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MATHEMATICS

ON THE MULTIPLIERSYSTEM OF THE RIEMANN-DEDEKIND FUNCTION η

BY

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1. *Introduction*

The function η defined by

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}) \text{ for } \operatorname{Im}\tau>0$$

satisfies the equations $\eta(\tau+1)=e^{\frac{\pi i}{12}}\eta(\tau)$ and $\eta\left(-\frac{1}{\tau}\right)=\sqrt{-i\tau}\,\eta(\tau)$ and therefore η is a modular form $\in\{\Gamma(1),-\frac{1}{2},v\}$ with a certain multiplier system v determined by the above equalities. In [1] and [2] v was determined by RADEMACHER using Dedekind sums. In [3] PETERSSON gave a simple formula for v. We shall find the same formula without using Dedekind sums. To do this we first determine the characters of the group $\Gamma(1)$. (By characters we mean characters of the first degree.) This problem is reduced to a problem on finite groups by proving that $\Gamma(12)$ is contained in the commutator subgroup of $\Gamma(1)$.

We use the well-known notation:

$\Gamma(1)$ is the group of matrices $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ with rational integral elements and $ad-bc=1$. We write $U=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $T=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and denote $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ by I.
$\Gamma(n)$ is the subgroup of $\Gamma(1)$ defined by $\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix} \pmod n$.
$\Gamma_0(n)$ is the subgroup of $\Gamma(1)$ defined by $c\equiv 0 \pmod n$.
$\mathfrak{M}(n)$ is the factor group $\Gamma(1)/\Gamma(n)$.
A multiplier system v of $\{\Gamma(1),-r,v\}$ is determined by $\lambda=v(U)=\zeta_6\cdot e^{\frac{\pi i r}{6}}$ in which ζ_6 denotes a 6-th root of unity. We have $\mu=v(T)=\lambda^{-3}$.

2. *Theorems about* v

In [4] we proved that for every matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ in $\Gamma(1)$ there is an integer $w\begin{pmatrix}a&b\\c&d\end{pmatrix}$ independent of r so that $v\begin{pmatrix}a&b\\c&d\end{pmatrix}=\lambda^{w\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}$.

If $f\in\{\Gamma(1),-r,v\}$ and we define F by $F(\tau)=f(p\tau)f^{-p}(\tau)$, then $F\in\{\Gamma_0(p),-r,v'\}$ with $v'\begin{pmatrix}a&b\\cp&d\end{pmatrix}=v\begin{pmatrix}a&bp\\c&d\end{pmatrix}v^{-p}\begin{pmatrix}a&b\\cp&d\end{pmatrix}=\lambda^{w\left(\begin{smallmatrix}a&bp\\c&d\end{smallmatrix}\right)-p\cdot w\left(\begin{smallmatrix}a&b\\cp&d\end{smallmatrix}\right)}$.

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For $f=\eta$, and p a prime larger than 3, Hecke proved

$$v'\begin{pmatrix} a & b \\ cp & d \end{pmatrix} = \left(\frac{d}{p}\right) \tag{2.1}$$

without using v. Here $\left(\frac{d}{p}\right)$ is the quadratic restsymbol (cf. [5]). If we replace v by v^3, the result (2.1) also holds for $p=3$. We therefore have $w\begin{pmatrix} a & bp \\ c & d \end{pmatrix} - pw\begin{pmatrix} a & b \\ cp & d \end{pmatrix} \equiv 0 \pmod{12}$.

Now consider $\{\Gamma(1), -r, v\}$ where r is an integer. Then λ is a 12-th root of unity and we have:

$$v\begin{pmatrix} a & bp \\ c & d \end{pmatrix} = v^p\begin{pmatrix} a & b \\ cp & d \end{pmatrix} \text{ and } v^p\begin{pmatrix} a & bp \\ c & d \end{pmatrix} = v\begin{pmatrix} a & b \\ cp & d \end{pmatrix} \text{ for } p>3. \tag{2.2}$$

