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solvability of the linear-quadratic control
problem without stability

by

A.H.W. Geerts

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

A NECESSARY AND SUFFICIENT CONDITION FOR SOLVABILITY
OF THE LINEAR-QUADRATIC CONTROL PROBLEM
WITHOUT STABILITY

ABSTRACT

In this short paper a necessary and sufficient
condition is given for the existence of a minimizing
control for the regular non-negative definite
linear-quadratic control problem without stability.

KEYWORDS

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1. Introduction

In order to solve the regular non-negative definite
linear-quadratic control problem without stability, one commonly
preassumes stabilizability of the system Σ, described by the
quadruple (A, B, C, D) ([1, Props. 9, 10]). The sufficiency of
(A, B)-stabilizability for the existence of the optimal input
being known ([1]), it is clear that this condition is not
necessary (take for instance a system with transfer function
identically zero).
Here we will show that for every initial state the infimum of
the cost criterion over the space of locally square integrable
inputs is finite if and only if $(\bar{A}_0, \bar{B})$ is stabilizable, where
$\bar{A}_0$ and $\bar{B}$ are the quotient maps of $A_0 = A - B(D'D)^{-1}D'C$ and B
w.r.t. the quotient space $\mathbb{R}^n/_{\mathcal{V}(\Sigma)}$ ($\mathbb{R}^n$ denotes the n-dimensional

state space). The linear subspace $\mathcal{V}(\Sigma)$ is the weakly
unobservable subspace (also called the output nulling subspace,
see e.g. [2, Def. 3.8] and [1]). In addition, it will turn out
that in case of $(\bar{A}_0, \bar{B})$-stabilizability the optimal control
exists and is unique.
Moreover it will follow directly from the foregoing that the
Algebraic Riccati Equation has a positive semi-definite solution
if and only if $(\bar{A}_0, \bar{B})$ is stabilizable. This result is of
interest since most articles dealing with these equations start
from sufficient conditions for the existence of a solution.
The condition presented here indeed covers such pathological
cases as the one mentioned above.

2. Preliminaries

Consider the finite-dimensional linear time-invariant system Σ:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{2.1a}$$

$$y = Cx + Du, \tag{2.1b}$$

with D left invertible, and the quadratic cost criterion

$$J(x_0, u) = \int_0^\infty y'y \, dt \quad . \tag{2.2}$$

Here $x(t) \in \mathbb{R}^n$ and $u(t)$, $y(t)$ are m-, r-dimensional real vectors, respectively. If $\ell_{2,loc}^m(\mathbb{R}^+)$ denotes the space of m-vectors whose components are locally square integrable over $\mathbb{R}^+$, then we state the non-negative definite linear-quadratic control problem without stability (LQCP)$^-$ as follows.

Define for every x_0

$$J^-(x_0) := \inf\{J(x_0, u) \mid u \in \ell_{2,loc}^m(\mathbb{R}^+)\} \quad . \tag{2.3}$$

(LQCP)$^-$: Determine for every initial state $J^-(x_0)$ such that it
       is finite

       and compute an optimal control, if it exists.

We will call (LQCP)$^-$ solvable if (2.3) is finite for all x_0.
Indeed it is well known (see e.g. [1]) that in case of
(A, B)-stabilizability the optimal control actually exists and
is unique. More precisely, if (A, B) is stabilizable then it
holds that

$$J^-(x_0) = x_0'K^-x_0 \tag{2.4}$$

with the real symmetric matrix K^- being the smallest positive
semi-definite solution of the Algebraic Riccati Equation

$$0 = C'C + A'K + KA - (KB + C'D)(D'D)^{-1}(B'K + D'C). \tag{2.5}$$

Furthermore, the optimal control exists for all x_0 and it is
given by the feedback law

$$u = -(D'D)^{-1}(B'K^- + D'C)x \quad . \tag{2.6}$$

See also [2], [5], [6].

In this paper we will "replace" the (A, B)-stabilizability by a "reduced order" stabilizability assumption that will prove to be both sufficient and necessary for solvability of (LQCP)⁻. In addition, we will see that we only have to deal with a "reduced order" Riccati Equation instead of with (2.5): our equation will be of dimension $(n - \dim(V(\Sigma))) \leq n$

We will need the next definitions. If $A: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and f is an A-invariant subspace, then we introduce the <u>induced</u> <u>map</u> $(\bar{X} := \mathbb{R}^n/_f)$ $\bar{A} : \bar{X} \to \bar{X}$ defined by

$$\bar{A} \bar{x} := \overline{Ax}$$

with $\bar{x} = x + f$.

