

Bounds for expected loss in Bayesian decision theory with imprecise prior probabilities

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F.P.A. Coolen

Eindhoven, September 1992
The Netherlands

Eindhoven University of Technology
Department of Mathematics and Computing Science
Probability theory, statistics, operations research and systems theory
P.O. Box 513
5600 MB Eindhoven - The Netherlands

Secretariate: Dommelbuilding 0.03
Telephone: 040-47 3130

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**BOUNDS FOR EXPECTED LOSS IN BAYESIAN DECISION THEORY
WITH IMPRECISE PRIOR PROBABILITIES**

F.P.A. Coolen

Eindhoven University of Technology

Department of Mathematics and Computing Science

P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Abstract

Classical Bayesian inference uses the expected value of a loss function with regard to a single prior distribution for a parameter to compare decisions, and an optimal decision minimizes the expected loss. Recently interest has grown in generalizations of this framework without specified priors, to allow imprecise prior probabilities. Within the Bayesian context the most promising method seems to be the intervals of measures method.

A major problem for the application of this method to decision problems seems to be the amount of calculation required, since for each decision there is no single value for expected loss, but a set of such values corresponding to all possible prior distributions. In this report the determination of lower and upper bounds for such a set of expected loss values with regard to a single decision is discussed, and general results are derived which show that the situation is less severe than would be expected at first sight. A simple algorithm to determine these bounds is described. The choice of a decision can be based on a comparison of the bounds of the expected loss per decision.

Key words and phrases

Bayesian decision theory, imprecise probabilities, intervals of measures, lower and upper bounds for expected loss.

AMS 1980 subject classifications: Primary 62C10; secondary 62A99.

1. Introduction

The Bayesian theory of statistical inference demands a prior distribution for some parameter. The prior represents information available before experimental data became available. Walley (1991) presents a clear and extensive discussion about the major drawbacks of describing a certain amount of information, or lack of information, by a single prior distribution. This leads to a generalization of the concept in which the single prior distribution is replaced by a set of prior distributions, using a generalized concept of probability, *imprecise probabilities*. Pericchi and Walley (1991) conclude that the *intervals of measures* method, introduced by DeRobertis and Hartigan (1981), is the most suitable form for a set of prior distributions in Bayesian inference.

The idea of imprecise probabilities goes back to Boole (1854), but little attention was paid to it until the contributions of Smith (1961), who proposed an axiom system for imprecise probabilities (Wolfenson and Fine, 1982) and Good (1962). Dempster (1968) proposed a framework of statistical inference that leads to unreasonable results in some cases (Walley, 1991; section 5.13). Walley (1991) provides a thorough mathematical foundation of a coherent theory of imprecise probabilities, where the interpretation of lower and upper probabilities is in terms of betting rates, strongly related to the ideas of de Finetti (1974).

In his book Walley pays little attention to application of imprecise probabilities to statistical decision theory, in which field the Bayesian concept is attractive (Lindley, 1973). Other interesting contributions to decision theory with imprecise probabilities are provided by Fishburn (1965), Gärdenfors and Sahlin (1982,1983), Levi (1982) and Wolfenson and Fine (1982). Smith (1961) also pays attention to decision making. Berger (1985) discusses a method of sensitivity analysis within statistical decision theory, called robust Bayesian inference. This method differs from the theory discussed by Walley (1991) in that Berger does not use imprecision in probabilities as a tool to report the amount of information on which the probabilities are based, the most important reason for introducing the generalization.

While these contributions develop the theory they do not provide an effective basis for use through a method of calculation. All these methods imply an exhaustive process of evaluating a loss function at each decision

and for all possible priors.

In this report it is shown that the amount of calculation is less than expected at first sight, when the set of prior probabilities is given through intervals of measures. The results make the practical application of imprecise prior probabilities to decision problems possible. Important additional aspects for such application, e.g. introduction of suitable statistical models and assessment of imprecise prior probabilities, are discussed by Coolen (1992a,b,1993).

In section 2 of this report Bayesian decision theory with imprecise prior probabilities is introduced and the restriction to intervals of measures is suggested. In section 3 results are given for possible practical application by reducing the required amount of calculation. An algorithm for general problems is described.

2. Bayesian decision theory with imprecise prior probabilities

Decision problems (Lindley, 1990) can, with a good deal of generality, be described as follows. Let X be the sample space of points x and Θ the parameter space of points θ . These are connected by $p(x|\theta)$, the probability density of X for a given θ (with respect to some measure). Let D be the decision space of points d , and $L(d, \theta)$ the loss in selecting d when θ obtains (we assume $L(d, \theta) \in \mathbb{R}$ for all d and θ). The Bayesian approach uses a prior probability density $\pi(\theta)$ over parameter space Θ and chooses as the optimum decision that d which minimizes the expected loss

$$EL(d, x) = \int_{\Theta} L(d, \theta) p(\theta|x) d\theta, \quad (1)$$

where $p(\theta|x)$ is the posterior probability density of θ , given x , obtained by Bayes' theorem. If no data x are available, the decision can be based on the prior, in which case the optimum decision is that d which minimizes the expected loss

$$EL(d) = \int_{\Theta} L(d, \theta) \pi(\theta) d\theta. \quad (2)$$

Without loss of generality we work with (2), since the posterior at one stage is just the prior for the next (Coolen, 1992a, 1993).

Suppose that the probability density $\pi(\theta)$ is not known precisely, but is only known to belong to a space Π of probability densities. This generalization is known in literature as *imprecise (prior) probabilities* (Walley, 1991). In contrast to a hierarchical Bayesian approach we do not define a probability density over Π . Assuming the loss function $L(d, \theta)$ is known, the expected loss for $\pi \in \Pi$ and $d \in D$ is

$$EL(d, \pi) = \int_{\Theta} L(d, \theta) \pi(\theta) d\theta. \quad (3)$$

For each decision $d \in D$ this leads to a set values of expected loss, denoted by

$$\mathcal{EL}(d, \Pi) = \{EL(d, \pi) \mid \pi \in \Pi\}. \quad (4)$$

To choose an optimal decision an additional criterion is needed, by which these sets $\mathcal{EL}(d, \Pi)$ can be compared for all $d \in D$. In this direction, two useful characteristics of $\mathcal{EL}(d, \Pi)$ are the lower expectation $\underline{\mathcal{EL}}(d, \Pi)$ and the upper expectation $\overline{\mathcal{EL}}(d, \Pi)$:

$$\underline{\mathcal{EL}}(d, \Pi) = \inf \mathcal{EL}(d, \Pi) \quad \text{and} \quad \overline{\mathcal{EL}}(d, \Pi) = \sup \mathcal{EL}(d, \Pi). \quad (5)$$

As remarked by Pericchi and Walley (1991), the *intervals of measures* method of DeRobertis and Hartigan (1981) is especially useful when imprecise probabilities are used for parameters, as in the Bayesian framework. Hence, in this paper we restrict Π to the form

$$\Pi = \left\{ \pi \mid \pi(\theta) = q(\theta)/C_q \text{ with } l(\theta) \leq q(\theta) \leq u(\theta) \text{ for all } \theta \in \Theta \right. \\ \left. \text{and } C_q = \int_{\Theta} q(\theta) d\theta \right\}, \quad (6)$$

where l and u are given and $0 \leq l(\theta) \leq u(\theta)$ for all $\theta \in \Theta$. Further, in this paper we restrict to continuous l and u with $\int_{\Theta} l(\theta) d\theta > 0$ and $\int_{\Theta} u(\theta) d\theta < \infty$. If $l(\theta) = u(\theta)$ for all $\theta \in \Theta$ then Π contains exactly one probability density and the standard situation without imprecision occurs.

This report is restricted to imprecise prior probabilities. Another possible generalization of the standard Bayesian decision theory is obtained by allowing imprecise loss functions. However, if it is only known that the loss function $L(d, \theta)$ is between two functions, say $\underline{L}(d, \theta) \leq L(d, \theta) \leq \bar{L}(d, \theta)$ for all $d \in D$ and $\theta \in \Theta$, then the analysis does not become much more difficult, as (3), (4) and (5) imply that the lower bound of the set of all possible values for the expected loss must be calculated using $\underline{L}(d, \theta)$, and the upper bound using $\bar{L}(d, \theta)$.