If in this case $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(12)$ we have $\mu \cdot v\begin{pmatrix} a & b \\ c & d \end{pmatrix} = v\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$ and as $(d, 6)=1$ we can apply (2.2):

$$\mu v\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} v^d\begin{pmatrix} b & -ad \\ 1 & -c \end{pmatrix} = \mu^d = \mu & (d>0), \\ v^{-d}\begin{pmatrix} b & ad \\ -1 & -c \end{pmatrix} = \mu^{-3d} = \mu & (d<0). \end{cases}$$

We see that $v\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. If we take $r=0$ we have

Theorem 1. If v is a character of $\Gamma(1)$ then $v=1$ on $\Gamma(12)$. This means that the commutator subgroup of $\Gamma(1)$ contains $\Gamma(12)$. (The commutator subgroup is generated by $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.)

Before continuing we give an alternative proof of (2.2) which does not use the theory of modular forms used by Hecke for his proof of (2.1).

Proof of (2.2):

Consider a character v of $\Gamma(1)$. We have to prove that the characters φ_1 and φ_2 of $\Gamma_0(p)$ defined by

$$\varphi_1\begin{pmatrix} a & b \\ cp & d \end{pmatrix} = v\begin{pmatrix} a & bp \\ c & d \end{pmatrix} \text{ and } \varphi_2\begin{pmatrix} a & b \\ cp & d \end{pmatrix} = v^p\begin{pmatrix} a & b \\ cp & d \end{pmatrix}$$

are equal for all primes $p>3$.

Let $p_1=2$, $p_2=3$, p_3, ... be the series of primes. Suppose the theorem is true for $3<p<q=p_{n+1}$. We shall prove the theorem for q. It is sufficient to prove that $\varphi_1=\varphi_2$ for a set of matrices generating $\Gamma_0(q)$.

The set of numbers $\pm p_3^{\alpha_3} p_4^{\alpha_4} \dots p_n^{\alpha_n}$ contains a set of representatives of the non-zero residue classes mod q. Call these $d_1, d_2, \dots d_{q-1}$ ($d_i=\varepsilon_i \delta_i$ with $\delta_i=|d_i|$). As a generating set of $\Gamma_0(q)$ we can choose

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} a_i & b_i \\ q & d_i \end{pmatrix} \quad (i,=1,2,\dots,q-1).$$

We have

$$v^q\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = v\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}; \quad v\begin{pmatrix} a_i & b_i q \\ 1 & d_i \end{pmatrix} = \lambda^{a_i+d_i-3}, \quad v\begin{pmatrix} a_i & b_i \\ q & d_i \end{pmatrix} = \lambda^3 v\begin{pmatrix} b_i & -a_i \\ d_i & -q \end{pmatrix}$$

and because the theorem holds for $3<p<q$ we have

$$v\begin{pmatrix} a_i & b_i \\ q & d_i \end{pmatrix} = \lambda^3 v^{\delta_i}\begin{pmatrix} b_i & -a_i\delta_i \\ \varepsilon_i & -q \end{pmatrix} = \lambda^{3+d_i(b_i-q-3)}.$$

Now

$$\lambda^{(b_i d_i - q d_i - 3d_i + 3)q - (a_i + d_i - 3)} = \lambda^{a_i d^2_i - a_i - q^2 d_i - 2d_i - 3d_i q + 3q + 3} = \lambda^{3(q+1)(1-d_i)} = 1$$

because $d_i^2 \equiv q^2 \equiv 1 \pmod{24}$ and $3(q+1)(1-d_i) \equiv 0 \pmod{12}$ and λ is a 12-th root of unity. Hence the characters are equal for the generators of $\Gamma_0(q)$.

To prove the theorem for all primes $p>3$ we now only have to show that it holds for $p=5$, i.e. for the generators of $\Gamma_0(5)$. This is checked by a simple calculation.

That the analogous theorem holds for v^3 and $p=3$ is proved in the same way.

3. *The characters of* $\Gamma(1)$

By theorem 1 the characters of $\Gamma(1)$ are the characters of $\mathfrak{M}(12)$. As $\mathfrak{M}(12)=\mathfrak{M}(3)\times\mathfrak{M}(4)$ a character of $\mathfrak{M}(12)$ is the product of a character of $\mathfrak{M}(3)$ and a character of $\mathfrak{M}(4)$.

