Asymptotically orthonormal sequences in Hilbert space

ASYMPTOTICALLY ORTHONORMAL SEQUENCES IN HILBERT SPACE

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Dedicated to the memory of L. FEJÉR.

1. Introduction. A sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ of vectors in Hilbert space will be called *asymptotically orthonormal* if to every $\varepsilon > 0$ there can be found an integer $N(\varepsilon)$ such that

$$|(\varphi_n, \varphi_m) - \delta_{nm}| < \varepsilon$$

for all $n > N(\varepsilon)$, $m > N(\varepsilon)$ (δ_{nm} is the Kronecker symbol).

We shall say that a sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ can be *orthonormally approximated* if there is an orthonormal sequence $\psi_1, \psi_2, \psi_3, \ldots$ such that $\varphi_n - \psi_n \to 0$ if $n \to \infty$.

B. SZ.-NAGY (in a letter to the author, June 18, 1958) proposed a question which, in our present terminology, can be phrased as follows: Can every asymptotically orthonormal sequence be orthonormally approximated? We shall show in this note (sec. **3**) that the answer is negative. On the other hand, we shall give (sec. **2**) a non-trivial example of an asymptotically orthonormal sequence for which the answer is positive.

In sec. **4** a number of remarks and some related open questions will be listed.

2. A positive example. As a fairly trivial example of an asymptotically orthonormal sequence we take a sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ of vectors such that $\varphi_2, \varphi_3, \varphi_4, \ldots$ is an orthonormal subsequence. We shall show that $\varphi_1, \varphi_2, \varphi_3, \ldots$ can be orthonormally approximated.

This statement is obvious if $\varphi_2, \varphi_3, \varphi_4, \ldots$ is an incomplete orthonormal system (for then we can take $\psi_2 = \varphi_2, \psi_3 = \varphi_3, \ldots$, and there is still room for an additional ψ_1 orthogonal to all other ψ's). If, however, the orthonormal system $\varphi_2, \varphi_3, \varphi_4, \ldots$ is complete, then the existence of the sequence $\psi_1, \psi_2, \psi_3, \ldots$ is less trivial. We shall prove it by an argument which even shows that there is a constant C and an orthonormal sequence $\psi_1, \psi_2, \psi_3, \ldots$ such that

$$\|\varphi_n - \psi_n\| < Cn^{-\frac{1}{2}} \qquad (n = 1, 2, \ldots). \tag{2.1}$$

As a preparation, we take a sequence of positive integers $h_1 < h_2 < h_3 < \ldots$, and we divide the set of integers $2, 3, 4, \ldots$ into groups: S_1 consists of the integers $2, 3, \ldots, h_1 + 1$, S_2 consists of $h_1 + 2, \ldots, h_1 + h_2 + 1$, S_3 consists of $h_1 + h_2 + 2, \ldots, h_1 + h_2 + h_3 + 1$, etc. By Φ_ν we denote the sum of the φ's with indices in the ν-th group:

$$\Phi_\nu = \sum_{k \in S_\nu} \varphi_k .$$

We now choose a sequence $\psi_1, \psi_2, \psi_3, \ldots$ as follows:

$$\psi_1 = h_1^{-\frac{1}{2}} \Phi_1,$$

and for each ν ($\nu = 1, 2, 3, \ldots$) the ψ's with index in the ν-th group will be defined by

$$\psi_n = \varphi_n - h_\nu^{-1} \Phi_\nu + (h_\nu h_{\nu+1})^{-\frac{1}{2}} \Phi_{\nu+1} \qquad (n \in S_\nu).$$

It is not difficult to evaluate the inner products of the ψ's. Using $(\Phi_\nu, \Phi_\mu) = h_\nu \delta_{\nu\mu}$, $(\varphi_n, \Phi_\nu) = 1$ $(n \in S_\nu)$, $(\varphi_n, \Phi_\nu) = 0$ $(n \notin S_\nu)$ we easily verify that $\psi_1, \psi_2, \psi_3, \ldots$ is an orthonormal sequence. And we have

$$\|\psi_n - \varphi_n\|^2 = 2\, h_\nu^{-1} \quad (n \in S_\nu).$$

Under the extra assumption that $h_\nu \to \infty$ if $\nu \to \infty$, it follows that $\|\psi_n - \varphi_n\| \to 0$. If we take, in particular, $h_\nu = 2^\nu$, we obtain (2.1).

