Solution to Problem 63-7 : On commutative rotations

Also solved by the proposer.

Problem 63-7, *On Commutative Rotations*, by JOEL BRENNER (Stanford Research Institute).

Show that if two nontrivial proper rotations in E_3 are commutative, then they are rotations about the same axis or else rotations through 180° about two mutually perpendicular axes.

Solution by THEODORE KATSANIS (NASA, Lewis Research Center).

Let A and B denote two nontrivial commutative rotations, and let x_A and x_B denote their respective axes. Let P be a plane containing x_A and parallel to x_B. If x_A is parallel to x_B, let O be any point on x_A; otherwise, let O be the intersection of the projection on P of x_B with x_A. In either case $A(O) = O$, so that $B(O) = BA(O) = AB(O)$. Since $B(O)$ is left fixed by A, it must lie on x_A. But if x_B does not intersect x_A, $B(O)$ cannot lie on x_A. Hence x_B intersects x_A and $B(O) = O$.

Consider now a point S on x_A, $S \neq O$. Then $A(S) = S$, so that $B(S) = BA(S) = AB(S)$, and $B(S)$ is left fixed by A. Hence $B(S)$ lies on x_A. This means that either $x_A = x_B$, or else x_B is perpendicular to x_A, in which case B must also be a 180° rotation.

Solution by K. A. POST (Technological University, Eindhoven, Netherlands).

Suppose A has the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

where c and s stand for $\cos \phi$ and $\sin \phi$ respectively, i.e., A represents a rotation about the x-axis through an angle ϕ. Let $B = (b_{ij})$ represent a rotation such that $AB = BA$. Equating both sides we get six equations

$$\begin{cases} (1 - c)b_{12} + sb_{13} = 0, \\ sb_{12} + (c - 1)b_{13} = 0, \end{cases} \qquad \begin{cases} sb_{31} + (c - 1)b_{21} = 0, \\ (1 - c)b_{31} + sb_{21} = 0, \end{cases}$$
$$s(b_{32} + b_{23}) = 0, \qquad s(b_{22} - b_{33}) = 0.$$

As $s^2 + (c - 1)^2 = 2 - 2c \neq 0$ (A is nontrivial), we obtain $b_{12} = b_{13} = b_{21} = b_{31} = 0$. Now there are two cases.

(1) $$b_{22} = b_{33} = c', \qquad b_{32} = -b_{23} = s', \qquad b_{11} = 1.$$

(Rotations about the same axis).

(2) $$s = 0, c = -1, \qquad b_{11} = -1, \qquad b_{22} = -b_{33} = c', \qquad b_{32} = b_{23} = s'.$$

(Rotations through 180° about mutually perpendicular axes).

Also solved by W. F. EBERLEIN (University of Rochester) by means of a coordinate free spinor analysis, A. MAYER (Reeves Instrument Corp.) in two ways and by the proposer in two ways.