Analogously, if B is a linear map from input space to state space, then we define the map $\bar{B}$ from input space to quotient space $\bar{X}$ by $\bar{B} u := \overline{Bu}$.

Finally, the weakly unobservable subspace $V(\Sigma)$ turns out to play an important role. We recall that it is the subspace of points x_0 for which there exists a smooth input such that the resulting output is identically zero ([2, Def. 3.8], [3, Def. 3.6]).

Our result now runs as follows. First we show that $V(\Sigma)$ is A_0-invariant with $A_0 = A - B(D'D)^{-1}D'C$. Thus $\bar{A}_0$ and $\bar{B}$ are defined w.r.t. $\mathbb{R}^n/_{V(\Sigma)}$ and our claim then reads:

(LQCP)⁻ is solvable $\iff$ $(\bar{A}_0, \bar{B})$ is stabilizable.

3. The result

We start with computing $V(\Sigma)$. We split up the output y in the following way. Write $D = UG$, with U left orthogonal and G invertible (Gram-Schmidt). If U_c is left orthogonal and such that $[U, U_c]$ is both orthogonal and invertible, then for $y_1 := U'y$ and $y_2 := U_c'y$ we find

$$y_1 = U'Cx + Gu, \qquad (3.1a)$$
$$y_2 = U_c'Cx \quad . \qquad (3.1b)$$

Applying the preliminary feedback

$$u = - G^{-1}U'Cx + v \qquad (3.2)$$

in (2.1a) and (3.1a) then yields

$$\dot{x} = A_0x + Bv \, , \, x_0 \, , \qquad (3.3a)$$
$$y_1 = Gv \, , \qquad (3.3b)$$
$$y_2 = U_c'Cx \, , \qquad (3.3c)$$

with $A_0 = A - BG^{-1}U'C = A - B(D'D)^{-1}D'C$. $\qquad (3.3d)$

Lemma.

$$V(\Sigma) = \langle \, \ker(C_0) \, | \, A_0 \, \rangle \qquad (3.4)$$

with $C_0 = (I - D(D'D)^{-1}D')C$. $\qquad (3.5)$

Proof. If $y_1 = 0$ and $y_2 = 0$ then $x_0 \in \langle \, \ker(U_c'C) \, | \, A_0 \, \rangle$ and conversely. Then the claim follows from the observation that $\ker(U_c') = \ker(U_c U_c')$ and $U_c U_c' = (I - UU') = (I - D(D'D)^{-1}D')$.

Corollary.

$V(\Sigma)$ is A_0-invariant.

Then define $\bar{X} := \mathbb{R}^n / V(\Sigma)$ and let $\bar{A}_0$, $\bar{B}$ be the induced maps as defined above.

THEOREM.

(LQCP)⁻ is solvable $\iff$ $(\bar{A}_0, \bar{B})$ is stabilizable.

Proof. Consider (3.3) and let X_2 be a subspace such that $V(\Sigma) \oplus X_2$ = $\mathbb{R}^n$. Then the equations in (3.3) transform into

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{011} & A_{012} \\ 0 & A_{022} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v , \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} , \tag{3.6a}$$

$$y_1 = Gv , \tag{3.6b}$$

$$y_2 = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \tag{3.6c}$$

with (C_2, A_{022}) observable. In addition,

$$J(x_0, u) = \int_0^\infty [v'G'Gv + x_2'C_2'C_2x_2]dt . \tag{3.6d}$$

Hence we establish from (3.6) that for solving (LQCP)⁻ we may confine ourselves to the subsystem

$$\dot{x}_2 = A_{022}x_2 + B_2v, \quad x_{02} , \tag{3.7a}$$

$$y_1 = Gv , \quad y_2 = C_2x_2 . \tag{3.7b}$$

Now assume that (A_{022}, B_2) is stabilizable. Then the solution of (LQCP)⁻ runs in the same manner as in (2.4)-(2.6):

$$J^-(x_0) = x_{02}'K_{22}^-x_{02} \tag{3.8}$$

with K_{22}^- the smallest positive semi-definite solution of the reduced order Riccati Equation (dimension $n - \dim(V(\Sigma))$

$$0 = C_2'C_2 + A_{022}'K_{22} + K_{22}A_{022} +$$
$$- K_{22}B_2(D'D)^{-1}B_2'K_{22} . \tag{3.9}$$

Moreover, $x_{02}'K_{22}^-x_{02} = x_0'K^-x_0$ and the optimal control u^- is given by the feedback

$$u^- = - (D'D)^{-1}D'Cx + v^- \tag{3.10}$$

with $v^- = - (D'D)^{-1}B_2'K_{22}^-x_2 = - (D'D)^{-1}B'K^-x$, recall (3.2). Also, due to the (C_2, A_{022})-observability, K_{22}^- is the only positive semi-definite solution of (3.9) and the resulting closed-loop matrix for x_2,

$$A_{022}^- := A_{022} - B_2(D'D)^{-1}B_2'K_{22}^- , \tag{3.11}$$

is asymptotically stable ([1], [2], [5]). Thus the optimal x_2, x_2^-, tends to zero for large t. Observe that even $K_{22}^- > 0$.