3. A negative example. We shall next construct a sequence which is asymptotically orthonormal but cannot be orthonormally approximated.

We first remark that, for every positive integer h, we can find $2h$ vectors $\varphi_1, \ldots, \varphi_{2h}$, all belonging to one and the same h-dimensional sub-space, such that

$$|(\varphi_\lambda, \varphi_\mu) - \delta_{\lambda\mu}| \leq h^{-\frac{1}{2}} \quad (\lambda, \mu = 1, \ldots, 2h). \tag{3.1}$$

In other words, if h is large, the φ's "almost" satisfy the orthonormality condition, though there are twice as many as there can be in any orthonormal system in that sub-space. A construction can be given as follows: $\varphi_1, \ldots, \varphi_h$ are taken strictly orthonormal, and $\varphi_{h+1}, \ldots \varphi_{2h}$ are linear combinations of $\varphi_1, \ldots, \varphi_h$:

$$\varphi_{h+\lambda} = \sum_{\nu=1}^{h} u_{\lambda\nu} \varphi_\nu \quad (\lambda = 1, \ldots, h).$$

For the $(u_{\lambda\nu})$ we take a unitary $h \times h$ matrix whose elements all have absolute value $h^{-\frac{1}{2}}$. Such a matrix exists:

$$u_{\lambda\nu} = h^{-\frac{1}{2}} e^{2\pi i \lambda\nu/h} \qquad (\lambda, \nu = 1, \ldots, h))$$

As $(u_{\lambda\nu})$ is unitary, the set $\varphi_{h+1}, \ldots, \varphi_{2h}$ is also orthonormal. Moreover, if $\lambda, \nu = 1, \ldots, h$, we have

$$|(\varphi_{h+\lambda}, \varphi_\nu)| = |u_{\lambda\nu}| = h^{-\frac{1}{2}},$$

and (3.1) follows.

We next choose in Hilbert space a sequence of mutually orthogonal finite-dimensional spaces $R_1, R_2, \ldots$, with dimensions $h_1, h_2, h_3, \ldots$, respectively. For each ν we take $2h_\nu$ vectors φ in R_ν, according to the above construction, and these φ's together form an infinite sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$. If we take care that $h_\nu \to \infty$ if $\nu \to \infty$, then this sequence is asymptotically orthonormal. For,

$(\varphi_n, \varphi_m) - \delta_{nm}$ vanishes if φ_n and φ_m are in different R's, and $|(\varphi_n, \varphi_m) - \delta_{nm}|$ is at most $h_\nu^{-\frac{1}{2}}$ if both n and m are in R_ν. We can easily be more specific: If we choose $h_1 = 2$, $h_2 = 2^2$, $h_3 = 2^3$, ..., there is a constant C such that

$$|(\varphi_n, \varphi_m) - \delta_{nm}| < C\{\max(n, m)\}^{-\frac{1}{2}} \quad (n, m = 1, 2, \ldots). \tag{3.2}$$

On the other hand we shall show that this sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ cannot be asymptotically approximated. In particular, we can show that for any orthonormal sequence $\psi_1, \psi_2, \psi_3, \ldots$ we have

$$\|\psi_n - \varphi_n\| \geq 2^{-\frac{1}{2}}.$$

for infinitely many values of n.

This immediately follows by taking $R = R_\nu$ ($\nu = 1, 2, 3, \ldots$), $M = 2h_\nu$, $N = h_\nu$ in the following elementary lemma[1]:

Lemma. Let N and M be integers, $0 < N < M$, let $\psi_1, \ldots, \psi_M$ be orthonormal vectors in Hilbert space, let R be an N-dimensional sub-space, and let Δ_k be the distance of ψ_k to R. Then we have

$$\sum_{k=1}^{M} \Delta_k^2 \geq M - N,$$

whence

$$\max \Delta_k \geq (M - N)^{\frac{1}{2}} M^{-\frac{1}{2}}. \tag{3.3}$$

Proof. Chose an orthonormal base $\chi_1, \ldots, \chi_N$ in R. Then we have

$$1 = (\chi_j, \chi_j) \geq \sum_{k=1}^{M} |(\psi_k, \chi_j)|^2 \qquad (j = 1, \ldots, N),$$

whence

$$N \geq \sum_{j=1}^{N} \sum_{k=1}^{M} |(\psi_k, \chi_j)|^2 = \sum_{k=1}^{M} \sum_{j=1}^{N} |(\psi_k, \chi_j)|^2 =$$

$$= \sum_{k=1}^{M} \{(\psi_k, \psi_k) - \Delta_k^2\} = M - \sum_{k=1}^{M} \Delta_k^2.$$

