Skew-symmetric matrices and the Euler equations of notational motion for the rigid systems

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0. Introduction

We consider a rigid system of n, $n \geq 1$, particles in three-dimensional space. The point $\mathcal{O}$ is a point of the system and the origin of an inertial cartesian coordinate system C. Furthermore, we introduce a cartesian coordinate system C' having $\mathcal{O}$ as origin, which is fixed in the rigid system and such that the coordinate axes of C' correspond to the principal axes of inertia of the rigid system relative to the point $\mathcal{O}$. We investigate the equations of rotational motion for the rigid system around the fixed point $\mathcal{O}$. The results of this investigation are the well-known Euler equations of motion for rigid systems with one point fixed. In this paper we derive these equations by use of the exponential function with skew-symmetric matrix argument.

1. The Eulerian angles

The orientation of the coordinate system C' relative to the inertial coordinate system C is specified by the so-called Eulerian angles ϕ, θ, ψ in the following way. The coordinates of a point relative to C and C' are denoted as $(x,\ y,\ z) = \underline{x}$ and $(x',\ y',\ z') = \underline{x}'$ respectively. The line of intersection of the xy-coordinate plane of C and the $x'y'$-plane of C' is referred to as the line of nodes. Rotate the reference frame of C through an angle ϕ about the z-axis such that the x-axis is along the line of nodes. The rotated reference frame corresponds to the cartesian $\xi\eta\zeta$-coordinate system. Rotate the reference frame of the $\xi\eta\zeta$-system through an angle θ about the ξ-axis such that the ζ-axis and the z'-axis of C' coincide with the same orientation. The rotated reference frame corresponds to the cartesian $\xi'\eta'\zeta'$-system. Rotate the reference frame of the $\xi'\eta'\zeta'$-system through an angle ψ about the ζ'-axis such that it coincides with the reference frame of the system C'. For a picture of the Eulerian angles we refer to [1] p. 107.

Let $\vec{e}_n$ and $\vec{e}_{z'}$ be two unit vectors in the direction of the line of nodes and the z'-axis of C' respectively, and such that the components $\underline{e}_n$ and $\underline{e}_{z'}$ relative to C of the vectors $\vec{e}_n$ and $\vec{e}_{z'}$ can be written as

$$
\begin{aligned}
&\underline{e}_n = (\cos(\phi),\ \sin(\phi), 0) \quad , \\
(1.1) \qquad & \\
&\underline{e}_{z'} = (\sin(\phi)\,\sin(\theta),\ -\cos(\phi)\,\sin(\theta),\ \cos(\theta)) \quad .
\end{aligned}
$$

Hence, the Eulerian angle θ gives the inclination of the z'-axis from the z-axis and the Eulerian angle ϕ measures the azimuth of the z'-axis about the z-axis. The Eulerian angle ψ describes the rotation of the rigid system about the z'- axis.

Let $\underline{x}_i'$, $1 \leq i \leq n$, be the coordinates of particle i relative to C'. Since the system

C' is fixed in the rigid system, the coordinates $\underline{x}'_i$ are independent of the time t. The coordinates $\underline{x}_i$ of particle i relative to the inertial system C can be written as

$$(1.2) \qquad \underline{x}_i = \exp(\phi A_3)\,\exp(\theta A_1)\,\exp(\psi A_3)\,\underline{x}'_i \ ,$$

where $\phi,\ \theta,\ \psi$ are the Eulerian angles, and the skew-symmetric matrices A_1, A_3 equal

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ ;$$

compare (1.13) in [2].
From (1.2) we infer that the Eulerian angles $\phi,\ \theta,\ \psi$ constitute a set of three generalized coordinates describing the rotational motion of the rigid system around the fixed point O relative to the inertial coordinate system C.

