An upper bound on the number of non-unique assignments in relaxed (mixed) integer linear programs of the assignment type

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Abstract.

An upper bound is given for the number of non-unique assignments when solving the linear programming relaxation of (mixed) integer linear programming problems in which the integer variables are governed by assignment type constraints.

Key-words. (mixed) integer linear programming, assignment problems, class-room scheduling.

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1. Introduction

We consider the (mixed) integer linear programming problem

$$(1.1) \quad Ax + By = b$$

$$(1.2) \quad Dy = e$$

$$(1.3) \quad x \geq 0$$

$$(1.4) \quad y = 0/1$$

$$(1.5) \quad p'x + q'y \quad \text{maximum} .$$

The mixed relations are arbitrary, the pure integer restrictions are either of the multiple choice type or of the assignment type. The interest is in the number of components of y which are not equal to 0 or 1 when solving the linear programming relaxation of this problem, obtained by replacing the condition $y = 0/1$ by the condition $y \geq 0$.

The case of the multiple choice type

$$(2) \qquad \sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1,\dots,m$$

has been considered already in [1], where it has been shown that in any basic feasible solution of the relaxed problem, the number of non-unique choices can not exceed the number of restrictions involved in (1.1) minus the number of non-zero x-components. A practical example of a location-allocation problem shows that this property may be useful in deriving near optimum solutions of mixed integer programming problems of type (1).

Here we will show that a similar bound can be given for the number of non-unique assignments of indices i to j in case the pure integer restrictions are of the assignment type

$$(3.1) \quad \sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1,\dots,m$$

$$(3.2) \quad \sum_{i=1}^{m} y_{ij} + y_{0j} = 1 \qquad j = 1,\dots,n .$$

An example is given of a class-room scheduling problem for which this property was helpful in devising a practical method for finding a feasible solution.

2. A bound on the number of non-unique assignments

The assignment restrictions (3) have the well-known property that each nonnegative integer solution is a basic feasible solution of the relaxed system obtained by replacing the non-zero condition of y by $y \geq 0$. Also any such basic solution is an integer solution of (3). This implies that any basic solution has exactly n non-zero components, all of them equal to 1.

Now, let $(\bar{x},\bar{y})$ be an arbitrary basic feasible solution of the relaxed problem (1) and let s be the number of non-zero $\bar{x}$-components.

For counting the non-zero $\bar{y}$-components, let, for $i \neq 0$,

- m_1 be the number of i-s, uniquely assigned to j-s,
- m_2 the number of i-s, non-uniquely assigned to j-s,
- λ the average number of non-zero $\bar{y}_{ij}$ for non-uniquely assigned i-s .
- n_1 the number of non-zero $\bar{y}_{0j}$ components.

Then the number of non-zero $\bar{y}$-components is

(4) $\quad m_1 + \lambda m_2 + n_1$

which by $m_1 + m_2 = m$, may be written in the form

(5) $\quad m + (\lambda - 1)m_2 + n_1 .$

The $s + m + (\lambda - 1)m_2 + n_1$ basic columns corresponding to non-zero $\bar{x}$- en $\bar{y}$-components are linearly independent. They constitute a matrix of the form

(6) $$\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$$

with a row-rank equal to $s + m + (\lambda - 1)m_2 + n_1$. Assuming that (1.1) involves r linear relations, the matrices P and Q are r-row submatrices of A and B whereas R is an (m + n)-row submatrix of D. Since $\bar{y}$ is a feasible solution of the relaxed system (3), the column rank of R equals n, hence also its row-rank. This implies that the row-rank of the matrix (6) cannot exceed n + r, hence

(7) $\quad s + m + (\lambda - 1)m_2 + n_1 \leq n + r$

or

(8) $\quad (\lambda - 1)m_2 \leq (r - s) + (n - m) - n_1 .$

It follows from (3) that

$$\sum_{j=1}^{n} \bar{y}_{0j} = n - m .$$

Since each of the n_1 non-zero components $\bar{y}_{0j}$ does not exceed 1, the number n_1 satisfies

(9) $\quad n_1 \geq n - m.$

Combination of (8) and (10) delivers

(10) $\quad (\lambda - 1)m_2 \leq r - s$

or

(11) $\quad m_2 \leq \left\lfloor \frac{r - s}{\lambda - 1} \right\rfloor \leq r - s$

the second inequality holding because of $\lambda \geq 2$.

The upper bound inequality (11) means that also in the case of assignment type restrictions, for any basic feasible solution $(\bar{x},\bar{y})$ of the relaxed problem (1), the number of non-uniquely assigned indices i to indices j does not exceed the number of restrictions involved in (1.1) minus the number of non-zero $\bar{x}$-components.

3. The clustering problem in class-room scheduling

In time table scheduling for secundary schools in the Netherlands a solution may be required for the so-called clustering problem.

There are s students each of which must choose a packet of p courses out of a set of c possible courses. The problem is then to make clusters of courses that are given at the same time. Each course chosen by a student must be assigned to one of these clusters, taking care of course that no student is scheduled twice in the same cluster and such that the class-room capacity m of the r available class-rooms is not exceeded.

Let

$x_{\sigma\gamma\kappa}$ be an assignment variable being equal to 1 if student σ, who has chosen course γ, is for this course assigned to cluster κ, and 0 otherwise;

$z_{\gamma\kappa}$ be an assignment variable being equal to 1 if course γ is assigned to cluster κ, and 0 otherwise;

$\ell_{\sigma\gamma}$ be equal to 1 if student σ has chosen course γ, and zero otherwise.