3. Bounds for expected loss

To choose an optimal decision the sets $\mathcal{EL}(d, \Pi)$ for all $d \in D$ must be compared. An attractive simplification is derived by replacing this problem by one of comparing intervals $[\underline{\mathcal{EL}}(d, \Pi), \overline{\mathcal{EL}}(d, \Pi)]$. This reduces, for each $d \in D$, the calculation of $EL(d, \pi)$ for all $\pi \in \Pi$ to the calculation of only two values, and seems to be reasonable as $\mathcal{EL}(d, \Pi) \subset [\underline{\mathcal{EL}}(d, \Pi), \overline{\mathcal{EL}}(d, \Pi)]$ whereas no smaller closed interval exists that contains $\mathcal{EL}(d, \Pi)$. However, it is still not clear that this reduces the amount of calculation. To this point, theorem 3.1 is an important result for Π of the form (6), as it shows that $\underline{\mathcal{EL}}(d, \Pi)$ and $\overline{\mathcal{EL}}(d, \Pi)$ are determined by a subspace of Π consisting of probability densities that depend on $L(d, \theta)$, $\ell(\theta)$ and $u(\theta)$, and that can be characterized by one parameter that belongs to a parameter space that is bounded if the loss function is bounded. To derive at theorem 3.1 several additional definitions and lemmas are needed.

For each decision $d \in D$ we define

$$L_{l,d} = \inf\{L(d, \theta) \mid \theta \in \Theta\} \quad \text{and} \quad L_{u,d} = \sup\{L(d, \theta) \mid \theta \in \Theta\}, \quad (7)$$

which may be equal to $-\infty$ or $+\infty$, and a partition of the parameter space Θ (for $u \in [L_{l,d}, L_{u,d}]$)

$$\begin{aligned} \Theta_u(d, u) &= \{\theta \in \Theta \mid L(d, \theta) > u\}, & \Theta_{=u}(d, u) &= \{\theta \in \Theta \mid L(d, \theta) = u\} \quad \text{and} \\ \Theta_l(d, u) &= \{\theta \in \Theta \mid L(d, \theta) < u\}. \end{aligned} \quad (8)$$

From this point we restrict to situations where the loss function $L(d, \theta)$ is such that

$$\int_{\Theta_u(d, u)} d\theta = 0 \quad \text{for all } u \in [L_{l,d}, L_{u,d}]. \quad (9)$$

After theorem 3.1 we will discuss the situation without this restriction.

Remark that, if $\Theta = \mathbb{R}$, this restriction implies that there is no interval in Θ on which $L(d, \theta)$ is constant.

For $d \in D$ we define, for $u \in [L_{l,d}, L_{u,d}]$:

$$C_{u,d}(u) = \int_{\Theta_u(d, u)} u(\theta) d\theta + \int_{\Theta_l(d, u)} \ell(\theta) d\theta \quad \text{and} \quad (10)$$

$$C_{1,d}^{(A)} = \int_{\Theta_u(d,A)} \ell(\theta) d\theta + \int_{\Theta_1(d,A)} u(\theta) d\theta. \quad (11)$$

We define two probability densities in Π that depend through A on the above partition of Θ :

$$\pi_{d,A}^u(\theta) = \begin{cases} u(\theta)/C_{u,d}^{(A)} & \text{for } \theta \in \Theta_u(d,A) \cup \Theta_{A^c}(d,A) \\ \ell(\theta)/C_{u,d}^{(A)} & \text{for } \theta \in \Theta_1(d,A) \end{cases} \quad \text{and} \quad (12)$$

$$\pi_{d,A}^1(\theta) = \begin{cases} \ell(\theta)/C_{1,d}^{(A)} & \text{for } \theta \in \Theta_u(d,A) \cup \Theta_{A^c}(d,A) \\ u(\theta)/C_{1,d}^{(A)} & \text{for } \theta \in \Theta_1(d,A) \end{cases}. \quad (13)$$

Next we define subspaces of Π by considering the normalizing constants of original densities between ℓ and u that determine the probability densities in Π . With

$$C_\ell = \int_{\Theta} \ell(\theta) d\theta \quad \text{and} \quad C_u = \int_{\Theta} u(\theta) d\theta, \quad (14)$$

we define, for $C_\ell \leq C \leq C_u$:

$$\Pi(C) = \{ \pi \in \Pi \mid \pi(\theta) = q(\theta)/C_q \text{ with } \ell(\theta) \leq q(\theta) \leq u(\theta) \text{ for all } \theta \in \Theta \\ \text{and } C_q = \int_{\Theta} q(\theta) d\theta = C \}. \quad (15)$$

Lemma 3.1

$$\Pi = \bigcup_{C=C_\ell}^{C_u} \Pi(C) = \bigcup_{A=L_{1,d}}^{L_{u,d}} \Pi(C_{u,d}^{(A)}), \quad (16)$$

and

$$\Pi = \bigcup_{C=C_\ell}^{C_u} \Pi(C) = \bigcup_{A=L_{1,d}}^{L_{u,d}} \Pi(C_{1,d}^{(A)}). \quad (17)$$

Proof

The first equality is an obvious consequence of (15).

The second equality of (16) holds because restriction (9) and continuity of $\ell(\theta)$ and $u(\theta)$ imply that $C_{u,d}^{(A)}$ is a continuous non-increasing function of A , with $C_{u,d}^{(L_{1,d})} = C_u$ and $C_{u,d}^{(L_{u,d})} = C_\ell$.

Analogously, the second equality of (17) holds because $C_{\ell,d}^{(A)}$ is a

continuous non-decreasing function of ω , with $C_{\ell, d}(L_{1, d}) = C_{\ell}$ and $C_{\ell, d}(L_{u, d}) = C_u$. □

Lemma 3.2 is an important result for the proof of theorem 3.1.

Lemma 3.2

Let $d \in D$, and let $\omega \in [L_{1, d}, L_{u, d}]$. Then the following relation holds for all probability densities $\pi \in \Pi(C_{u, d}(\omega))$:

$$EL(d, \pi) \leq EL(d, \pi_{d, \omega}^u). \quad (18)$$

And for all $\pi \in \Pi(C_{1, d}(\omega))$:

$$EL(d, \pi) \geq EL(d, \pi_{d, \omega}^l). \quad (19)$$

Proof

We prove (18).

Let $\pi \in \Pi(C_{u, d}(\omega))$ and let $q_{\pi}(\theta)$ be a corresponding function between ℓ and u , so $\pi(\theta) = q_{\pi}(\theta)/C_{u, d}(\omega)$ with $\ell(\theta) \leq q_{\pi}(\theta) \leq u(\theta)$ for all $\theta \in \Theta$. Then

$$\begin{aligned} EL(d, \pi) &= \int_{\Theta} L(d, \theta) \pi(\theta) d\theta = \int_{\Theta} L(d, \theta) q_{\pi}(\theta) d\theta / C_{u, d}(\omega) \\ &= \int_{\Theta_u(d, \omega)} L(d, \theta) q_{\pi}(\theta) d\theta / C_{u, d}(\omega) + \int_{\Theta_l(d, \omega)} L(d, \theta) q_{\pi}(\theta) d\theta / C_{u, d}(\omega) \\ &\leq \int_{\Theta_u(d, \omega)} L(d, \theta) u(\theta) d\theta / C_{u, d}(\omega) + \int_{\Theta_l(d, \omega)} L(d, \theta) \ell(\theta) d\theta / C_{u, d}(\omega) \\ &= \int_{\Theta} L(d, \theta) \pi_{d, \omega}^u(\theta) d\theta = EL(d, \pi_{d, \omega}^u). \end{aligned}$$

