \right) \xrightarrow{d} N(0; 4m^2), \quad n \rightarrow \infty.$$

Proof. We have $\sqrt{k_n} \left(T_n - \frac{\bar{F}^{*m}(x_n)}{\bar{F}(x_n)} \right) = \frac{n}{\sqrt{k_n}} (U_n(X_{(n-k_n)}) - \bar{F}^{*m}(x_n))$ and following Randles (1982) we write

$$U_n(X_{(n-k_n)}) - \bar{F}^{*m}(x_n) = A_{1n} + A_{2n}$$

where A_{1n} and A_{2n} are given by

$$(4.3) \quad A_{1n} = U_n(x_n) - \bar{F}^{*m}(x_n) + \bar{F}^{*m}(X_{(n-k_n)}) - \bar{F}^{*m}(x_n)$$

$$(4.4) \quad A_{2n} = U_n(X_{(n-k_n)}) - \bar{F}^{*m}(X_{(n-k_n)}) - U_n(x_n) + \bar{F}^{*m}(x_n).$$

Each of the terms A_{1n} and A_{2n} will be considered in a separate lemma and the proof of Theorem 2 follows from a combination of the two.

LEMMA 1

Under the conditions of Theorem 2,

$$\frac{n}{\sqrt{k_n}} A_{1n} \xrightarrow{d} N(0; 4m^2), \quad n \rightarrow \infty.$$

Proof. $U_n(x)$ as given by (2.2) is a U -statistic with kernel (depending on a parameter x) defined by

$$(4.5) \quad h(x_1, \dots, x_m; x) = I(x_1 + \dots + x_m > x)$$

and with mean

$$(4.6) \quad \theta(x) = E[h(X_1, \dots, X_m; x)] = \overline{F^{*m}}(x) .$$

Let $\tilde{U}_n(x)$ be the projection of the U -statistic $U_n(x)$ (see Serfling (1980), p. 187)

$$\tilde{U}_n(x) = \frac{m}{n} \sum_{i=1}^n g(X_i; x)$$

where

$$g(x_1; x) = \int \dots \int h(x_1, \dots, x_m; x) dF(x_2) \dots dF(x_m) - \theta(x) = \overline{F^{*(m-1)}}(x - x_1) - \theta(x) .$$

With the method of Serfling (1980, p. 182), one easily calculates that for each $n \geq m$

$$\begin{aligned} & E[(U_n(x) - \theta(x) - \tilde{U}_n(x))^2] \\ &= \binom{n}{m}^{-1} \{ \overline{F^{*m}}(x) + (m-1) (\overline{F^{*m}}(x))^2 - m E[(\overline{F^{*(m-1)}}(x - X_2))^2] \} \\ &= \binom{n}{m}^{-1} O(\overline{F}(x)), \quad x \rightarrow \infty . \end{aligned}$$

The last step follows from the fact that $f \in SD$ and Lemma 3.1.1 in Omey and Willekens (1987).

As a consequence we have

$$(4.7) \quad U_n(x_n) - \theta(x_n) = \tilde{U}_n(x_n) + O_P(n^{-m/2} (\overline{F}(x_n))^{\frac{1}{2}}) = \tilde{U}_n(x_n) + O_P(n^{-(m+1)/2} k_n^{\frac{1}{2}}) .$$

For the last two terms in expression (4.3) we prove

$$(4.8) \quad \overline{F^{*m}}(X_{(n-k_n)}) - \overline{F^{*m}}(x_n) = mf(x_n) (X_{(n-k_n)} - x_n) + o_P\left(\frac{\sqrt{k_n}}{n}\right) .$$

Since from Corollary 1,

$$(4.9) \quad X_{(n-k_n)} - x_n = O_P\left(\frac{\sqrt{k_n}}{nf(x_n)}\right), \quad n \rightarrow \infty$$

and using the monotonicity of $\overline{F^{*m}}$, (4.8) will be proved if we show that for every constant $c > 0$,

$$\frac{n}{\sqrt{k_n}} \left\{ \overline{F^{*m}}\left(x_n \pm c \frac{\sqrt{k_n}}{nf(x_n)}\right) - \overline{F^{*m}}(x_n) \mp cm \frac{\sqrt{k_n}}{n} \right\} \rightarrow 0, \quad n \rightarrow \infty .$$

