

## Solution to Problem 63-7 : On commutative rotations

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Also solved by the proposer.

*Problem 63-7, On Commutative Rotations*, by JOEL BRENNER (Stanford Research Institute).

Show that if two nontrivial proper rotations in  $E_3$  are commutative, then they are rotations about the same axis or else rotations through  $180^\circ$  about two mutually perpendicular axes.

Solution by THEODORE KATSANIS (NASA, Lewis Research Center).

Let  $A$  and  $B$  denote two nontrivial commutative rotations, and let  $x_A$  and  $x_B$  denote their respective axes. Let  $P$  be a plane containing  $x_A$  and parallel to  $x_B$ . If  $x_A$  is parallel to  $x_B$ , let  $O$  be any point on  $x_A$ ; otherwise, let  $O$  be the intersection of the projection on  $P$  of  $x_B$  with  $x_A$ . In either case  $A(O) = O$ , so that  $B(O) = BA(O) = AB(O)$ . Since  $B(O)$  is left fixed by  $A$ , it must lie on  $x_A$ . But if  $x_B$  does not intersect  $x_A$ ,  $B(O)$  cannot lie on  $x_A$ . Hence  $x_B$  intersects  $x_A$  and  $B(O) = O$ .

Consider now a point  $S$  on  $x_A$ ,  $S \neq O$ . Then  $A(S) = S$ , so that  $B(S) = BA(S) = AB(S)$ , and  $B(S)$  is left fixed by  $A$ . Hence  $B(S)$  lies on  $x_A$ . This means that either  $x_A = x_B$ , or else  $x_B$  is perpendicular to  $x_A$ , in which case  $B$  must also be a  $180^\circ$  rotation.

Solution by K. A. POST (Technological University, Eindhoven, Netherlands).  
Suppose  $A$  has the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

where  $c$  and  $s$  stand for  $\cos \phi$  and  $\sin \phi$  respectively, i.e.,  $A$  represents a rotation about the  $x$ -axis through an angle  $\phi$ . Let  $B = (b_{ij})$  represent a rotation such that  $AB = BA$ . Equating both sides we get six equations

$$\begin{cases} (1-c)b_{12} + sb_{13} = 0, & sb_{31} + (c-1)b_{21} = 0, \\ sb_{12} + (c-1)b_{13} = 0, & (1-c)b_{31} + sb_{21} = 0, \\ s(b_{32} + b_{23}) = 0, & s(b_{22} - b_{33}) = 0. \end{cases}$$

As  $s^2 + (c-1)^2 = 2 - 2c \neq 0$  ( $A$  is nontrivial), we obtain  $b_{12} = b_{13} = b_{21} = b_{31} = 0$ . Now there are two cases.

$$(1) \quad b_{22} = b_{33} = c', \quad b_{32} = -b_{23} = s', \quad b_{11} = 1.$$

(Rotations about the same axis).

$$(2) \quad s = 0, c = -1, \quad b_{11} = -1, \quad b_{22} = -b_{33} = c', \quad b_{32} = b_{23} = s'.$$

(Rotations through  $180^\circ$  about mutually perpendicular axes).

Also solved by W. F. EBERLEIN (University of Rochester) by means of a coordinate free spinor analysis, A. MAYER (Reeves Instrument Corp.) in two ways and by the proposer in two ways.