

# On the construction of a (0,2)-interpolating deficient quintic spline function

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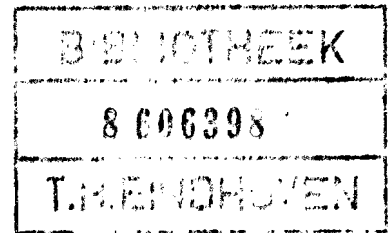
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On the construction of a  $(0,2)$ -interpolating  
deficient quintic spline function

by

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### Abstract

Given an arbitrary odd positive integer  $n$  and arbitrary data  $\{y_\nu\}_{\nu=0}^n$ ,  $\{y_\nu''\}_{\nu=0}^n$ ,  $y_0'''$ ,  $y_n'''$ , Meir and Sharma [2] proved that there exists a unique deficient quintic spline function  $s \in C^3[0,n]$ , with nodes at  $0,1,2,\dots,n$ , such that  $s(\nu) = y_\nu$  ( $\nu = 0,1,\dots,n$ ),  $s''(\nu) = y_\nu''$  ( $\nu = 0,1,\dots,n$ ),  $s'''(0) = y_0'''$  and  $s'''(n) = y_n'''$ . Using generating functions we give a simple construction of this deficient quintic spline.

## 1. Introduction

Let there be given an arbitrary positive integer  $n$ . A deficient quintic spline function  $s(x)$ , defined on the interval  $[0, n]$ , is a function satisfying the following conditions:

- a)  $s \in C^3[0, n]$ ;
- b) In each interval  $[v, v+1]$  ( $v = 0, 1, \dots, n-1$ )  $s$  is a polynomial of degree at most five.

In [2] Meir and Sharma proved that under various boundary conditions there exists a unique deficient quintic spline function, that interpolates a given function  $f$  and its second derivative at the nodes  $0, 1, \dots, n$ . This kind of interpolation is called  $(0, 2)$ -interpolation. The method of proof in [2] can be used to derive an algorithm for the evaluation of the deficient spline on the basis of the given data. This algorithm involves the solution of a  $(2n) \times (2n)$  system of linear equations.

The calculation of the deficient quintic spline we propose in this note is based on a method due to Greville [1], who uses generating functions for the construction of a natural cubic spline function interpolating a given function at equidistant nodes. We remark that Greville's method is also used by Metz [3] for the calculation of an interpolating spline function of arbitrary odd degree. The advantage of the method prescribed in this note, as compared to the one suggested in [2], is that the deficient quintic spline can be obtained explicitly, without the necessity of solving a  $(2n) \times (2n)$  system of linear equations.

## 2. The Euler-Fröbenius polynomials

In what follows we will need a class of polynomials that is well known in the theory of spline approximation. These polynomials, denoted by  $\Pi_p(t)$ , can be defined by the expansion

$$(2.1) \quad \Pi_p(t) = (1 - t)^{p+1} \sum_{j=0}^{\infty} (1 + j)^p t^j \quad (|t| < 1; p = 0, 1, 2, \dots) .$$

It is easy to verify that they satisfy the recurrence relation

$$(2.2) \quad \Pi_{p+1}(t) = (1 + pt)\Pi_p(t) + t(1 - t)\Pi'_p(t) ,$$

with  $\Pi_0(t) = 1$ .

From this relation one deduces that

$$\begin{aligned} \Pi_0(t) &= 1 , & \Pi_3(t) &= t^2 + 4t + 1 , \\ \Pi_1(t) &= 1 , & \Pi_4(t) &= t^3 + 11t^2 + 11t + 1 , \\ \Pi_2(t) &= t + 1 , & \Pi_5(t) &= t^4 + 26t^3 + 66t^2 + 26t + 1 . \end{aligned}$$

The polynomials  $\Pi_p(t)$  are called the Euler-Fröbenius polynomials (cf. [4], p. 22).

### 3. Statement of the problem

Given a set of  $2n + 4$  real numbers  $y_0, y_1, \dots, y_n, y_0'', \dots, y_n'', y_0''', y_n'''$ , the problem is to determine a deficient quintic spline function  $s$  in such a way that

$$(3.1) \quad \left\{ \begin{array}{l} \text{i) } s(v) = y_v \quad (v = 0, 1, \dots, n), \\ \text{ii) } s''(v) = y_v'' \quad (v = 0, 1, \dots, n), \\ \text{iii) } s'''(0) = y_0''' , \\ \text{iv) } s'''(n) = y_n''' . \end{array} \right.$$

Assuming that  $n$  is an odd positive integer, it is established in [2] that the above problem (3.1) has a unique solution. As is remarked in [2], the boundary conditions iii) and iv) can be replaced by similar ones, without destroying the existence and uniqueness of the solution. Our method of construction as will be developed in section 4, also applies to these cases.