2. The angular velocity

Let P be a point fixed in the coordinate system C'. The coordinates of P relative to C and C' are denoted as $\underline{x}$ and $\underline{x}'$ respectively. Hence, the coordinates $\underline{x}'$ are independent of the time t. We have

$$(2.1) \qquad \underline{x} = D\underline{x}' \ ,$$

where

$$D = \exp(\phi A_3)\,\exp(\theta A_1)\,\exp(\psi A_3) \ ;$$

compare (1.2).
Let $\vec{v}$ be the velocity vector of P relative to C. The components $\underline{v}$ with respect to C of the vector $\vec{v}$ can be written as

$$(2.2) \qquad \underline{v} = \dot{\underline{x}} = \dot{D}\underline{x}' \ .$$

The matrix D is the product of three matrices. Each of these matrices is an exponential function of an Eulerian angle times a constant skew-symmetric matrix. Hence, the time derivative $\dot{D}$ of D can easily be calculated and the matrix $\dot{D}$ can be put in the form

$$(2.3) \qquad \dot{D} = \Omega' D \ ,$$

where the skew-symmetric matrix

$$(2.4)\qquad \Omega' = \begin{bmatrix} 0 & -\omega'_z & \omega'_y \\ \omega'_z & 0 & -\omega'_x \\ -\omega'_y & \omega'_x & 0 \end{bmatrix} ,$$

where

$$\omega'_x = \dot\theta\, \cos(\phi) + \dot\psi\, \sin(\theta)\, \sin(\phi) \quad ,$$

$$\omega'_y = \dot\theta\, \sin(\phi) - \dot\psi\, \sin(\theta)\, \cos(\phi) \quad ,$$

$$\omega'_z = \dot\phi + \dot\psi\, \cos(\theta) \quad .$$

Hence,

$$(2.5)\qquad \underline{v} = \dot{D}\underline{x}' = \Omega' D\underline{x}' = \Omega'\underline{x} = \underline{\omega}' \times \underline{x} \quad ,$$

where

$$\underline{\omega}' = (\omega'_x,\, \omega'_y,\, \omega'_z) \quad .$$

From (2.5) we conclude that $\underline{\omega}'$ are the components with respect to the system C of the angular velocity of the system C' relative to C.

The transformation

$$(2.6)\qquad \begin{array}{ll} \phi \mapsto -\psi \quad , & \dot\phi \mapsto -\dot\psi \quad , \\ \theta \mapsto -\theta \quad , & \dot\theta \mapsto -\dot\theta \quad , \\ \psi \mapsto -\phi \quad , & \dot\psi \mapsto -\dot\phi \quad , \end{array}$$

takes the matrix Ω' into the matrix

$$(2.7)\qquad \Omega = \begin{bmatrix} 0 & \omega_{z'} & -\omega_{y'} \\ -\omega_{z'} & 0 & \omega_{x'} \\ \omega_{y'} & -\omega_{x'} & 0 \end{bmatrix} ,$$

where

$$\omega_{x'} = \dot\phi\, \sin(\theta)\, \sin(\psi) + \dot\theta\, \cos(\psi) \quad ,$$

$$\omega_{y'} = \dot\phi\, \sin(\theta)\, \cos(\psi) - \dot\theta\, \sin(\psi) \quad ,$$

$$\omega_{z'} = \dot\phi\, \cos(\theta) + \dot\psi \quad .$$

Furthermore, the transformation (2.6) takes the matrices D and $\dot{D}$ into the matrices D^{-1} and $(D^{-1})^\bullet$ respectively. From (2.3) we infer

$$(2.8) \qquad (D^{-1})^\bullet = \Omega D^{-1} \quad .$$

Use (2.3) and $(D^{-1})^\bullet D = -D^{-1}\dot{D}$ to obtain

$$\Omega = (D^{-1})^\bullet D = -D^{-1}\dot{D} = -D^{-1}\Omega' D \quad .$$

Hence

$$(2.9) \qquad D\Omega = -\Omega' D \quad .$$

Let P be a point fixed in the coordinate system C. The coordinates of P relative to C and C' are denoted as $\underline{x}$ and $\underline{x}'$ respectively. Hence, the coordinates $\underline{x}$ are independent of the time t. Furthermore, the coordinates $\underline{x}$ and $\underline{x}'$ satisfy the relation (2.1). Let $\vec{v}'$ be the velocity of P relative to C'. The components $\underline{v}'$ with respect to C' of the vector $\vec{v}'$ can be written as

$$(2.10) \qquad \underline{v}' = \dot{\underline{x}}' = (D^{-1})^\bullet \underline{x} \quad .$$