We first determine the characters of $\mathfrak{M}(3)$. These have been calculated by many different authors (a.o. [6], [7]). We wish to express them as functions of the elements a, b, c, d of the matrices of $\mathfrak{M}(3)$. The characters are equal to 1 on the commutator subgroup of $\mathfrak{M}(3)$. This subgroup has 8 elements. These are all the third powers of elements of $\mathfrak{M}(3)$. I, U and U^2 are the representatives of the classes of the factor group. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of the commutator subgroup we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^3 \equiv \begin{pmatrix} \alpha\delta(\delta-\alpha)-(a+\delta) & \beta(\alpha+\delta+1)(\alpha+\delta-1) \\ \gamma(\alpha+\delta+1)(\alpha+\delta-1) & -\alpha\delta(\delta-\alpha)-(a+\delta) \end{pmatrix}.$$

We see that the elements of the commutator subgroup are characterized by: $a+d\equiv 0 \pmod 3$ or $b\equiv c\equiv 0 \pmod 3$. We can write this as $(a+d)c+\varrho b(c^2-1)\equiv 0 \pmod 3$ where ϱ will be chosen later.

Now let χ be a character of $\mathfrak{M}(3)$. Then $\chi(U)=\zeta$ is a third root of unity and we have $\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\zeta^n$ if $\begin{pmatrix} a-nc & b-nd \\ c & d \end{pmatrix}$ is in the commutator subgroup, i.e. if $(a+d)c+\varrho b(c^2-1)\equiv n[c^2+\varrho d(c^2-1)] \pmod 3$. If we choose $\varrho=-d$ then $[c^2+\varrho d(c^2-1)]\equiv 1 \pmod 3$ and hence the character can be written as

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \zeta^{(a+d)c-bd(c^2-1)}. \tag{3.1}$$

We now determine the characters of $\mathfrak{M}(4)$ in the same way. This group has 48 elements; the commutator subgroup consists of the 12 fourth powers of the elements. These are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

As we see, the elements with $c \equiv 1 \pmod 2$ are characterized by $a+d \equiv 3 \pmod 4$. This enables us to use the same method we used for $\mathfrak{M}(3)$. We again consider the function $(a+d)c-bd(c^2-1)=f(a, b, c, d)$. By the above, the matrices with $c \equiv 1 \pmod 2$ which are in the commutator subgroup of $\mathfrak{M}(4)$ are characterized by $f(a, b, c, d) \equiv 3c \pmod 4$. By a simple calculation we see that for the fourth powers with $c \equiv 0 \pmod 2$ we have $b \equiv c+d-1 \pmod 4$. Hence $bd \equiv cd+1-d \pmod 4$. If $c \equiv 0 \pmod 2$ we have $(a+d)c-bd(c^2-1) \equiv bd \pmod 4$. Therefore we have $f(a, b, c, d) \equiv 3d-3-3cd \pmod 4$ if $c \equiv 0 \pmod 2$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the commutator subgroup of $\mathfrak{M}(4)$.

We define

$$(3.2) \qquad g(a, b, c, d) = \begin{cases} f(a, b, c, d)-3c & \text{if } c \equiv 1 \pmod 2, \\ f(a, b, c, d)+3d-3-3cd & \text{if } c \equiv 0 \pmod 2. \end{cases}$$

If χ is a character of $\mathfrak{M}(4)$ and $\chi(U)=\zeta$ where ζ is a fourth root of unity, then $\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\zeta^n$ if $g(a-nc, b-nd, c, d) \equiv 0 \pmod 4$. We can write this as $\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\zeta^{g(a,b,c,d)}$. In the formula (3.1) we can replace the exponent by $g(a, b, c, d)$ because multiples of 3 do not change the value of the power in that case. Therefore

$$(3.3) \qquad \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \zeta^{g(a,b,c,d)}$$

is a uniform expression for the characters of $\mathfrak{M}(3)$ and $\mathfrak{M}(4)$.

We have proved

Theorem 2. If v is a character of $\mathfrak{M}(12)$, i.e. a character of $\Gamma(1)$, then $v(U)=\zeta$ is a twelfth root of unity and

$$v\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \zeta^{g(a,b,c,d)},$$

where $g(a, b, c, d)$ is defined by (3.2).