### 4. Construction of the deficient quintic spline function

It is clear that every deficient quintic spline function satisfying  $s(0) = y_0$ ,  $s''(0) = y_0''$  and  $s'''(0) = y_0'''$  can be represented in the form

$$(4.1) \quad s(x) = y_0 + \frac{1}{2}y_0''x^2 + \frac{1}{6}y_0'''x^3 + \gamma x + \sum_{j=0}^{n-1} (\alpha_j(x-j)_+^4 + \beta_j(x-j)_+^5) ,$$

where, as usual, the truncated power function  $z_+^m$  is defined by

$$z_+^m = \begin{cases} z^m & (z \geq 0) , \\ 0 & (z < 0) . \end{cases}$$

Using (4.1) the second derivative of  $s(x)$  is given by

$$(4.2) \quad s''(x) = y_0'' + y_0'''x + 12 \sum_{j=0}^{n-1} \alpha_j (x-j)_+^2 + 20 \sum_{j=0}^{n-1} \beta_j (x-j)_+^3 .$$

Now our problem is to determine the parameters  $\gamma, \alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1}$  in such a way that

$$\begin{cases} s(v) = y_v & (v = 1, \dots, n) , \\ s''(v) = y_v'' & (v = 1, \dots, n) , \\ s'''(n) = y_n''' . \end{cases}$$

Substituting  $x = k + 1$  ( $k = 0, 1, 2, \dots$ ) in (4.1) and (4.2), we obtain

$$(4.3) \quad \begin{cases} s(k+1) = y_0 + \frac{1}{2} y_0''(k+1)^2 + \frac{1}{6} y_0'''(k+1)^3 + \gamma(k+1) + \\ \quad \quad \quad + \sum_{j=0}^{\min(k, n-1)} (\alpha_j (k+1-j)^4 + \beta_j (k+1-j)^5) \\ s''(k+1) = y_0'' + y_0'''(k+1) + 12 \sum_{j=0}^{\min(k, n-1)} \alpha_j (k+1-j)^2 + \\ \quad \quad \quad + 20 \sum_{j=0}^{\min(k, n-1)} \beta_j (k+1-j)^3 . \end{cases}$$

Let  $H_p(t)$  denote the infinite series

$$(4.4) \quad H_p(t) = \sum_{k=0}^{\infty} (k+1)^p t^k \quad (p = 0, 1, \dots) ,$$

that converges in the interior of the unit circle.

We further introduce

$$(4.5) \quad \begin{cases} \sigma(t) = \sum_{k=0}^{\infty} (s(k+1) - y_0 - \frac{1}{2} y_0''(k+1)^2 - \frac{1}{6} y_0'''(k+1)^3) t^k, \\ \rho(t) = \sum_{k=0}^{\infty} (s''(k+1) - y_0'' - y_0'''(k+1)) t^k. \end{cases}$$

Since  $s(x)$  is a polynomial of degree at most five for  $x \geq n$ , these series also converge inside the unit circle.

Moreover, we denote by  $A(t)$  and  $B(t)$  the polynomials

$$(4.6) \quad A(t) = \sum_{k=0}^{n-1} \alpha_k t^k, \quad B(t) = \sum_{k=0}^{n-1} \beta_k t^k.$$

As a consequence of (4.3), (4.4), (4.5) and (4.6) one has the identities

$$(4.7) \quad \begin{cases} \sigma(t) = \gamma H_0(t) + A(t)H_4(t) + B(t)H_5(t), \\ \rho(t) = 12A(t)H_2(t) + 20B(t)H_3(t). \end{cases}$$

In view of (4.1) there follows

$$s'''(x) = y_0''' + 24 \sum_{j=0}^{n-1} \alpha_j (x-j)_+ + 60 \sum_{j=0}^{n-1} \beta_j (x-j)_+^2.$$

Substituting  $x = n$  in this equation we obtain

$$(4.8) \quad y_n''' - y_0''' = 24 \sum_{j=0}^{n-1} \alpha_j (n-j) + 60 \sum_{j=0}^{n-1} \beta_j (n-j)^2.$$

In this connection we note that the coefficient of  $t^{n-1}$  in the expansion of

$$24H_1(t)A(t) + 60H_2(t)B(t)$$

is also equal to the right-hand side of (4.8).