To prove the inequality herein we use the fact that

$$C_{u, d}(\omega) = \int_{\Theta_u(d, \omega)} u(\theta) d\theta + \int_{\Theta_l(d, \omega)} \ell(\theta) d\theta = \int_{\Theta_u(d, \omega)} q_{\pi}(\theta) d\theta + \int_{\Theta_l(d, \omega)} q_{\pi}(\theta) d\theta,$$

$$\text{so } \int_{\Theta_u(d, \omega)} [u(\theta) - q_{\pi}(\theta)] d\theta = \int_{\Theta_l(d, \omega)} [q_{\pi}(\theta) - \ell(\theta)] d\theta,$$

which, in combination with (8), leads to

$$\begin{aligned}
& \int_{\Theta_u(d, \mathfrak{w})} L(d, \theta) u(\theta) d\theta + \int_{\Theta_l(d, \mathfrak{w})} L(d, \theta) \ell(\theta) d\theta - \int_{\Theta_u(d, \mathfrak{w})} L(d, \theta) q_\pi(\theta) d\theta - \int_{\Theta_l(d, \mathfrak{w})} L(d, \theta) q_\pi(\theta) d\theta \\
&= \int_{\Theta_u(d, \mathfrak{w})} L(d, \theta) [u(\theta) - q_\pi(\theta)] d\theta + \int_{\Theta_l(d, \mathfrak{w})} L(d, \theta) [\ell(\theta) - q_\pi(\theta)] d\theta \\
&\geq \int_{\Theta_u(d, \mathfrak{w})} \mathfrak{w} [u(\theta) - q_\pi(\theta)] d\theta + \int_{\Theta_l(d, \mathfrak{w})} L(d, \theta) [\ell(\theta) - q_\pi(\theta)] d\theta \\
&= \int_{\Theta_l(d, \mathfrak{w})} \mathfrak{w} [q_\pi(\theta) - \ell(\theta)] d\theta + \int_{\Theta_l(d, \mathfrak{w})} L(d, \theta) [\ell(\theta) - q_\pi(\theta)] d\theta \\
&= \int_{\Theta_l(d, \mathfrak{w})} [\mathfrak{w} - L(d, \theta)] [q_\pi(\theta) - \ell(\theta)] d\theta \geq 0.
\end{aligned}$$

Remark that restriction (9) is necessary.

The proof of (19) is analogous. \square

Now theorem 3.1 can be proved. This is an essential result since it shows that for a given decision $d \in D$, one need consider only probability densities of the form $\pi_{d, \mathfrak{w}}^u(\theta)$ to determine $\overline{\mathcal{E}\mathcal{L}}(d, \Pi)$, and probability densities of the form $\pi_{d, \mathfrak{w}}^l(\theta)$ to determine $\underline{\mathcal{E}\mathcal{L}}(d, \Pi)$.

We define

$$\Pi_d^u = \{ \pi_{d, \mathfrak{w}}^u \in \Pi \mid \mathfrak{w} \in [L_{l, d}, L_{u, d}] \}; \quad (20)$$

$$\mathcal{E}\mathcal{L}(d, \Pi_d^u) = \{ EL(d, \pi) \mid \pi \in \Pi_d^u \}; \quad (21)$$

$$\overline{\mathcal{E}\mathcal{L}}(d, \Pi_d^u) = \sup \mathcal{E}\mathcal{L}(d, \Pi_d^u); \quad (22)$$

$$\Pi_d^l = \{ \pi_{d, \mathfrak{w}}^l \in \Pi \mid \mathfrak{w} \in [L_{l, d}, L_{u, d}] \}; \quad (23)$$

$$\mathcal{E}\mathcal{L}(d, \Pi_d^l) = \{ EL(d, \pi) \mid \pi \in \Pi_d^l \} \quad \text{and} \quad (24)$$

$$\underline{\mathcal{E}\mathcal{L}}(d, \Pi_d^l) = \inf \mathcal{E}\mathcal{L}(d, \Pi_d^l). \quad (25)$$

Theorem 3.1

Let $d \in D$ and let the loss function $L(d, \theta)$ satisfy restriction (9), then

$$\overline{\mathcal{E}\mathcal{L}}(d, \Pi) = \overline{\mathcal{E}\mathcal{L}}(d, \Pi_d^u) \quad (26)$$

and

$$\underline{\mathcal{E}\mathcal{L}}(d, \Pi) = \underline{\mathcal{E}\mathcal{L}}(d, \Pi_d^l). \quad (27)$$

Proof

We prove (26).

$$\begin{aligned} \overline{\mathcal{E}\mathcal{L}}(d, \Pi) &= \sup \{EL(d, \pi) \mid \pi \in \Pi\} = \sup \{ \sup \{EL(d, \pi) \mid \pi \in \Pi(C)\} \mid C \in [C_l, C_u] \} = \\ &= \sup \{ \sup \{EL(d, \pi) \mid \pi \in \Pi(C_{u,d}(\vartheta))\} \mid \vartheta \in [L_{l,d}, L_{u,d}] \} = \\ &= \sup \{EL(d, \pi_{d,\vartheta}^u) \mid \vartheta \in [L_{l,d}, L_{u,d}]\} = \sup \mathcal{E}\mathcal{L}(d, \Pi_d^u) = \overline{\mathcal{E}\mathcal{L}}(d, \Pi_d^u). \end{aligned}$$

The second and third equality are based on lemma 3.1, the fourth on lemma 3.2. The other equalities follow from (4), (5), (20), (21) and (22).

The proof of (27) is analogous. □

In theorem 3.1 restriction (9) needs to be satisfied. Of course loss functions that do not satisfy this restriction can be of interest, so the above theory needs to be generalized. We first consider what happens if restriction (9) is not satisfied, so if

$$\exists \vartheta \in [L_{l,d}, L_{u,d}] \text{ such that } \int_{\Theta_{\vartheta}(d, \vartheta)} d\theta > 0. \quad (28)$$

Then the second equalities of (16) and (17) do not hold, as $C_{u,d}(\vartheta)$ and $C_{l,d}(\vartheta)$ are no longer continuous functions of ϑ . In fact, if

$$C_{u,d}(\vartheta^-) = \lim_{\vartheta \uparrow \vartheta} C_{u,d}(\vartheta) \quad \text{and} \quad C_{u,d}(\vartheta^+) = \lim_{\vartheta \downarrow \vartheta} C_{u,d}(\vartheta) \quad (29)$$

and

$$C_{l,d}(\vartheta^-) = \lim_{\vartheta \uparrow \vartheta} C_{l,d}(\vartheta) \quad \text{and} \quad C_{l,d}(\vartheta^+) = \lim_{\vartheta \downarrow \vartheta} C_{l,d}(\vartheta) \quad (30)$$

exist, then

$$C_{u,d}^{(\mathfrak{u}^-)} = \int_{\Theta_u(d,\mathfrak{u})} u(\theta) d\theta + \int_{\Theta_{\mathfrak{u}}(d,\mathfrak{u})} u(\theta) d\theta + \int_{\Theta_l(d,\mathfrak{u})} l(\theta) d\theta; \quad (31)$$

$$C_{u,d}^{(\mathfrak{u}^+)} = \int_{\Theta_u(d,\mathfrak{u})} u(\theta) d\theta + \int_{\Theta_l(d,\mathfrak{u})} l(\theta) d\theta + \int_{\Theta_{\mathfrak{u}}(d,\mathfrak{u})} l(\theta) d\theta. \quad (32)$$

$$C_{l,d}^{(\mathfrak{u}^-)} = \int_{\Theta_u(d,\mathfrak{u})} l(\theta) d\theta + \int_{\Theta_{\mathfrak{u}}(d,\mathfrak{u})} l(\theta) d\theta + \int_{\Theta_l(d,\mathfrak{u})} u(\theta) d\theta; \quad (33)$$

$$C_{l,d}^{(\mathfrak{u}^+)} = \int_{\Theta_u(d,\mathfrak{u})} l(\theta) d\theta + \int_{\Theta_l(d,\mathfrak{u})} u(\theta) d\theta + \int_{\Theta_{\mathfrak{u}}(d,\mathfrak{u})} u(\theta) d\theta. \quad (34)$$

If $\int_{\Theta_{\mathfrak{u}}(d,\mathfrak{u})} [u(\theta)-l(\theta)]d\theta > 0$, then $\pi \notin \bigcup_{\mathfrak{u}=L_{1,d}}^{L_{u,d}} \Pi(C_{u,d}^{(\mathfrak{u})})$ for $\pi \in \Pi(C)$ if

$C \in (C_{u,d}^{(\mathfrak{u}^-)}, C_{u,d}^{(\mathfrak{u}^+)})$ and $\pi \notin \bigcup_{\mathfrak{u}=L_{1,d}}^{L_{u,d}} \Pi(C_{l,d}^{(\mathfrak{u})})$ for $\pi \in \Pi(C)$ if

$C \in (C_{l,d}^{(\mathfrak{u}^-)}, C_{l,d}^{(\mathfrak{u}^+)})$, whereas $(C_{u,d}^{(\mathfrak{u}^-)}, C_{u,d}^{(\mathfrak{u}^+)})$ and $(C_{l,d}^{(\mathfrak{u}^-)}, C_{l,d}^{(\mathfrak{u}^+)})$ are a non-empty intervals. This implies that lemma 3.1 does not hold anymore.