Corollary: The commutator subgroup of $\Gamma(1)$ consists of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $g(a, b, c, d) \equiv 0 \pmod{12}$.

4. *The multiplier system v of the function η.*

If v is the multiplier system of η then $v^2(U)=\lambda^2=e^{\frac{\pi i}{6}}$ is a twelfth root of unity and hence v^2 is a character of $\Gamma(1)$. Therefore we have:

$$(4.1) \qquad v\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{\frac{\pi i}{12} g(a,b,c,d)},$$

where $\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1$ or -1.

Now

$$v\begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix} = \varepsilon\begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix} e^{\frac{\pi i}{12} g(a,b,c,d)}\, e^{\frac{\pi i}{12} n} = v\begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{\frac{\pi i}{12} n}$$

and hence ε only depends on c and d. We write $\varepsilon = \varepsilon(c, d)$. We define:

$$\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right) \text{ if } c \neq 0,$$

$$\left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right)(-1)^{\frac{\operatorname{sgn} c-1}{2} \cdot \frac{\operatorname{sgn} d-1}{2}} \text{ if } c \neq 0,$$

$$\left(\frac{0}{\pm 1}\right)^* = \left(\frac{0}{\pm 1}\right)_* = 1.$$

Consider $p > 3$, $p \nmid d$ and $c \equiv 1 \pmod 2$ and apply (2.1):

$$v\begin{pmatrix} a & b \\ cp & d \end{pmatrix} = \varepsilon(cp, d)\, e^{\frac{\pi i}{12} g(a, b, cp, d)},$$

$$v\begin{pmatrix} a & bp \\ c & d \end{pmatrix} = \varepsilon(c\ \ , d)\, e^{\frac{\pi i}{12} g(a, bp, c, d)}.$$

$g(a, bp, c, d) \equiv pg(a, b, cp, d) \pmod{24}$ because $p^2 \equiv 1 \pmod{24}$. By (2.1) we therefore have

$$\varepsilon(c, d) = \left(\frac{d}{p}\right) \varepsilon(cp, d) \qquad \text{for } p > 3.$$

If we replace v by v^3 and use the fact that $\varepsilon^3 = \varepsilon$ we can prove the same result for $p = 3$.

By calculating the value of v for the matrices $\begin{pmatrix} 0 & \mp 1 \\ \pm 1 & d \end{pmatrix}$ we find $\varepsilon(\pm 1, d) = 1$. Therefore we have

$$(4.2) \qquad \varepsilon(c, d) = \left(\frac{d}{c}\right)^* \qquad \text{for } c \equiv 1 \pmod 2.$$

For $c \equiv 0 \pmod 2$ we find in the same way as above

$$\varepsilon(cp, d) = \left(\frac{p}{d}\right) \varepsilon(c, d) \qquad \text{for } p \geqslant 3.$$

Using the results we have found, we can calculate the multiplier of $\begin{pmatrix} b & -a \\ d & \mp 2^n \end{pmatrix} = \begin{pmatrix} a & b \\ \pm 2^n & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In this way we find $\varepsilon(\pm 2^n, d)$. The result is

$$\varepsilon(\pm 2^n, d) = \left(\frac{\pm 2^n}{d}\right)_*.$$

We easily find $\varepsilon(0, \pm 1) = 1$. So we finally have:

$$(4.3) \qquad \varepsilon(c, d) = \left(\frac{c}{d}\right)_* \qquad \text{for } c \equiv 0 \pmod 2.$$

Combining (4.1), (4.2) and (4.3) we get

Theorem 3. The multiplier system of η is given by

$$v\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{c}\right)^* e^{\frac{\pi i}{12}\{(a+d)c - bd(c^2-1) - 3c\}} & \text{if } c \equiv 1 \pmod 2, \\ \left(\frac{c}{d}\right)_* e^{\frac{\pi i}{12}\{(a+d)c - bd(c^2-1) + 3d - 3 - 3cd\}} & \text{if } c \equiv 0 \pmod 2. \end{cases}$$

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