Using the mean value theorem, this requires that

$$\frac{f^{\times m}(x_n + O(\frac{\sqrt{k_n}}{nf(x_n)}))}{f(x_n)} \rightarrow m, n \rightarrow \infty,$$

which in its turn is guaranteed by the fact that $f \in SD'$. Indeed, $\frac{\sqrt{k_n}}{nf(x_n)} = \frac{1}{\sqrt{k_n}} \frac{\bar{F}(x_n)}{f(x_n)} < M \frac{x_n^p L(x_n)}{\sqrt{k_n}} = o(x_n^p L(x_n)), n \rightarrow \infty$. Hence, from (4.8) and the representation in Theorem 1, we obtain

$$(4.10) \quad \overline{F^{*m}}(X_{(n-k_n)}) - \overline{F^{*m}}(x_n) = \frac{m}{n} \sum_{i=1}^n (I(X_i > x_n) - k_n/n) + o_p(\frac{\sqrt{k_n}}{n}).$$

Combining (4.7) and (4.10) gives the representation

$$\begin{aligned} \frac{n}{\sqrt{k_n}} A_{1n} &= \frac{m}{\sqrt{k_n}} \sum_{i=1}^n [g(X_i; x_n) + I(X_i > x_n) - k_n/n] + o_p(1) \\ &\equiv \sum_{i=1}^n Z_{in} + o_p(1). \end{aligned}$$

The result now follows by application of Liapunov's central limit theorem for triangular arrays and noting that

$$\begin{aligned} \text{Var}(Z_{in}) &= \frac{m^2}{k_n} \{E[(\overline{F^{*(m-1)}}(x_n - X_1))^2] + 3\bar{F}(x_n) - [\overline{F^{*m}}(x_n) + \bar{F}(x_n)]^2\} \\ &- 4 \frac{m^2}{k_n} \bar{F}(x_n) = \frac{4m^2}{n}, n \rightarrow \infty. \end{aligned}$$

LEMMA 2

Under the conditions of Theorem 2,

$$\frac{n}{\sqrt{k_n}} A_{2n} \xrightarrow{P} 0, n \rightarrow \infty.$$

Proof. Put

$$Q_n(s) = \frac{n}{\sqrt{k_n}} \binom{n}{m}^{-1} \sum_{C_m} [\tilde{h}(X_{i_1}, \dots, X_{i_m}; x_n + \frac{\sqrt{k_n}}{nf(x_n)} s) - \tilde{h}(X_{i_1}, \dots, X_{i_m}; x_n)]$$

where $\tilde{h}(x_1, \dots, x_m; x) = h(x_1, \dots, x_m; x) - \theta(x)$, with h and θ as before (see (4.5) and (4.6)). In this notation we have to show

$$Q_n \left(\frac{nf(x_n)}{\sqrt{k_n}} (X_{(n-k_n)} - x_n) \right) \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

By (4.9) it is therefore enough to show that for every bounded interval C ,

$$(4.11) \quad P \left(\sup_{s \in C} |Q_n(s)| > \varepsilon \right) \rightarrow 0, \quad n \rightarrow \infty,$$

for every $\varepsilon > 0$.

The way to show (4.11) follows more or less the same lines as the proof of Theorem 3.1 in Sukhatme (1958).

We first investigate the differences of the kernel \tilde{h} : for $0 \leq s < t$,

$$(4.12) \quad \begin{aligned} & E \left| \tilde{h} \left(X_1, \dots, X_m; x_n + \frac{\sqrt{k_n}}{nf(x_n)} t \right) - \tilde{h} \left(X_1, \dots, X_m; x_n + \frac{\sqrt{k_n}}{nf(x_n)} s \right) \right| \\ & \leq E \left[I \left(x_n + \frac{\sqrt{k_n}}{nf(x_n)} s \leq X_1 + \dots + X_m \leq x_n + \frac{\sqrt{k_n}}{nf(x_n)} t \right) \right] \\ & \quad + F^{*m} \left(x_n + \frac{\sqrt{k_n}}{nf(x_n)} t \right) - F^{*m} \left(x_n + \frac{\sqrt{k_n}}{nf(x_n)} s \right) \\ & = 2 \frac{\sqrt{k_n}}{nf(x_n)} \int_s^t f^{*m} \left(x_n + \frac{\sqrt{k_n}}{nf(x_n)} u \right) du \leq 2m c_1 \frac{\sqrt{k_n}}{n} (t - s) \end{aligned}$$

where $c_1 > 0$ is some absolute constant.