Use (2.1) and (2.8) to write

$$(2.11) \qquad \underline{v}' = \Omega D^{-1}\underline{x} = \Omega \underline{x}' = \underline{\omega} \times \underline{x}' \quad ,$$

where

$$\underline{\omega} = (\omega_{x'},\ \omega_{y'},\ \omega_{z'}) \quad ;$$

compare (2.7). From (2.11) we conclude that $\underline{\omega}$ are the components with respect to the system C' of the angular velocity of the system C relative to C'; compare formula (4-103) on p. 134 in [1].

3. The kinetic energy

The coordinates of particle i, $1 \le i \le n$, in the rigid system with respect to C and C' are denoted as $\underline{x}_i$ and $\underline{x}_i'$ respectively. The velocity vector of particle i relative to C is denoted as $\vec{v}_i$. The components $\underline{v}_i$ with respect to C of the vector $\vec{v}_i$ satisfy (use (2.1, 2, 3, 9))

$$(3.1) \qquad \underline{v}_i = \dot{\underline{x}}_i = \dot{D}\underline{x}_i' = \Omega' D \underline{x}_i' = -D\Omega \underline{x}_i' \quad .$$

Let m_i be the mass of particle i. The kinetic energy T of the rigid system relative to C can be written as

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i(\underline{v}_i,\ \underline{v}_i) =$$

$$= \frac{1}{2}\sum_{i=1}^{n} m_i(D\Omega\underline{x}'_i,\ D\Omega\underline{x}'_i) =$$

$$= \frac{1}{2}\sum_{i=1}^{n} m_i(\Omega\underline{x}'_i,\ \Omega\underline{x}'_i) =$$

$$= \frac{1}{2}\sum_{i=1}^{n} m_i(\underline{\omega} \times \underline{x}'_i,\ \underline{\omega} \times \underline{x}'_i) \quad ,$$

where

$$\underline{\omega} = (\omega_{x'},\ \omega_{y'},\ \omega_{z'}) \quad ;$$

compare (2.7). From (3.4) and (3.6) in [2] we conclude

$$(3.2) \qquad T = \tfrac{1}{2}(\underline{\omega},\ I\underline{\omega}) \quad ,$$

where I is the matrix representation relative to C' of the inertia tensor of the rigid system with respect to the fixed point $\mathcal{O}$. The coordinate axes of C' correspond to the principal axes of inertia of the rigid system relative to $\mathcal{O}$. Hence,

$$(3.3) \qquad T = \tfrac{1}{2}(I_{x'}\,\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2) \quad ,$$

where $I_{x'}$, $I_{y'}$, $I_{z'}$ are the principal moments of inertia (eigenvalues of the inertia tensor) of the rigid system relative to the fixed point $\mathcal{O}$.

4. The components of the generalized force

Let $\vec{F}_i$ be the force vector acting on particle i and let $\vec{M}_i$ be the corresponding torque vector on particle i relative to the fixed point $\mathcal{O}$. The components $\underline{F}_i$ and $\underline{M}_i$ with respect to C of the vectors $\vec{F}_i$ and $\vec{M}_i$ satisfy the relation

$$(4.1) \qquad \underline{M}_i = \underline{x}_i \times \underline{F}_i \quad ,$$

where $\underline{x}_i$ are the coordinates relative to C of particle i. The vector

$$\vec{M} = \sum_{i=1}^{n} \vec{M}_i$$

is the torque vector relative to $\mathcal{O}$ on the rigid system. For the components $\underline{M}$ relative to C of the vector $\vec{M}$ we get

$$(4.2) \qquad \underline{M} = \sum_{i=1}^{n} \underline{M}_i = \sum_{i=1}^{n} \underline{x}_i \times \underline{F}_i \ .$$