To solve this problem, we define (for $\mathfrak{u} \in [L_{1,d}, L_{u,d}]$ and $q \in [0,1]$)

$$C_{u,d}^*(\mathfrak{u}, q) = C_{u,d}^{(\mathfrak{u})} + \int_{\Theta_{\mathfrak{u}}(d,\mathfrak{u})} [qu(\theta)+(1-q)l(\theta)]d\theta \quad (35)$$

and

$$C_{l,d}^*(\mathfrak{u}, q) = C_{l,d}^{(\mathfrak{u})} + \int_{\Theta_{\mathfrak{u}}(d,\mathfrak{u})} [ql(\theta)+(1-q)u(\theta)]d\theta. \quad (36)$$

Of course, if $\int_{\Theta_{\mathfrak{u}}(d,\mathfrak{u})} d\theta = 0$ then $C_{u,d}^*(\mathfrak{u}, q) = C_{u,d}^{(\mathfrak{u})}$ and

$C_{l,d}^*(\mathfrak{u}, q) = C_{l,d}^{(\mathfrak{u})}$ for all $q \in [0,1]$.

Further, we define

$$\begin{aligned} \Pi(C_{u,d}^*(\omega, q)) &= \{ \pi \in \Pi \mid \pi(\theta) = q(\theta)/C_q \text{ with } \ell(\theta) \leq q(\theta) \leq u(\theta) \\ &\text{for all } \theta \in \Theta, \quad C_q = \int_{\Theta} q(\theta) d\theta = C_{u,d}^*(\omega, q) \text{ and} \\ &\int_{\Theta_{\omega}(d, \omega)} q(\theta) d\theta = \int_{\Theta_{\omega}(d, \omega)} [qu(\theta) + (1-q)\ell(\theta)] d\theta \}, \end{aligned} \quad (37)$$

and

$$\begin{aligned} \Pi(C_{l,d}^*(\omega, q)) &= \{ \pi \in \Pi \mid \pi(\theta) = q(\theta)/C_q \text{ with } \ell(\theta) \leq q(\theta) \leq u(\theta) \\ &\text{for all } \theta \in \Theta, \quad C_q = \int_{\Theta} q(\theta) d\theta = C_{l,d}^*(\omega, q) \text{ and} \\ &\int_{\Theta_{\omega}(d, \omega)} q(\theta) d\theta = \int_{\Theta_{\omega}(d, \omega)} [q\ell(\theta) + (1-q)u(\theta)] d\theta \}. \end{aligned} \quad (38)$$

Lemma 3.3 is a generalized form of lemma 3.1.

Lemma 3.3

$$\Pi = \bigcup_{C=C_l}^{C_u} \Pi(C) = \bigcup_{\omega=L_{l,d}}^{L_{u,d}} \bigcup_{q=0}^1 \Pi(C_{u,d}^*(\omega, q)) \quad (39)$$

and

$$\Pi = \bigcup_{C=C_l}^{C_u} \Pi(C) = \bigcup_{\omega=L_{l,d}}^{L_{u,d}} \bigcup_{q=0}^1 \Pi(C_{l,d}^*(\omega, q)). \quad (40)$$

Proof

The second equality of (39) follows from

$$\{ C_{u,d}^*(\omega, q) \mid \omega \in [L_{l,d}, L_{u,d}], q \in [0, 1] \} = [C_l, C_u], \quad (41)$$

where again continuity of $\ell(\theta)$ and $u(\theta)$ plays a role.

The proof of (40) is analogous. \square

Definitions (12) and (13) must also be replaced in this generalized situation. Here we use the fact that we know that $L(d, \theta) = u$ for all $\theta \in \Theta_u(d, u)$, so to determine $\overline{\mathcal{E}\mathcal{L}}(d, \Pi)$ and $\underline{\mathcal{E}\mathcal{L}}(d, \Pi)$ we do not need to specify probability densities analogous to definitions (12) and (13) for all $\theta \in \Theta$. We define, for all $q \in [0, 1]$

$$\pi_{d, u, q}^{u, *}(\theta) = \begin{cases} u(\theta)/C_{u, d}^*(u, q) & \text{for } \theta \in \Theta_u(d, u) \\ l(\theta)/C_{u, d}^*(u, q) & \text{for } \theta \in \Theta_l(d, u) \end{cases}, \quad (42)$$

whereas we do not need to specify $\pi_{d, u, q}^{u, *}(\theta)$ exactly for $\theta \in \Theta_{u_0}(d, u)$, but these values need to be such that

$$\begin{aligned} \pi_{d, u, q}^{u, *}(\theta) &= q(\theta)/C_{u, d}^*(u, q) \text{ for } \theta \in \Theta_{u_0}(d, u), \text{ where } q \text{ is some density} \\ &\text{between } l \text{ and } u, \text{ so } l(\theta) \leq q(\theta) \leq u(\theta), \text{ with} \\ \int_{\Theta_{u_0}(d, u)} q(\theta) d\theta &= \int_{\Theta_{u_0}(d, u)} [qu(\theta) + (1-q)l(\theta)] d\theta. \end{aligned} \quad (43)$$

Analogously, we define, for all $q \in [0, 1]$

$$\pi_{d, u, q}^{l, *}(\theta) = \begin{cases} l(\theta)/C_{l, d}^*(l, q) & \text{for } \theta \in \Theta_l(d, u) \\ u(\theta)/C_{l, d}^*(l, q) & \text{for } \theta \in \Theta_u(d, u) \end{cases}, \quad (44)$$

and the values of $\pi_{d, u, q}^{l, *}(\theta)$ for $\theta \in \Theta_{u_0}(d, u)$ need to be such that

$$\begin{aligned} \pi_{d, u, q}^{l, *}(\theta) &= q(\theta)/C_{l, d}^*(l, q) \text{ for } \theta \in \Theta_{u_0}(d, u), \text{ where } q \text{ is some density} \\ &\text{between } l \text{ and } u, \text{ so } l(\theta) \leq q(\theta) \leq u(\theta), \text{ with} \\ \int_{\Theta_{u_0}(d, u)} q(\theta) d\theta &= \int_{\Theta_{u_0}(d, u)} [ql(\theta) + (1-q)u(\theta)] d\theta. \end{aligned} \quad (45)$$

For both (43) and (45) it is possible to derive such $q(\theta)$ for each $q \in [0, 1]$; for q we could even choose continuous functions as l and u are continuous. However, we do not need to specify these q .

Now a generalized form of lemma 3.2 is given.

Lemma 3.4

Let $d \in D$, $\mathfrak{u} \in [L_{1,d}, L_{u,d}]$, and $q \in [0, 1]$. Then the following relation holds for all probability densities $\pi \in \Pi(C_{u,d}^*(\mathfrak{u}, q))$:

$$EL(d, \pi) \leq EL(d, \pi_{d, \mathfrak{u}, q}^{u, *}). \quad (46)$$

And for all $\pi \in \Pi(C_{1,d}^*(\mathfrak{u}, q))$:

$$EL(d, \pi) \geq EL(d, \pi_{d, \mathfrak{u}, q}^{1, *}). \quad (47)$$

Proof

We prove (46).