For $\delta > 0$ and integer r to be specified later, define

$$Q_{n,r}(s) = \frac{n}{\sqrt{k_n}} \binom{n}{m}^{-1} \sum_{C_m} E \left\{ \left[\tilde{h} \left(X_{i_1}, \dots, X_{i_m}; x_n + \frac{\sqrt{k_n}}{nf(x_n)} s \right) - \tilde{h} \left(X_{i_1}, \dots, X_{i_m}; x_n + \frac{\sqrt{k_n}}{nf(x_n)} r\delta \right) \right] \right\}.$$

Then,

$$(4.13) \quad Q_n(s) = Q_{n,r}(s) + Q_{n,0}(r\delta).$$

First consider $Q_{n,0}(r\delta)$. Then

$$\begin{aligned} E(Q_{n,0}^2(r\delta)) &= \frac{n^2}{k_n} \binom{n}{m}^{-2} \sum_{C_m} \sum_{C_m} E \left\{ \left[\tilde{h} \left(X_{i_1}, \dots, X_{i_m}; x_n + \frac{\sqrt{k_n}}{nf(x_n)} r\delta \right) - \tilde{h} \left(X_{i_1}, \dots, X_{i_m}; x_n \right) \right] \right. \\ & \quad \left. \cdot \left[\tilde{h} \left(X_{j_1}, \dots, X_{j_m}; x_n + \frac{\sqrt{k_n}}{nf(x_n)} r\delta \right) - \tilde{h} \left(X_{j_1}, \dots, X_{j_m}; x_n \right) \right] \right\}. \end{aligned}$$

Consider all terms with $1 \leq l \leq m$ equal components. Then by the boundedness of \tilde{h} and (4.12), the contribution of these terms will be smaller than

$$c_2 \frac{n^2}{k_n} \binom{n}{m}^{-2} \binom{n}{2m-l} \frac{\sqrt{k_n}}{n} r\delta - c_2 n^{1-l} \frac{\delta r}{\sqrt{k_n}}, \quad n \rightarrow \infty.$$

If $l = 0$, obviously the expectation of the product is zero. Thus we have $E(Q_{n,0}^2(r\delta)) \leq c_3 \frac{\delta r}{\sqrt{k_n}}$,

which implies

$$(4.14) \quad Q_{n,0}(r\delta) \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

We now deal with $Q_{n,r}(s)$. Denote

$$\begin{aligned} H_{r,n}(x_1, \dots, x_m) &= I(x_n + \frac{\sqrt{k_n}}{nf(x_n)} r\delta \leq x_1 + \dots + x_m \leq x_n + \frac{\sqrt{k_n}}{nf(x_n)} (r+1)\delta) \\ &\quad + F^{*m}(x_n + \frac{\sqrt{k_n}}{nf(x_n)} (r+1)\delta) - F^{*m}(x_n + \frac{\sqrt{k_n}}{nf(x_n)} r\delta). \end{aligned}$$

Then by (4.12),

$$(4.15) \quad \sup_{r\delta \leq s < (r+1)\delta} |Q_{n,r}(s)| \leq \frac{n}{\sqrt{k_n}} \binom{n}{m}^{-1} \sum_{C_m} H_{r,n}(X_{i_1}, \dots, X_{i_m}) \leq D_n + 2m c_1 \delta$$

where

$$D_n = \frac{n}{\sqrt{k_n}} \binom{n}{m}^{-1} \sum_{C_m} \{H_{r,n}(X_{i_1}, \dots, X_{i_m}) - E[H_{r,n}(X_{i_1}, \dots, X_{i_m})]\}.$$

In the same way as for $Q_{n,0}(r\delta)$ one can show that $E(D_n^2) \xrightarrow{P} 0$ so that $D_n \xrightarrow{P} 0, n \rightarrow \infty$.

Now let C be any bounded set in \mathbb{R} and let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon/8mc_1$, then $C \subseteq \bigcup_{r \in K} [r\delta, (r+1)\delta]$, with K a finite set of integers. By (4.13),

$$\sup_{s \in C} |Q_n(s)| \leq \sup_{r \in K} \{ \sup_{r\delta \leq s < (r+1)\delta} |Q_{n,r}(s)| + |Q_{n,0}(r\delta)| \}$$

such that

$$P(\sup_{s \in C} |Q_n(s)| > \varepsilon) \leq (\#K) [P(\sup_{r\delta \leq s < (r+1)\delta} |Q_{n,r}(s)| > \varepsilon/2) + P(|Q_{n,0}(r\delta)| > \varepsilon/2)].$$

Then, by (4.14) and (4.15), $\sup_{s \in C} |Q_n(s)| \xrightarrow{P} 0$ and the lemma is proved.