Conversely, suppose that for every x_0 a control $u*$ exists such that $J(x_0, u*) < \infty$.

Then instead of the infinite horizon case consider the finite horizon problem $(T > 0)$

Find

$$J(x_0, T) := \inf\{J(x_0, u, T) \mid u \in \mathcal{L}_2^m([0, T])\} ,$$

and compute, if it exists, the optimal input.

Here, of course, $J(x_0, u, T) = \int_0^T y'y \, dt$. For every T the solution of this problem is ([5], [6])

$$J(x_0, T) = x_{02}'K_{22}(T)x_{02}$$

with $K_{22}(T)$ the solution of the Riccati Differential Equation

$$\dot{K}_{22}(t) = C_2'C_2 + A_{022}'K_{22}(t) + K_{22}(t)A_{022} +$$
$$- K_{22}(t)B_2(D'D)^{-1}B_2'K_{22}(t) ,$$

$$K_{22}(0) = 0 ,$$

see (3.7). Now

$$x_{02}'K_{22}(T)x_{02} \leq J(x_0, u*, T) \leq J(x_0, u*)$$

and since $K_{22}(t) \geq 0$ $(t \geq 0)$ and non-decreasing for increasing t, we may conclude that $\tilde{K}_{22} := \lim_{T \to \infty} K_{22}(T)$ exists. This means that $\tilde{K}_{22}$ is a positive semi-definite solution of (3.9). Now let K_{22} be an arbitrary solution of (3.9). Then, by completing the square, one easily establishes that ((3.6))

$$J(x_0, u, T) = \int_0^T [v'G'Gv + x_2'C_2'C_2x_2]dt =$$
$$\int_0^T [v' + x_2'K_{22}B_2(D'D)^{-1}](D'D)[v + (D'D)^{-1}B_2'K_{22}x_2]dt$$
$$+ x_{02}'K_{22}x_{02} - x_2'(T)K_{22}x_2(T) .$$

Thus, for $u = -(D'D)^{-1}D'Cx + v$ with $v = -(D'D)^{-1}B_2'K_{22}x_2$, $J(x_0, u, T) = x_{02}'K_{22}x_{02} - x_2'(T)K_{22}x_2(T) \geq x_{02}'K_{22}(T)x_{02}$ from which we conclude:

(a) the feedback law with $K_{22} = \tilde{K}_{22}$ is optimal for (LQCP)⁻;

(b) if K_{22} is an arbitrary positive semi-definite solution of (3.9), then $K_{22} \geq \tilde{K}_{22}$ $(x_2'(T)K_{22}x_2(T)$ always $\geq 0)$.

Hence we again conlude from the (C_2, A_{022})-observability that there is a feedback law for v such that the resulting closed-loop matrix for x_2 is asymptotically stable (namely $v = -(D'D)^{-1}B_2'\tilde{K}_{22}x_2)$. Or, in other words, (A_{022}, B_2) is stabilizable.

This completes the proof (I am indebted to Professor M.L.J. Hautus for the last part).

COROLLARY 1.

Let $(\bar{A}_0, \bar{B})$ be stabilizable. Then
$$J^-(x_0) = \min\{J(x_0, u) \mid u \in \mathcal{L}^m_{2,loc}(\mathbb{R}^+), \ u \text{ such that}$$
$$(x/_{V(\Sigma)})(\omega) = 0\} .$$

Proof. See the first part of the proof for the Theorem.

COROLLARY 2.

There exists a positive semi-definite solution of the Algebraic Riccati Equation (2.5) $\iff$ $(\bar{A}_0, \bar{B})$ is stabilizable.

Proof. To start, observe that every solution K_{22} of (3.9) corresponds to a solution K of (2.5) with matrix representation
$$\begin{bmatrix} 0 & 0 \\ 0 & K_{22} \end{bmatrix}$$ w.r.t. the new basis (see (3.6)).