Let $\pi \in \Pi(C_{u,d}^*(\mathfrak{u}, q))$ and let $q_\pi(\theta)$ be a corresponding function between ℓ and u , so $\pi(\theta) = q_\pi(\theta)/C_{u,d}^*(\mathfrak{u}, q)$ with $\ell(\theta) \leq q_\pi(\theta) \leq u(\theta)$ for all $\theta \in \Theta$, and

also with $\int_{\Theta_{\mathfrak{u}}(d, \mathfrak{u})} q_\pi(\theta) d\theta = \int_{\Theta_{\mathfrak{u}}(d, \mathfrak{u})} [qu(\theta) + (1-q)\ell(\theta)] d\theta$. Then

$$\begin{aligned} EL(d, \pi) &= \int_{\Theta} L(d, \theta) \pi(\theta) d\theta = \int_{\Theta} L(d, \theta) q_\pi(\theta) d\theta / C_{u,d}^*(\mathfrak{u}, q) \\ &= \int_{\Theta_u(d, \mathfrak{u})} L(d, \theta) q_\pi(\theta) d\theta / C_{u,d}^*(\mathfrak{u}, q) + \int_{\Theta_1(d, \mathfrak{u})} L(d, \theta) q_\pi(\theta) d\theta / C_{u,d}^*(\mathfrak{u}, q) \\ &\quad + \int_{\Theta_{\mathfrak{u}}(d, \mathfrak{u})} \mathfrak{u} q_\pi(\theta) d\theta / C_{u,d}^*(\mathfrak{u}, q) \\ &= \int_{\Theta_u(d, \mathfrak{u})} L(d, \theta) q_\pi(\theta) d\theta / C_{u,d}^*(\mathfrak{u}, q) + \int_{\Theta_1(d, \mathfrak{u})} L(d, \theta) q_\pi(\theta) d\theta / C_{u,d}^*(\mathfrak{u}, q) \\ &\quad + \int_{\Theta_{\mathfrak{u}}(d, \mathfrak{u})} \mathfrak{u} [qu(\theta) + (1-q)\ell(\theta)] d\theta / C_{u,d}^*(\mathfrak{u}, q) \\ &\leq \int_{\Theta_u(d, \mathfrak{u})} L(d, \theta) u(\theta) d\theta / C_{u,d}^*(\mathfrak{u}, q) + \int_{\Theta_1(d, \mathfrak{u})} L(d, \theta) \ell(\theta) d\theta / C_{u,d}^*(\mathfrak{u}, q) \\ &\quad + \int_{\Theta_{\mathfrak{u}}(d, \mathfrak{u})} \mathfrak{u} [qu(\theta) + (1-q)\ell(\theta)] d\theta / C_{u,d}^*(\mathfrak{u}, q) \\ &= \int_{\Theta} L(d, \theta) \pi_{d, \mathfrak{u}, q}^{u, *}(\theta) d\theta = EL(d, \pi_{d, \mathfrak{u}, q}^{u, *}). \end{aligned}$$

The inequality is proved analogously to the proof of lemma 3.2.

The proof of (47) is analogous. □

Now a generalized form of theorem 3.1 can be proved. We define

$$\Pi_d^{u,*} = \{ \pi_{d,\omega,q}^{u,*} \in \Pi \mid \omega \in [L_{1,d}, L_{u,d}], q \in [0,1] \}; \quad (48)$$

$$\mathcal{EL}(d, \Pi_d^{u,*}) = \{ EL(d, \pi) \mid \pi \in \Pi_d^{u,*} \}; \quad (49)$$

$$\overline{\mathcal{EL}}(d, \Pi_d^{u,*}) = \sup \mathcal{EL}(d, \Pi_d^{u,*}); \quad (50)$$

$$\Pi_d^{l,*} = \{ \pi_{d,\omega,q}^{l,*} \in \Pi \mid \omega \in [L_{1,d}, L_{u,d}], q \in [0,1] \}; \quad (51)$$

$$\mathcal{EL}(d, \Pi_d^{l,*}) = \{ EL(d, \pi) \mid \pi \in \Pi_d^{l,*} \} \quad \text{and} \quad (52)$$

$$\underline{\mathcal{EL}}(d, \Pi_d^{l,*}) = \inf \mathcal{EL}(d, \Pi_d^{l,*}). \quad (53)$$

Theorem 3.2

Let $d \in D$, then

$$\overline{\mathcal{EL}}(d, \Pi) = \overline{\mathcal{EL}}(d, \Pi_d^{u,*}), \quad (54)$$

and

$$\underline{\mathcal{EL}}(d, \Pi) = \underline{\mathcal{EL}}(d, \Pi_d^{l,*}). \quad (55)$$

Proof

We prove (54).

$$\begin{aligned} \overline{\mathcal{EL}}(d, \Pi) &= \sup \{ EL(d, \pi) \mid \pi \in \Pi \} = \sup \{ \sup \{ EL(d, \pi) \mid \pi \in \Pi(C) \} \mid C \in [C_l, C_u] \} = \\ &= \sup \{ \sup \{ EL(d, \pi) \mid \pi \in \Pi(C_{u,d}^*(\omega, q)) \} \mid \omega \in [L_{1,d}, L_{u,d}], q \in [0,1] \} = \\ &= \sup \{ EL(d, \pi_{d,\omega,q}^{u,*}) \mid \omega \in [L_{1,d}, L_{u,d}], q \in [0,1] \} = \sup \mathcal{EL}(d, \Pi_d^{u,*}) = \overline{\mathcal{EL}}(d, \Pi_d^{u,*}). \end{aligned}$$

The second and third equality are based on lemma 3.3, the fourth on lemma 3.4. The other equalities follow from (4), (5), (48), (49) and (50).

The proof of (55) is analogous. \square

The above theory suggests that, to determine $\overline{\mathcal{EL}}(d, \Pi)$ and $\underline{\mathcal{EL}}(d, \Pi)$, we need to calculate $EL(d, \pi_{d, \mathfrak{w}, q}^{u, *})$ and $EL(d, \pi_{d, \mathfrak{w}, q}^{l, *})$ for all $\mathfrak{w} \in [L_{l, d}, L_{u, d}]$ and $q \in [0, 1]$. However, (see remark after (36)), if $\int_{\Theta_{\mathfrak{w}}(d, \mathfrak{w})} d\theta = 0$ then

$EL(d, \pi_{d, \mathfrak{w}, q}^{u, *}) = EL(d, \pi_{d, \mathfrak{w}}^u)$ and $EL(d, \pi_{d, \mathfrak{w}, q}^{l, *}) = EL(d, \pi_{d, \mathfrak{w}}^l)$ for all $q \in [0, 1]$. Next we show that also for $\mathfrak{w} \in [L_{l, d}, L_{u, d}]$ with $\int_{\Theta_{\mathfrak{w}}(d, \mathfrak{w})} d\theta > 0$

it is not necessary to calculate $EL(d, \pi_{d, \mathfrak{w}, q}^{u, *})$ and $EL(d, \pi_{d, \mathfrak{w}, q}^{l, *})$ for all $q \in [0, 1]$. This will lead to theorem 3.3, a more useful result than theorem 3.2.

Lemma 3.5

Let $d \in D$, and let $\mathfrak{w} \in [L_{l, d}, L_{u, d}]$. Then, for all $q \in [0, 1]$

$$EL(d, \pi_{d, \mathfrak{w}, q}^{u, *}) \leq \sup \{ EL(d, \pi_{d, \mathfrak{w}, q}^{u, *}) \mid q \in [0, 1] \} \tag{56}$$

and

$$EL(d, \pi_{d, \mathfrak{w}, q}^{l, *}) \geq \inf \{ EL(d, \pi_{d, \mathfrak{w}, q}^{l, *}) \mid q \in [0, 1] \}. \tag{57}$$

Proof

We prove (56). Let $q \in [0, 1]$, then

$$EL(d, \pi_{d, \mathfrak{w}, q}^{u, *}) = \int_{\Theta_{\mathfrak{w}}(d, \mathfrak{w})} L(d, \theta) \pi_{d, \mathfrak{w}, q}^{u, *}(\theta) d\theta = \frac{A + \mathfrak{w}h(q)}{B + h(q)},$$

with $A = \int_{\Theta_u(d, \mathfrak{w})} L(d, \theta) u(\theta) d\theta + \int_{\Theta_l(d, \mathfrak{w})} L(d, \theta) l(\theta) d\theta;$

$$B = \int_{\Theta_u(d, \mathfrak{w})} u(\theta) d\theta + \int_{\Theta_l(d, \mathfrak{w})} l(\theta) d\theta;$$

and $h(q) = \int_{\Theta_{\mathfrak{w}}(d, \mathfrak{w})} [qu(\theta) + (1-q)l(\theta)] d\theta.$

If $\mathfrak{w} > A/B$ then the function $\frac{A + \mathfrak{w}h(q)}{B + h(q)}$ is an increasing function of $h(q)$, and since $h(q)$ is a non-decreasing function of q it follows that

$$\frac{A + \mathfrak{w}h(q)}{B + h(q)} \leq \frac{A + \mathfrak{w}h(1)}{B + h(1)} \text{ for all } q \in [0, 1]. \tag{58}$$

If $\omega < A/B$ then the function $\frac{A + \omega h(q)}{B + h(q)}$ is a decreasing function of $h(q)$ so $\frac{A + \omega h(q)}{B + h(q)} \leq \frac{A + \omega h(0)}{B + h(0)}$ for all $q \in [0,1]$. (59)

Finally, if $\omega = A/B$ then the function $\frac{A + \omega h(q)}{B + h(q)}$ is a constant function of $h(q)$.