5. SPINOFFS AND EXTENSIONS

5.1. The choice of k_n . For practical purposes it would be interesting to know whether the ratio $\overline{F^{**m}}(x_n)/\overline{F}(x_n)$ in Theorem 2 could be replaced by its limit m . It is clear from this viewpoint that k_n has to be chosen such that

$$(5.1) \quad \sqrt{k_n} \left(\frac{\overline{F^{**m}}(x_n)}{\overline{F}(x_n)} - m \right)$$

remains bounded as $n \rightarrow \infty$. To describe the behaviour of (5.1), we need second order theory for subexponential distribution functions as established in Omeij and Willekens (1986, 1987). From Omeij and Willekens (1987), we know that for a large subclass of S , $\overline{F^{**m}}(x)/\overline{F}(x) - m - 2\mu \begin{bmatrix} m \\ 2 \end{bmatrix} f(x)/\overline{F}(x)$, $x \rightarrow \infty$, where $\mu = \int_0^\infty x dF(x)$. In order to get a decent rate in Theorem 2, it follows from the discussion above that an appropriate choice for $\{k_n\}$ is a sequence of integers such that

$$(5.2) \quad \frac{\sqrt{k_n} f(x_n)}{\overline{F}(x_n)} \rightarrow c, \quad n \rightarrow \infty,$$

where $c > 0$ is some constant. Although this choice of k_n causes an asymptotic bias, it also provides the best possible rate for the convergence to zero of the approximate mean square error $E[(T_n - m)^2]$.

In general we do not have information on the failure rate f/\overline{F} appearing in (5.2), but to give some idea on the order of k_n we specify this condition to the following examples (cf. examples of the class SD'): (i) $\overline{F}(x) = \exp(-x^\alpha)$, ($x > 0, 0 < \alpha < 1$) and (ii) $\overline{F}(x) = x^{-\alpha}$ ($x > 1, \alpha > 1$). In case (i) we obtain $k_n \sim c_1 \cdot (\log n)^\beta$ with $\beta = 2(1 - \alpha)/\alpha$ and in case (ii) we obtain $k_n \sim c_2 \cdot n^\beta$ with $\beta = 2/(\alpha + 2)$.

5.2. It is well known that the class S can be embedded in the family $\{S(\gamma), \gamma \geq 0\}$ with $S(\gamma)$ containing all distribution functions which satisfy (cf. Chover et al (1973a))

$$(i) \quad \frac{\overline{F}(x-y)}{\overline{F}(x)} \rightarrow e^{-y} \quad (x \rightarrow \infty), \quad \text{for every } y \in \mathbb{R}$$

and

$$(ii) \quad \frac{\overline{F^{**m}}(x)}{\overline{F}(x)} \rightarrow m \hat{f}^{m-1}(-\gamma) \quad (x \rightarrow \infty), \quad \text{for some } m \geq 2.$$

Here $\hat{f}(s) = \int_0^\infty e^{-sx} dF(x)$.

The class S is then identified as $S(0)$.

Densities with corresponding properties to (i) and (ii) (with $m = 2$) are gathered in the class $SD(\gamma)$. As examples of distribution functions with densities in $SD(\gamma)$, we mention the generalized inverse Gaussian distributions, cf. Embrechts (1983).

Using exactly the same techniques as in Sections 3 and 4, we can extend Theorem 2 to the classes $SD(\gamma)$, $\gamma > 0$. The general result then reads as follows:

suppose X_1, \dots, X_n are i.i.d. random variables from a distribution function F and denote $F' = f$. Let $k_n (1 \leq k_n < n)$ be a sequence of integers such that $k_n \rightarrow \infty$ and $k_n = o(n)$ ($n \rightarrow \infty$) and put $F(x_n) = \frac{k_n}{n}$. Let $\gamma \geq 0$ and suppose $f \in SD(\gamma)$ if $\gamma > 0$, $f \in SD'$ if $\gamma = 0$. Then

$$\sqrt{k_n} \left(T_n - \frac{\bar{F}^{*m}(x_n)}{\bar{F}(x_n)} \right) \xrightarrow{d} N(0, a(m, \gamma))$$

with

$$a(m, \gamma) = m^2 \left\{ 4 + 2(\hat{f}^{m-1}(\gamma) - 1) - \gamma \int_0^\infty \{1 - (F^{*(m-1)}(y))^2\} e^{-y} dy \right\} .$$

Similarly as in 5.1, we may be interested in replacing $\frac{\bar{F}^{*m}(x_n)}{\bar{F}(x_n)}$ by $m\hat{f}^{m-1}(-\gamma)$. This will involve second order theory for the class $S(\gamma)$. We refer to Willekens (1986).

ACKNOWLEDGEMENT

Part of this work was done while the first author was visiting the University of North Carolina at Chapel Hill. He likes to thank the Department of Statistics for their kind hospitality and financial support.

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