Following the definitions (5.12) in [2] we define the components Q_ϕ, Q_θ, Q_ψ of the generalized force on the rigid system relative to C as

$$(4.3) \qquad \begin{aligned} Q_\phi &= \sum_{i=1}^{n} (\underline{F}_i, \frac{\partial \underline{x}_i}{\partial \phi}) \ , \\ Q_\theta &= \sum_{i=1}^{n} (\underline{F}_i, \frac{\partial \underline{x}_i}{\partial \theta}) \ , \\ Q_\psi &= \sum_{i=1}^{n} (\underline{F}_i, \frac{\partial \underline{x}_i}{\partial \psi}) \ . \end{aligned}$$

Introduce the following notations

$$\begin{aligned} \underline{e}_z &= (0,\ 0,\ 1) \ , \\ \underline{e}_n &= (\cos(\phi),\ \sin(\phi),\ 0) \ , \\ \underline{e}_{z'} &= (\sin(\phi)\sin(\theta),\ -\cos(\phi)\sin(\theta),\ \cos(\theta)) \ ; \end{aligned}$$

compare (1.1). From (2.1) and elementary calculations we get

$$(4.4) \qquad \begin{aligned} \frac{\partial \underline{x}_i}{\partial \phi} &= \frac{\partial D}{\partial \phi} \underline{x}_i' = A_3 D \underline{x}_i' = \underline{e}_z \times \underline{x}_i \ , \\ \frac{\partial \underline{x}_i}{\partial \theta} &= \frac{\partial D}{\partial \theta} \underline{x}_i' = \exp(\phi A_3) A_1 \exp(-\phi A_3) D \underline{x}_i' = \underline{e}_n \times \underline{x}_i \ , \\ \frac{\partial \underline{x}_i}{\partial \psi} &= \frac{\partial D}{\partial \psi} \underline{x}_i' = \exp(\phi A_3) \exp(\theta A_1) A_3 \exp(-\theta A_1) \exp(-\phi A_3) D \underline{x}_i' = \\ &= \underline{e}_{z'} \times \underline{x}_i \ . \end{aligned}$$

From (4.2, 3, 4) we obtain

$$(4.5) \qquad \begin{aligned} &Q_\phi = \sum_{i=1}^{n} (\underline{F}_i,\ \underline{e}_z \times \underline{x}_i) = (\underline{e}_z,\ \sum_{i=1}^{n} \underline{x}_i \times \underline{F}_i) = (\underline{e}_z,\ \underline{M}) \ , \\ &Q_\theta = (\underline{e}_n,\ \underline{M}) \ \text{ and } \ Q_\psi = (\underline{e}_{z'},\ \underline{M}) \ . \end{aligned}$$

Suppose that the forces on the particles are conservative, i.e. there exists a potential function $V(x_1,\ y_1,\ z_1, \ldots,\ x_n,\ y_n,\ z_n)$ from $\mathbb{R}^{3n}$ to $\mathbb{R}$ such that

$$(4.6) \qquad \underline{F}_i = -(\frac{\partial V}{\partial x_i}(\underline{x}_1, ..., \underline{x}_n), \frac{\partial V}{\partial y_i}(\underline{x}_1, ..., \underline{x}_n), \frac{\partial V}{\partial z_i}(\underline{x}_1, ..., \underline{x}_n))$$

$$=: -(\text{grad}_i V)\ (\underline{x}_1, ..., \underline{x}_n) \quad ,$$

where $\underline{x}_i$ are the coordinates relative to C of particle i.
The potential energy V of the system is defined as

$$(4.7) \qquad V = V(\phi,\ \theta,\ \psi) = V(\underline{x}_1(\phi,\ \theta,\ \psi), ..., \underline{x}_n(\phi,\ \theta,\ \psi)) \quad .$$