The proof of (57) is analogous. \square

To derive a result that reduces the amount of calculation compared to theorem 3.2, we define

$$\Pi_d^{u, **} = \{ \pi_{d, \omega, q}^{u, *}, \pi \in \Pi \mid \omega \in [L_{l, d}, L_{u, d}], q \in \{0,1\} \}; \quad (60)$$

$$\mathcal{EL}(d, \Pi_d^{u, **}) = \{ EL(d, \pi) \mid \pi \in \Pi_d^{u, **} \}; \quad (61)$$

$$\overline{\mathcal{EL}}(d, \Pi_d^{u, **}) = \sup \mathcal{EL}(d, \Pi_d^{u, **}); \quad (62)$$

$$\Pi_d^{l, **} = \{ \pi_{d, \omega, q}^{l, *}, \pi \in \Pi \mid \omega \in [L_{l, d}, L_{u, d}], q \in \{0,1\} \}; \quad (63)$$

$$\mathcal{EL}(d, \Pi_d^{l, **}) = \{ EL(d, \pi) \mid \pi \in \Pi_d^{l, **} \} \quad \text{and} \quad (64)$$

$$\underline{\mathcal{EL}}(d, \Pi_d^{l, **}) = \inf \mathcal{EL}(d, \Pi_d^{l, **}). \quad (65)$$

Theorem 3.3

Let $d \in D$, then

$$\overline{\mathcal{EL}}(d, \Pi) = \overline{\mathcal{EL}}(d, \Pi_d^{u, **}), \quad (66)$$

and

$$\underline{\mathcal{EL}}(d, \Pi) = \underline{\mathcal{EL}}(d, \Pi_d^{l, **}). \quad (67)$$

Proof

Equalities (66) and (67) are based on lemma 3.5 and theorem 3.2 together with definitions (60)-(65). \square

For each $d \in D$, the problem of determining $\overline{\mathcal{EL}}(d, \Pi)$ and $\underline{\mathcal{EL}}(d, \Pi)$ can, as a result of theorem 3.3, be replaced by the problem of determining $\overline{\mathcal{EL}}(d, \Pi_d^{u, **})$ and $\underline{\mathcal{EL}}(d, \Pi_d^{l, **})$. To solve these optimization problems we must consider $EL(d, \pi_{d, \omega, q}^{u, *})$ and $EL(d, \pi_{d, \omega, q}^{l, *})$ for all $\omega \in [L_{1, d}, L_{u, d}]$ and $q \in \{0, 1\}$. We next show that these optimization problems are much simpler than they seem to be, because of the form of $EL(d, \pi_{d, \omega, q}^{u, *})$ and $EL(d, \pi_{d, \omega, q}^{l, *})$ as functions of ω .

Lemma 3.6 is an important result on which the proof of the remarkable theorem 3.4 is based. We introduce a new notation:

$$\begin{aligned} \Theta_{u\omega}(d, \omega) &= \{\theta \in \Theta \mid L(d, \theta) \geq \omega\} = \Theta_u(d, \omega) \cup \Theta_\omega(d, \omega) \quad \text{and} \\ \Theta_{l\omega}(d, \omega) &= \{\theta \in \Theta \mid L(d, \theta) \leq \omega\} = \Theta_l(d, \omega) \cup \Theta_\omega(d, \omega). \end{aligned} \quad (68)$$

Lemma 3.6

Let $d \in D$ and $\omega \in [L_{1, d}, L_{u, d}]$. For $EL(d, \pi_{d, \omega, q}^{u, *})$ the following relations hold:

If $\min\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\} \geq \omega$, then for all $\omega \in [L_{1, d}, \omega]$:

$$\max\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\} \leq \max\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\}. \quad (69)$$

If $\max\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\} \leq \omega$, then for all $\omega \in (\omega, L_{u, d}]$:

$$\max\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\} \leq \max\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\}. \quad (70)$$

And, for $EL(d, \pi_{d, \omega, q}^{l, *})$ analogous relations are:

If $\min\{EL(d, \pi_{d, \omega, q}^{l, *}) \mid q \in \{0, 1\}\} \geq \omega$, then for all $\omega \in [L_{1, d}, \omega]$:

$$\min\{EL(d, \pi_{d, \omega, q}^{l, *}) \mid q \in \{0, 1\}\} \geq \min\{EL(d, \pi_{d, \omega, q}^{l, *}) \mid q \in \{0, 1\}\}. \quad (71)$$

If $\max\{EL(d, \pi_{d, \omega, q}^{l, *}) \mid q \in \{0, 1\}\} \leq \omega$, then for all $\omega \in (\omega, L_{u, d}]$:

$$\min\{EL(d, \pi_{d, \omega, q}^{l, *}) \mid q \in \{0, 1\}\} \geq \min\{EL(d, \pi_{d, \omega, q}^{l, *}) \mid q \in \{0, 1\}\}. \quad (72)$$

Proof

We prove (69) and (70).

For the proof of (69), let $u, w \in [L_{l,d}, L_{u,d}]$ with $w < u$. We need to compare $EL(d, \pi_{d,u}^{u,*}, q_{u^*})$ with $EL(d, \pi_{d,w}^{u,*}, q_{w^*})$ for all four possible combinations of $q_{u^*} \in \{0,1\}$ and $q_{w^*} \in \{0,1\}$. Let $q_{u^*} = q_{w^*} = 0$, then the fact that the following relations hold:

$$\begin{aligned} \Theta_{u^*}(d, u) \subseteq \Theta_u(d, u), \quad \Theta_{l^*}(d, w) \subseteq \Theta_l(d, w) \quad \text{and} \\ \Theta_u(d, w) \setminus \Theta_u(d, u) = \{\theta \in \Theta \mid w < L(d, \theta) \leq u\} = \Theta_{l^*}(d, w) \setminus \Theta_{l^*}(d, u), \end{aligned} \quad (73)$$

leads to

$$\begin{aligned} EL(d, \pi_{d,w}^{u,*}, 0) \leq EL(d, \pi_{d,u}^{u,*}, 0) &\iff \\ \frac{\int_{\Theta_u(d, w)} L(d, \theta) u(\theta) d\theta + \int_{\Theta_{l^*}(d, w)} L(d, \theta) l(\theta) d\theta}{\int_{\Theta_u(d, w)} u(\theta) d\theta + \int_{\Theta_{l^*}(d, w)} l(\theta) d\theta} \leq EL(d, \pi_{d,u}^{u,*}, 0) &\iff \\ \frac{\int_{\Theta_u(d, u)} L(d, \theta) u(\theta) d\theta + \int_{\Theta_{l^*}(d, w)} L(d, \theta) l(\theta) d\theta + \int_{\Theta_u(d, w) \setminus \Theta_u(d, u)} L(d, \theta) [u(\theta) - l(\theta)] d\theta}{\int_{\Theta_u(d, u)} u(\theta) d\theta + \int_{\Theta_{l^*}(d, w)} l(\theta) d\theta + \int_{\Theta_u(d, w) \setminus \Theta_u(d, u)} [u(\theta) - l(\theta)] d\theta} & \\ \leq \frac{\int_{\Theta_u(d, u)} L(d, \theta) u(\theta) d\theta + \int_{\Theta_{l^*}(d, w)} L(d, \theta) l(\theta) d\theta}{\int_{\Theta_u(d, u)} u(\theta) d\theta + \int_{\Theta_{l^*}(d, w)} l(\theta) d\theta} &\iff \\ \frac{\int_{\Theta_u(d, w) \setminus \Theta_u(d, u)} L(d, \theta) [u(\theta) - l(\theta)] d\theta}{\int_{\Theta_u(d, w) \setminus \Theta_u(d, u)} [u(\theta) - l(\theta)] d\theta} \leq EL(d, \pi_{d,u}^{u,*}, 0). & \quad (74) \end{aligned}$$

From the fact that $L(d, \theta) \leq w$ for all $\theta \in \Theta_u(d, w) \setminus \Theta_u(d, u)$, we conclude that (74) holds if $EL(d, \pi_{d,u}^{u,*}, 0) \geq w$.