4. Some remarks. (i) It follows from the example of sec. **2** that the orthonormal approximability of a sequence is not affected if we add or omit a finite number of vectors at the beginning. For example, if $\varphi_1, \varphi_2, \varphi_3, \ldots$ can be orthonormally approximated by an orthonormal sequence $\psi_1, \psi_2, \psi_3, \ldots$, then from the fact that $0, \psi_1, \psi_2, \psi_3, \ldots$ can be orthonormally approximated according to sec. **2**, we infer that $0, \varphi_1, \varphi_2, \varphi_3, \ldots$ can be orthonormally approximated.

[1] Prof. B. Sz.-Nagy kindly pointed out to me that a similar lemma occurs in his paper Approximation properties of orthogonal expansion, *Acta Sci. Math., Szeged,* **15**(1953), 31—37.

(ii) Formula (2.1) admits some improvement. It was shown by Mr. K. A. POST *(Eindhoven)* that if $f(n) > 0$, $\sum_1^\infty f(n) = \infty$, and if again $\varphi_2, \varphi_3 \ldots$ is a complete orthonormal sequence, then there exists an orthonormal sequence $\psi_1, \psi_2, \psi_3, \ldots$ with

$$\|\varphi_n - \psi_n\|^2 = O\,[f(n)].$$

On the other hand, it was shown by Prof. J. H. VAN LINT *(Eindhoven)* that there does not exist an orthonormal sequence $\psi_1, \psi_2, \ldots$ for which $\Sigma\|\varphi_n - \psi_n\|^2$ converges.

(iii) The inequality (3.1) is certainly not best possible. For example, if we take

$$\psi_\lambda = \left(1 + h^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \varphi_\lambda \qquad (\lambda = 1, \ldots, 2h),$$

with the φ_λ's of sec. **3**, we obtain

$$|(\psi_\lambda, \psi_\mu) - \delta_{\lambda\mu}| \le h^{-\frac{1}{2}}\left(1 + h^{-\frac{1}{2}}\right)^{-1} \qquad (\lambda = 1, \ldots, 2h).$$

This certainly is best possible if $h = 1$, but the author does not know what the situation is if $h > 1$.

(iv) If $\varphi_1, \varphi_2, \varphi_3, \ldots$ is a sequence of vectors in Hilbert space, then by d_n we shall denote the dimension of the space spanned by $\varphi_1, \ldots, \varphi_n$. It follows from sec. **2** that there exists a sequence of φ's which can be orthonormally approximated though $d_n = n - 1$ for all $n > 1$. It is not difficult to prove the following stronger assertion:

Let $a_1, a_2, \ldots$ be a sequence of integers, $0 \le a_1 \le a_2 \le a_3 \le \ldots$, and assume that $a_1 \le 1$, $a_{n+1} \le a_n + 1$ for all n, and that a_n/n tends to 1 if $n \to \infty$. Then there exists a sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ which can be orthonormally approximated, though it satisfies $d_n = a_n$ $(n = 1, 2, 3, \ldots)$.

A proof can be given along the following lines. First replace the a_n's by integers b_n, with $0 \le b_1 \le b_2 \le \ldots$, $b_n \le a_n$, with the additional property that the n's with $b_n = b_{n-1}$ (exceptional n's) are rare in the sense that the length of the gap between two successive exceptional n's tends to infinity if we run through the integers. Next we take an orthonormal sequence $\psi_1, \psi_2, \psi_3, \ldots$. We split this sequence up into groups such that each exceptional index indicates the end of a group. We now replace each ψ_i by φ_i, where φ_i arises from ψ_i by substracting the average of all elements of the group to which ψ_i belongs (if ψ_i belongs to an infinite group, we do not substract anything). In this way we get a sequence of φ's for which $d_n = b_n$, and which can be orthonormally approximated (by the ψ's). Finally, the enlarging of the dimensions from b_n to a_n can be achieved by very small alterations of the φ's which do not disturb the relation $\varphi_n - \psi_n \to 0$.