Then the problem can be formulated as a pure integer programming problem. Determine a feasible solution of the system

$$(12.1) \quad \forall_{\sigma\gamma} \quad \sum_{\kappa=1}^{k} x_{\sigma\gamma\kappa} = \ell_{\sigma\gamma}$$

$$(12.2) \quad \forall_{\sigma\kappa} \quad \sum_{\gamma=1}^{c} x_{\sigma\gamma\kappa} + x_{\sigma\kappa} = 1$$

$$(12.3) \quad \forall_{\gamma\kappa} \quad \sum_{\sigma=1}^{s} x_{\sigma\gamma\kappa} - mz_{\gamma\kappa} \leq 0$$

$$(12.4) \quad \forall_{\kappa} \quad \sum_{\gamma=1}^{c} z_{\gamma\kappa} \leq r$$

$$(12.5) \quad x_{\sigma\gamma\kappa} = 0/1$$

$$(12.6) \quad z_{\gamma\kappa} = 0/1$$

A detailed study of this problem will appear in a separate paper [ ]. Here we restrict ourselves to the case that the clusters are known already beforehand, hence that the variables $z_{\gamma\kappa}$ have already been assigned values 0 or 1, satisfying of course the relations (12.4). The structure of this problem is visualised in Figure 1. It is obviously of the type (1), the matrix A being a negative unit matrix corresponding to the slack variables in the class-room capacity restrictions (12.3).

Applying the upper bound relation (11), it follows that the total number of courses that are not uniquely assigned to clusters, does not exceed the number of fully occupied class-rooms.

References.

[1] J.F. Benders, J.A.E.E. van Nunen, A property of assignment type mixed integer linear programming problems (accepted for publication in Operations Research Letters).

[2] J.F. Benders, A linear programming approach to the clustering problem in class-room scheduling. (Memorandum COSOR in preparation).

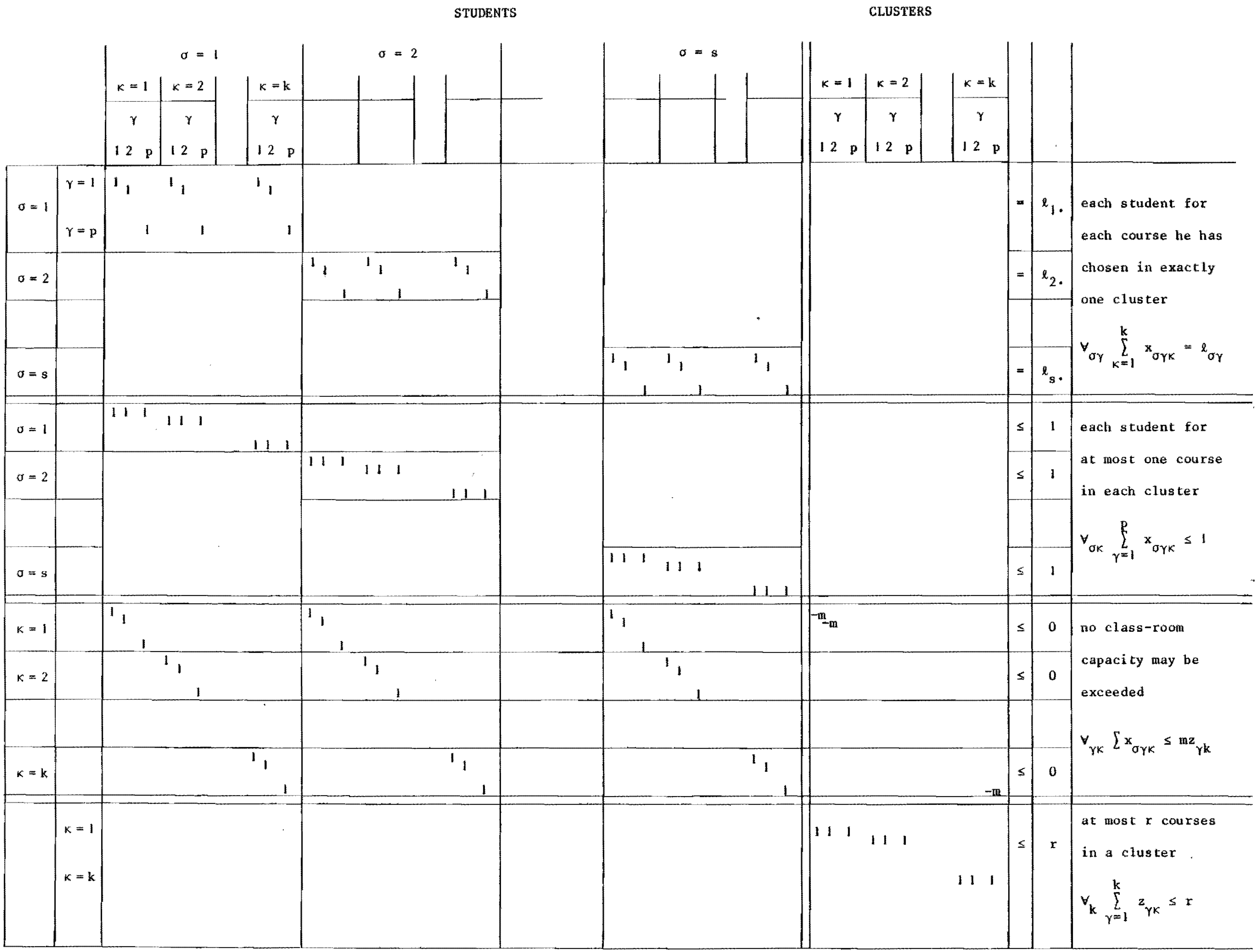

FIGURE 1. THE TECHNOLOGY MATRIX