The proof is similar for the other three possible combinations of q_{u^*} and q_{w^*} ,

for which one has to replace the partitions of Θ but for which analogous relations as (74) appear. This completes the proof of (69), as

$$\begin{aligned} \max\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\} &\leq \min\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\} \\ &\leq \max\{EL(d, \pi_{d, \omega, q}^{u, *}) \mid q \in \{0, 1\}\}. \end{aligned} \quad (75)$$

The proof of (70) is quite similar to that of (69). Let $\omega, \omega \in [L_{1, d}, L_{u, d}]$ with $\omega > \omega$ and let $q_{\omega} = q_{\omega} = 0$, then analogous relations as (73) are used to prove that:

$$\begin{aligned} EL(d, \pi_{d, \omega, 0}^{u, *}) &\leq EL(d, \pi_{d, \omega, 0}^{u, *}) \quad \Leftrightarrow \\ &\frac{\int_{\Theta_u(d, \omega)} L(d, \theta) u(\theta) d\theta + \int_{\Theta_{1\omega}(d, \omega)} L(d, \theta) l(\theta) d\theta - \int_{\Theta_u(d, \omega) \setminus \Theta_u(d, \omega)} L(d, \theta) [u(\theta) - l(\theta)] d\theta}{\int_{\Theta_u(d, \omega)} u(\theta) d\theta + \int_{\Theta_{1\omega}(d, \omega)} l(\theta) d\theta - \int_{\Theta_u(d, \omega) \setminus \Theta_u(d, \omega)} [u(\theta) - l(\theta)] d\theta} \\ &\leq \frac{\int_{\Theta_u(d, \omega)} L(d, \theta) u(\theta) d\theta + \int_{\Theta_{1\omega}(d, \omega)} L(d, \theta) l(\theta) d\theta}{\int_{\Theta_u(d, \omega)} u(\theta) d\theta + \int_{\Theta_{1\omega}(d, \omega)} l(\theta) d\theta} \quad \Leftrightarrow \\ &\frac{\int_{\Theta_u(d, \omega) \setminus \Theta_u(d, \omega)} L(d, \theta) [u(\theta) - l(\theta)] d\theta}{\int_{\Theta_u(d, \omega) \setminus \Theta_u(d, \omega)} [u(\theta) - l(\theta)] d\theta} \geq EL(d, \pi_{d, \omega, 0}^{u, *}). \end{aligned} \quad (76)$$

From the fact that $\omega < L(d, \theta)$ for all $\theta \in \Theta_u(d, \omega) \setminus \Theta_u(d, \omega)$, we conclude that (76) certainly holds if $EL(d, \pi_{d, \omega, 0}^{u, *}) \leq \omega$. Again the proof is analogous for the other three possible combinations of q_{ω} and q_{ω} , for which one has to replace the partitions of Θ but for which analogous relations as (76) appear. This completes the proof of (70), again by use of (75).

The proofs of relations (71) and (72) are analogous. □

Theorem 3.4 is a useful result for the optimization process necessary to calculate $\overline{\mathcal{E}\mathcal{L}}(d, \Pi)$ and $\underline{\mathcal{E}\mathcal{L}}(d, \Pi)$, as it gives us as well a strong tool to delete a part of Θ while searching for these optima (after one calculation) as a sufficient condition for a calculated value to be such an optimum.

We define

$$\Pi_{d, \omega+}^{u, **} = \{ \pi_{d, \omega, q}^{u, *}, \pi_{d, \omega, q}^{u, **} \in \Pi \mid \omega \in [\omega, L_{u, d}], q \in \{0, 1\} \}; \quad (77)$$

$$\mathcal{E}\mathcal{L}(d, \Pi_{d, \omega+}^{u, **}) = \{ EL(d, \pi) \mid \pi \in \Pi_{d, \omega+}^{u, **} \}; \quad (78)$$

$$\overline{\mathcal{E}\mathcal{L}}(d, \Pi_{d, \omega+}^{u, **}) = \sup \mathcal{E}\mathcal{L}(d, \Pi_{d, \omega+}^{u, **}); \quad (79)$$

$$\Pi_{d, \omega-}^{u, **} = \{ \pi_{d, \omega, q}^{u, *}, \pi_{d, \omega, q}^{u, **} \in \Pi \mid \omega \in [L_{1, d}, \omega], q \in \{0, 1\} \}; \quad (80)$$

$$\mathcal{E}\mathcal{L}(d, \Pi_{d, \omega-}^{u, **}) = \{ EL(d, \pi) \mid \pi \in \Pi_{d, \omega-}^{u, **} \}; \quad (81)$$

$$\overline{\mathcal{E}\mathcal{L}}(d, \Pi_{d, \omega-}^{u, **}) = \sup \mathcal{E}\mathcal{L}(d, \Pi_{d, \omega-}^{u, **}); \quad (82)$$

$$\Pi_{d, \omega+}^{l, **} = \{ \pi_{d, \omega, q}^{l, *}, \pi_{d, \omega, q}^{l, **} \in \Pi \mid \omega \in [\omega, L_{u, d}], q \in \{0, 1\} \}; \quad (83)$$

$$\mathcal{E}\mathcal{L}(d, \Pi_{d, \omega+}^{l, **}) = \{ EL(d, \pi) \mid \pi \in \Pi_{d, \omega+}^{l, **} \}; \quad (84)$$

$$\underline{\mathcal{E}\mathcal{L}}(d, \Pi_{d, \omega+}^{l, **}) = \inf \mathcal{E}\mathcal{L}(d, \Pi_{d, \omega+}^{l, **}); \quad (85)$$

$$\Pi_{d, \omega-}^{l, **} = \{ \pi_{d, \omega, q}^{l, *}, \pi_{d, \omega, q}^{l, **} \in \Pi \mid \omega \in [L_{1, d}, \omega], q \in \{0, 1\} \}; \quad (86)$$

$$\mathcal{E}\mathcal{L}(d, \Pi_{d, \omega-}^{l, **}) = \{ EL(d, \pi) \mid \pi \in \Pi_{d, \omega-}^{l, **} \}; \quad (87)$$

$$\underline{\mathcal{E}\mathcal{L}}(d, \Pi_{d, \omega-}^{l, **}) = \inf \mathcal{E}\mathcal{L}(d, \Pi_{d, \omega-}^{l, **}); \quad (88)$$

Theorem 3.4

Let $d \in D$ and $\omega \in [L_{l,d}, L_{u,d}]$.

If $\min\{EL(d, \pi_{d,\omega,q}^{u,*}) \mid q \in (0,1)\} \geq \omega$, then

$$\overline{\mathcal{E}\mathcal{L}}(d, \Pi) = \overline{\mathcal{E}\mathcal{L}}(d, \Pi_{d,\omega+}^{u,**}). \quad (89)$$

If $\max\{EL(d, \pi_{d,\omega,q}^{u,*}) \mid q \in (0,1)\} \leq \omega$, then

$$\overline{\mathcal{E}\mathcal{L}}(d, \Pi) = \overline{\mathcal{E}\mathcal{L}}(d, \Pi_{d,\omega-}^{u,**}). \quad (90)$$

If $\min\{EL(d, \pi_{d,\omega,q}^{l,*}) \mid q \in (0,1)\} \geq \omega$, then

$$\underline{\mathcal{E}\mathcal{L}}(d, \Pi) = \underline{\mathcal{E}\mathcal{L}}(d, \Pi_{d,\omega+}^{l,**}). \quad (91)$$

If $\max\{EL(d, \pi_{d,\omega,q}^{l,*}) \mid q \in (0,1)\} \leq \omega$, then

$$\underline{\mathcal{E}\mathcal{L}}(d, \Pi) = \underline{\mathcal{E}\mathcal{L}}(d, \Pi_{d,\omega-}^{l,**}). \quad (92)$$

If $\min\{EL(d, \pi_{d,\omega,q}^{u,*}) \mid q \in (0,1)\} = \max\{EL(d, \pi_{d,\omega,q}^{u,*}) \mid q \in (0,1)\} = \omega$, then

$$\overline{\mathcal{E}\mathcal{L}}(d, \Pi) = \omega. \quad (93)$$

If $\min\{EL(d, \pi_{d,\omega,q}^{l,*}) \mid q \in (0,1)\} = \max\{EL(d, \pi_{d,\omega,q}^{l,*}) \mid q \in (0,1)\} = \omega$, then

$$\underline{\mathcal{E}\mathcal{L}}(d, \Pi) = \omega. \quad (94)$$

Proof

Relations (89)-(92) are based on theorem 3.3, lemma 3.6 and definitions (77)-(88). Relation (93) is a result of combination of (89), (90) and lemma 3.6, and (94) of combination of (91), (92) and lemma 3.6. \square

For a special loss function $L(d, \theta)$ corollary 3.1 is a useful result of theorem 3.4.