(v) We have the following converse of the result of the previous remark: If $\varphi_1, \varphi_2, \varphi_3, \ldots$ is a sequence for which d_n/n does not tend to 1, then the sequence cannot be orthonormally approximated. This statement is a generalization of the contents of the end of sec. **3**, and can be proved along similar lines: As

d_n/n does not tend to 1, there is a number β, with $0 < \beta < 1$, such that $d_m < \beta m$ infinitely often. It follows that, for every n, there is an M such that $\varphi_{n+1}, \ldots, \varphi_{n+M}$ span a space of dimension $< \beta M$. According to the lemma of sec. **3**, there is, for every arbitrary orthonormal sequence $\psi_1, \psi_2, \psi_3, \ldots$, at least one k $(n+1 \leq k \leq n+M)$ satisfying $\|\psi_k - \varphi_k\| \geq (1-\beta)^{\frac{1}{2}}$. As n is arbitrary, we infer that the φ's are not orthonormally approximated by the ψ's.

(vi) It was shown in sec. **3** that (3.2) does not guarantee that the φ_n's can be orthonormally approximated. Such a guarantee can be attained if one replaces (3.2) by

$$|(\varphi_n, \varphi_m) - \delta_{nm}| \leq f\,[\max(n, m)] \quad (n, m = 1, 2, \ldots)$$

with a suitable positive function f tending to zero sufficiently rapidly. (Such a function f can be constructed by analyzing the Schmidt orthonormalization procedure). It would be interesting to know a function f which is optimal for this purpose (in a sense still to be specified).

It seems more difficult to prove or disprove the existence of a positive function g with the property that

$$|(\varphi_n, \varphi_m) - \delta_{nm}| \leq g\,[\min(n, m)] \quad (n, m = 1, 2, \ldots)$$

guarantees that the φ_n's can be orthonormally approximated. (It is the contents of remark (i) that g has this property if it is zero from a certain index onward, and therefore the method of sec. **2** may have some bearing on the existence of a *positive* function g with that property.)

(vii) Prof. B. Sz.-Nagy remarked that, by a minor modification of the construction of sec. **3**, we can obtain a sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ with additional property of being a linearly independent sequence of elements with norm 1. In fact linear independence can always be attained by arbitrary small perturbations.

(viii) Throughout the paper, we have been dealing with complex Hilbert space. The results remain true, however, in the real case. The only difference appears in the beginning of sec. **3**, where we have to take a real orthogonal matrix $u_{\lambda\nu}$ whose elements all have the same absolute value. This is certainly possible if h is a power of 2 (where such matrices were given by Sylvester), and nothing is lost in sec. **3** if we restrict ourselves to these special values of h.

(ix) Inequality (3.3) is best possible: If M and N are given $(0 < N < M)$ we can find an N-dimensional sub-space such that $\max \Delta_k = (M-N)^{\frac{1}{2}} M^{-\frac{1}{2}}$, and it follows that $\Delta_1 = \ldots = \Delta_M$. This is achieved by taking the sub-space spanned by

$$u_{i1}\psi_1 + \ldots + u_{iM}\psi_M \qquad (i = 1, \ldots, N),$$

where (u_{ij}) is a unitary matrix whose elements all have absolute value 1 (see sec. **2**), for then we have

$$1 - (\Delta_k)^2 = |u_{1k}|^2 + \ldots + |u_{Nk}|^2 = (M - N)/M.$$

(x) Dr. P. J. VAN ALBADA *(Eindhoven)* showed that (3.3) is also best possible in the case of real euclidean space, constructing fo every pair M, N $(0 < N < M)$ an N-dimensional sub-space with $\Delta_1 = \ldots = \Delta_M$.

Note added in proof. The results mentioned in remark (ii) have been published meanwhile: J. H. VAN LINT and K. A. POST, A Problem in Hilbert Space, *Proc. Ned. Akad. Wetensch. Amsterdam, A* **63** (= *Indagationes Math.* **22**) (1960), 409—411.

Dr. VAN ALBADA's result [see remark (x)] shall appear as a problem in *Nieuw Archief v. Wiskunde* (Solution in Wiskundige Opgaven met de Oplossingen).