Hence if $(\bar{A}_0, \bar{B})$ is stabilizable, then $\begin{bmatrix} 0 & 0 \\ 0 & K_{22}^- \end{bmatrix}$, with K_{22}^- the unique positive definite solution of (3.9), corresponds to a positive semi-definite solution of (2.5).

Conversely, for every real symmetric K satisfying (2.5) we may
write for the finite horizon cost criterion (see the second part
of the proof for the Theorem)

$$J(x_0, u, T) = \int_0^T [u' + x'(C'D + KB)(D'D)^{-1}](D'D) \cdot$$

$$[u + (D'D)^{-1}(B'K + D'C)x]dt +$$

$$x_0'Kx_0 - x'(T)Kx(T) ,$$

from which we conclude that for $K \geq 0$ there is an input u* such
that $J(x_0, u*) < \infty$ (namely $u* = - (D'D)^{-1}(B'K + D'C)x$; note
again that $x'(T)Kx(T)$ is always ≥ 0). But then, by following the
line of the second part of the proof for the Theorem, we
establish that $(\overline{A}_0, \overline{B})$ is stabilizable.

Remarks.

1.   Corollary 1 is also proven in [4, Prop. 3.3].

2.   Note that the smallest positive semi-definite solution K^-
      of (2.5) has w.r.t. the new basis the matrix representation
      $\begin{bmatrix} 0 & 0 \\ 0 & K_{22}^- \end{bmatrix}$ since $V(\Sigma) = \ker K^-$ (K_{22}^- is the unique positive
      definite solution of (3.9)).

3   For standard problems (C'D = 0) it follows that $V(\Sigma) =$
      $< \ker(C) | A >$ and $A_0 = A$. Thus $\overline{X} = \mathbb{R}^n /_{< \ker(C) | A >}$ and the
      conditon in the Theorem becomes: $(\overline{A}, \overline{B})$ is stabilizable.

4.   From Corollary 2 we establish as a by-result that (2.5) has
      both a positive semi-definite and a negative semi-definite
      solution if and only if $(\overline{A}_0, \overline{B})$ is controllable. Note that
      indeed even for a Riccati Equation of the form

          $0 = A'K + KA - KBB'K$

      (which has both a solution ≥ 0 and ≤ 0, namely $K \equiv 0$) this
      condition is trivially satisfied.

5. The Theorem stated in this article for a regular (LQCP)⁻ (ker(D) = 0) can be generalized for the singular (LQCP)⁻, i.e. the case that ker(D) ≠ 0. Using the terminology in [3], we have the following.

If $V_d(\Sigma) = V(\Sigma) + W(\Sigma)$, where the latter subspace is defined in [3, Def. 3.3], and $\bar{X} = \mathbb{R}^n/V_d(\Sigma)$, then the necessary and sufficient condition for solvability of the (LQCP)⁻ with D not left invertible is

$$(\bar{A}_{\alpha_D}, \bar{B}_{\alpha_D}) \text{ is stabilizable} \quad .$$

Here the matrices A_{α_D} and $\bar{B}_{\alpha_D}$ are yielded by the generalized dual structure algorithm, see [3, sec. 4].

Conclusion.

A necessary and sufficient condition for solvability of the
linear-quadratic control problem without stability has been
derived. This condition is given in terms of the original system
coefficients and is furthermore necessary and sufficient for the
existence of a positive semi-definite solution of the Algebraic
Riccati Equation.

References.

[1]   J.C. Willems, A. Kitapçi & L.M. Silverman, "Singular
      optimal control: a geometric approach", SIAM J. Contr. &
      Opt., vol. 24, pp. 323 -337, 1986.

[2]   M.L.J. Hautus & L.M. Silverman, "System structure and
      singular control", Lin. Alg. & Appl., vol. 50, pp. 369 -
      402, 1983.

[3]   A.H.W. Geerts, "All optimal controls for the singular
      linear-quadratic problem without stability; a new
      interpretation of the optimal cost", Memorandum COSOR
      87-14, Eindhoven University of Technology, 1987, submitted
      for publication.

[4]   A.H.W. Geerts, "Continuity properties of the cheap control
      problem without stability", Memorandum COSOR 87-17,
      Eindhoven University of Technology, 1987, submitted for
      publication.

[5]   H. Kwakernaak & R. Sivan, Linear Optimal Control Systems,
      Wiley, New York, 1972.

[6]   B.D.O. Anderson & J.B. Moore, Linear Optimal Control,
      Prentice-Hall, Englewood Cliffs, New Jersey, 1971.