Corollary 3.1

Let $d \in D$ and let $L(d, \theta)$ be continuous, bounded (so $L_{1,d} > -\infty$ and $L_{u,d} < \infty$) and such that restriction (9) holds. As assumed before, let $\ell(\theta)$ and $u(\theta)$ also be continuous.

Then there exist $\omega_0, \omega_1 \in [L_{1,d}, L_{u,d}]$ such that

$$\overline{\mathcal{E}\mathcal{L}}(d, \Pi) = EL(d, \pi_{d, \omega_0}^u) = \omega_0 \quad \text{and} \quad (95)$$

$$\underline{\mathcal{E}\mathcal{L}}(d, \Pi) = EL(d, \pi_{d, \omega_1}^l) = \omega_1. \quad (96)$$

Proof

We prove (95).

Let $\omega_0 \in [L_{1,d}, L_{u,d}]$, then, as restriction (9) holds, theorem 3.4 states that a sufficient condition for $\overline{\mathcal{E}\mathcal{L}}(d, \Pi) = \omega_0$ is $EL(d, \pi_{d, \omega_0}^u) = \omega_0$.

For $\omega \in [L_{1,d}, L_{u,d}]$:

$$EL(d, \pi_{d, \omega}^u) - \omega = 0 \quad \Leftrightarrow$$

$$k_1(\omega) = \int_{\Theta_u(d, \omega)} [L(d, \theta) - \omega] u(\theta) d\theta = \int_{\Theta_l(d, \omega)} [\omega - L(d, \theta)] \ell(\theta) d\theta = k_2(\omega). \quad (97)$$

In (97) $k_1(\omega)$ is a continuous non-increasing function of ω , with

$$k_1(L_{1,d}) = \int_{\Theta} [L(d, \theta) - L_{1,d}] u(\theta) d\theta \geq 0 \quad \text{and} \quad k_1(L_{u,d}) = 0, \quad \text{and} \quad k_2(\omega) \text{ is a}$$

continuous non-decreasing function of ω , with $k_2(L_{1,d}) = 0$ and

$$k_2(L_{u,d}) = \int_{\Theta} [L_{u,d} - L(d, \theta)] \ell(\theta) d\theta \geq 0. \quad \text{This implies that there is at least}$$

one $\omega_0 \in [L_{1,d}, L_{u,d}]$ for which $k_1(\omega_0) = k_2(\omega_0)$, which proves (95).

The proof of (96) is analogous. □

The theory in this section allows us to describe a simple algorithm to calculate $\underline{\mathcal{E}\mathcal{L}}(d, \Pi)$ and $\overline{\mathcal{E}\mathcal{L}}(d, \Pi)$ for a decision $d \in D$, if Π is of the form (6). Hereto, we assume that D , Θ and $L(d, \theta)$ are known ($d \in D$, $\theta \in \Theta$) and that Π is of the form (6) with continuous $\ell(\theta)$ and $u(\theta)$ given, such that $\int_{\Theta} \ell(\theta) d\theta > 0$ and $\int_{\Theta} u(\theta) d\theta < \infty$.

If Θ is bounded then it is possible to define a partition $\{\Theta_1, \dots, \Theta_n\}$ of Θ such that all Θ_i have equal measure (so $\int_{\Theta_i} d\theta = \int_{\Theta_j} d\theta$ for all

$i, j \in \{1, \dots, n\}$). The integral over Θ_i for a function of interest is approximated by the value of that function in one point $\theta_i \in \Theta_i$ multiplied by $\int_{\Theta_i} d\theta$. If n is large this approximation will be reasonable. If Θ is

unbounded, we must define bounds such that the integral over the bounded subspace is a good approximation of the integral over Θ . This is a standard method of numerical integration.

Quite informally an algorithm for the determination of $\overline{\mathcal{E}\mathcal{L}}(d, \Pi)$ for a decision $d \in D$ can be described by the following steps (given d):

1. Determine $L_{l,d}$ and $L_{u,d}$ (in many cases approximations can be used).

2. Take a $\omega \in [L_{l,d}, L_{u,d}]$. Calculate $\int_{\Theta} L(d, \theta) \pi_{d, \omega, q}^{u, *}(d, \theta) d\theta$, if necessary

numerically by using a partition $\{\Theta_i | i=1, \dots, n\}$ as described above, calculating $L(d, \theta_i)$ for some $\theta_i \in \Theta_i$ for all i , multiplying this by $u(\theta_i)$ if $L(d, \theta_i) > \omega$ or by $\ell(\theta_i)$ if $L(d, \theta_i) < \omega$, and approximating $EL(d, \pi_{d, \omega, q}^{u, *})$ by the sum of these terms divided by the sum of the $u(\theta_i)$ and $\ell(\theta_i)$ used herein. If $L(d, \theta_i) = \omega$, then we have to perform two calculations, one using $u(\theta_i)$ and one using $\ell(\theta_i)$, and afterwards take the largest of the corresponding results.

3. Compare the (approximate) value of $\int_{\Theta} L(d, \theta) \pi_{d, \omega, q}^{u, *}(d, \theta) d\theta$ to ω and use the

results of theorem 3.4 to determine whether these steps must be repeated using a new value $\omega \in (\omega, L_{u,d})$ or $\omega \in [L_{l,d}, \omega)$, or the calculated value is the maximum we are looking for (or a good approximation to it). If the maximum is not yet found, repeat steps 2 and 3 until the maximum is found (or an approximation).

During this calculation of $\overline{\mathcal{E}\mathcal{L}}(d, \Pi)$ we can also perform the necessary steps to calculate $\underline{\mathcal{E}\mathcal{L}}(d, \Pi)$.

To end this section, we present a simple example of the above theory.

Example 3.1

Let, for a given decision $d \in D$, the loss function be equal to $L(d, \theta) = \theta(1-\theta)$, with $\theta \in [0, 1]$. We want to determine, for this decision d , $\underline{\mathcal{E}}\mathcal{L}(d, \Pi)$ and $\overline{\mathcal{E}}\mathcal{L}(d, \Pi)$, with Π given according to (6), with $\ell(\theta) = 1$ and $u(\theta) = 2$ for all $\theta \in [0, 1]$. As $L(d, \theta)$ is not constant on an interval of positive length, this loss function satisfies restriction (9) and theorem 3.1 holds. Therefore, we can restrict to probability densities of the form (12) to determine $\overline{\mathcal{E}}\mathcal{L}(d, \Pi)$ and of the form (13) to determine $\underline{\mathcal{E}}\mathcal{L}(d, \Pi)$.

For this example, we have $L_{1,d} = 0$, $L_{u,d} = 1/4$ and, for $\omega \in [0, 1/4]$,

$$\Theta_u(d, \omega) = ((1-\sqrt{1-4\omega})/2, (1+\sqrt{1-4\omega})/2),$$

$$\Theta_1(d, \omega) = [0, (1-\sqrt{1-4\omega})/2) \cup ((1+\sqrt{1-4\omega})/2, 1] \text{ and}$$

$$\Theta_\omega(d, \omega) = \{(1-\sqrt{1-4\omega})/2, (1+\sqrt{1-4\omega})/2\}, \text{ from which it is clear that indeed restriction (9) holds.}$$

Further, $C_{u,d}(\omega) = 1+\sqrt{1-4\omega}$ and $C_{1,d}(\omega) = 2-\sqrt{1-4\omega}$. Using theorem 3.1 we

can determine $\overline{\mathcal{E}}\mathcal{L}(d, \Pi)$ by maximization of $EL(d, \pi_{d,\omega}^u) = \frac{1 + (1+2\omega)\sqrt{1-4\omega}}{6(1+\sqrt{1-4\omega})}$,

leading to $\overline{\mathcal{E}}\mathcal{L}(d, \Pi) = 0.1875$ (for $\omega=0.1875$) and $\underline{\mathcal{E}}\mathcal{L}(d, \Pi)$ by minimization of

$$EL(d, \pi_{d,\omega}^1) = \frac{2 - (1+2\omega)\sqrt{1-4\omega}}{6(2-\sqrt{1-4\omega})}, \text{ leading to } \underline{\mathcal{E}}\mathcal{L}(d, \Pi) = 0.1435 \text{ (for}$$

$\omega=0.1435$).

The fact that the optima are equal to the values of ω for which these optima are adopted is in agreement with theorem 3.4.

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