We have

$$\frac{\partial V}{\partial \phi} = \sum_{i=1}^{n}((\text{grad}_i V)(\underline{x}_1, ..., \underline{x}_n), \frac{\partial \underline{x}_i}{\partial \phi}) =$$

$$= -\sum_{i=1}^{n}(\underline{F}_i, \frac{\partial \underline{x}_i}{\partial \phi}) = -Q_\phi \quad .$$

Hence,

$$(4.8) \qquad Q_\phi = -\frac{\partial V}{\partial \phi},\ Q_\theta = -\frac{\partial V}{\partial \theta},\ Q_\psi = -\frac{\partial V}{\partial \psi} \quad .$$

5. The Euler equations

The Lagrangian equations of motion for the rigid system relative to the inertial coordinate system C can be written as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = Q_\phi \quad ,$$

$$(5.1) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = Q_\theta \quad ,$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} = Q_\psi \quad ;$$

compare (5.13) in [2]. We investigate the third equation in (5.1). Use (3.2) and the symmetry of I to obtain

$$\frac{\partial T}{\partial \dot{\psi}} = (\frac{\partial \underline{\omega}}{\partial \dot{\psi}},\ I\underline{\omega}) \quad ,$$

and from (2.7) we get

$$\frac{\partial \underline{\omega}}{\partial \dot{\psi}} = (0,\ 0,\ 1) \quad .$$

Hence,

$$\frac{\partial T}{\partial \dot{\psi}} = I_{z'}\omega_{z'} \quad . \tag{5.2}$$

Furthermore,

$$\frac{\partial T}{\partial \psi} = (\frac{\partial \underline{\omega}}{\partial \psi}, \; I\underline{\omega}) \quad ,$$

and from (2.7)

$$\frac{\partial \underline{\omega}}{\partial \psi} = (\omega_{y'}, \; -\omega_{x'}, \; 0) \quad .$$

Hence,

$$\frac{\partial T}{\partial \psi} = \omega_{x'}\omega_{y'}(I_{x'} - I_{y'}) \quad . \tag{5.3}$$

From (4.5) we conclude

$$Q_{\psi} = M_{z'} \quad , \tag{5.4}$$

where $(M_{x'}, \; M_{y'}, \; M_{z'})$ are the components relative to C' of the torque vector $\vec{M}$ on the rigid system. By use of (5.2, 3, 4) the third equation in (5.1) can be written as

$$I_{z'} \; \dot{\omega}_{z'} - \omega_{x'}\omega_{y'}(I_{x'} - I_{y'}) = M_{z'} \quad .$$

Hence, by cyclic permutation,

$$\begin{aligned} I_{x'}\dot{\omega}_{x'} - \omega_{y'}\omega_{z'}(I_{y'} - I_{z'}) &= M_{x'} \quad , \\ I_{y'}\dot{\omega}_{y'} - \omega_{z'}\omega_{x'}(I_{z'} - I_{x'}) &= M_{y'} \quad , \\ I_{z'}\dot{\omega}_{z'} - \omega_{x'}\omega_{y'}(I_{x'} - I_{y'}) &= M_{z'} \quad . \end{aligned} \tag{5.5}$$

The equations (5.5) are the so-called Euler equations of rotational motion for rigid systems with one point fixed; compare formula (5-34) on p. 158 in [1].

References

[1] H. Goldstein: Classical Mechanics, Addison-Wesley Publishing Co., Reading, Massachusetts, 1950.

[2] D.A. Overdijk: Skew-symmetric matrices in classical mechanics, Memorandum COSOR 89-23, T.U. Eindhoven, 1989.

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M 90-03	February	I.J.B.F. Adan J. Wessels W.H.M. Zijm	Analysis of the assymmetric shortest queue problem Part II: Numerical analysis
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M 90-23	July	A. Dekkers J. van der Wal	Mean value analysis of priority stations without preemption
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