From (4.7) it follows that

$$A(t) = \frac{w_1(t)}{w(t)}, \quad B(t) = \frac{w_2(t)}{w(t)},$$

where

$$w(t) = 20H_3(t)H_4(t) - 12H_2(t)H_5(t) ,$$

$$w_1(t) = 20H_3(t)(\sigma(t) - \gamma H_1(t)) - \rho(t)H_5(t) ,$$

$$w_2(t) = \rho(t)H_4(t) - 12H_2(t)(\sigma(t) - \gamma H_1(t)) .$$

Taking into account formula (2.1) one gets

$$(4.9) \quad w(t) = 4(1-t)^{-9}(5\Pi_3\Pi_4 - 3\Pi_2\Pi_5) = 8(1-t)^{-5}(1+t) .$$

In order to calculate the parameter  $\gamma$  in the representation (4.1), it turns out to be convenient to write

$$(4.10) \quad 24H_1(t)A(t) + 60H_2(t)B(t) = \frac{1}{w(t)} (\omega_1(t)\gamma + \omega_2(t)\sigma(t) + \omega_3(t)\rho(t)) ,$$

where

$$\omega_1(t) = 240(1-t)^{-8}(3\Pi_2^2 - 2\Pi_1\Pi_3) = 240(1-t)^{-6} ,$$

$$\omega_2(t) = 240(1-t)^{-6}(2\Pi_1\Pi_3 - 3\Pi_2^2) = -240(1-t)^{-4} ,$$

$$\omega_3(t) = 12(1-t)^{-8}(5\Pi_2\Pi_4 - 2\Pi_1\Pi_5) = 12(1-t)^{-6}(3t^2 + 14t + 3) .$$

In view of this and formulae (4.9), (4.10) it follows that

$$(4.11) \quad 24H_1(t)A(t) + 60H_2(t)B(t) = \frac{30}{1-t^2}\gamma - \frac{30(1-t)}{1+t}\sigma(t) + \\ + \frac{9t^2 + 42t + 9}{2(1-t^2)}\rho(t) .$$

As a consequence of formula (4.8) and the remark immediately following (4.8), the coefficient of  $t^{n-1}$  in the expansion of the right-hand side of (4.11) must be equal to  $y_n''' - y_0'''$ . Assuming that  $n$  is an odd positive integer, one thus obtains an equation for the parameter  $\gamma$ . Hence, in this case,  $\gamma$  is uniquely determined. If, however,  $n$  is even, then the coefficient of  $t^{n-1}$  in the expansion of  $(1-t^2)$  is equal to zero and  $\gamma$  cannot be evaluated. We note that this phenomenon is completely in agreement with the existence theorem as given in [2]. So, from now on we will assume that  $n$  is an odd positive integer.



Putting

$$\frac{1-t}{1+t} = \sum_{\ell=0}^{\infty} a_{\ell} t^{\ell}, \quad \frac{9t^2 + 42t + 9}{2(1-t^2)} = \sum_{\ell=0}^{\infty} b_{\ell} t^{\ell},$$

we easily obtain that

$$(4.12) \quad \begin{cases} a_0 = 1, a_{\ell} = 2(-1)^{\ell}, (\ell = 1, 2, \dots), \\ b_0 = 9, b_1 = 42, b_2 = 18, b_{\ell} = b_{\ell-2}, (\ell = 3, 4, \dots). \end{cases}$$

By our above remarks and taking into account formulae (4.11), (4.12) and the conditions (3.1), a simple calculation yields

$$(4.13) \quad y_n''' - y_0''' = 30\gamma - 30 \sum_{k=0}^{n-1} (y_{n-k} - y_0 - \frac{(n-k)^2}{2} y_0'' - \frac{(n-k)^3}{6} y_0''') a_k + \\ + \frac{1}{2} \sum_{k=0}^{n-1} (y_{n-k}'' - y_0'' - (n-k)y_0''') b_k,$$

which enables us to give an explicit expression for the parameter  $\gamma$ . What remains is the calculation of the parameters  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1}$ . Substituting  $x = 1$  in (4.1) and (4.2), we get two equations for the two parameters  $\alpha_0$  and  $\beta_0$ , from which these parameters can be easily determined. Next we substitute  $x = 2$  in (4.1) and (4.2) to calculate the parameters  $\alpha_1$  and  $\beta_1$ . We continue this procedure up to and including  $x = n$ . This completes the construction of the deficient quintic spline function.

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