Approximations and two-sample tests based on P-P and Q-Q of the Kaplan-Meier estimators of lifetime distributions

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Approximations and Two-Sample Tests Based on $\mathcal{P}$ - $\mathcal{P}$ and $\mathcal{Q}$ - $\mathcal{Q}$ Plots of the Kaplan - Meier Estimators of Lifetime Distributions

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Approximations and Two-Sample Tests Based on $P - P$ and $Q - Q$ Plots of the Kaplan-Meier Estimators of Lifetime Distributions

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Abstract.
Let F_n and G_n denote the usual Kaplan-Meier product-limit estimators of lifetime distributions based on two independent samples, and let F_n^{inv} and G_n^{inv} denote their quantile functions. We consider the corresponding $P-P$ plot $F_n(G_n^{\text{inv}})$ and $Q-Q$ plot $F_n^{\text{inv}}(G_n)$, and establish strong approximations of empirical processes based on these $P-P$ and $Q-Q$ plots by appropriate sequences of Gaussian processes. It is shown that the rates of approximation we obtain are the best which can be achieved by this method. We apply these results to obtain the limiting distributions of test statistics which are functionals of $F_n\left(G_n^{\text{inv}}(s)\right) - s$, $G_n\left(F_n^{\text{inv}}(s)\right) - s$, and $F_n\left(G_n^{\text{inv}}(s)\right) + G_n\left(F_n^{\text{inv}}(s)\right) - 2s$, and propose solutions to the problem of testing the assumption that the underlying lifetime distributions F and G are equal, in the case where the censoring distributions are arbitrary and unknown.

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1. Introduction

In this paper, we are concerned with nonparametric comparative statistical inference for survival distributions in the two-sample case. Namely, if F_m and G_n denote Kaplan-Meier estimators of lifetime (or failure) distribution functions based on independent samples of sizes m and n respectively, we consider statistical comparison procedures of the unknown distributions F and G based on F_m and G_n. We start our discussion of this problem by some notation and the statement of relevant results in the literature (we refer to Chapter 8 in M. Csörgő (1983) for further details).

Let $\{X_i, i \geq 1\}$, $\{Y_i, i \geq 1\}$, $\{U_i, i \geq 1\}$ and $\{V_i, i \geq 1\}$ be four independent sequences of i.i.d. positive random variables with distribution functions denoted respectively by $F(x) = P(X_i \leq x)$, $G(x) = P(Y_i \leq x)$, $H(x) = P(U_i \leq x)$ and $K(x) = P(V_i \leq x)$ for $i \geq 1$ and $-\infty < x \leq \infty$. The meaning of these random variables is as follows. For each $i \geq 1$, X_i (resp. Y_i) denotes the uncensored lifetime of the i-th individual of the first (resp. the second) sample. Likewise, U_i (resp. V_i) denotes the censoring time at which the i-th individual of the first (resp. the second) sample is withdrawn from the investigation. Set further $Z_i' = \min(X_i, U_i)$, $Z_i'' = \min(Y_i, V_i)$, $\delta_i' = 1_{\{X_i \leq U_i\}}$ and $\delta_i'' = 1_{\{Y_i \leq V_i\}}$ for $i \geq 1$. In the classical model with random censorship from the right, one observes the sequences $\{(Z_i', \delta_i'), 1 \leq i \leq m\}$ and $\{(Z_i'', \delta_i''), 1 \leq i \leq n\}$. The corresponding product-limit (PL) estimators introduced by Kaplan and Meier (1958) are then given by

$$F_m(x) = 1 - \prod_{Z'_{(i)} \leq x} \left(1 - \delta'_{(i)}/(m - i + 1)\right) \quad \text{for } -\infty < x \leq \infty , \qquad (1.1)$$

and

$$G_n(x) = 1 - \prod_{Z''_{(i)} \leq x} \left(1 - \delta''_{(i)}/(n - i + 1)\right) \quad \text{for } -\infty < x \leq \infty , \qquad (1.2)$$

where $0 < Z'_{(1)} \leq \ldots \leq Z'_{(m)}$ (resp. $0 < Z''_{(1)} \leq \ldots \leq Z''_{(n)}$) are the order statistics of $\{Z_i', 1 \leq i \leq m\}$ (resp. $\{Z_i'', 1 \leq i \leq n\}$), $\{\delta'_{(i)}, 1 \leq i \leq m\}$ (resp. $\{\delta''_{(i)}, 1 \leq i \leq n\}$) being the corresponding values of the δ_i' (resp. of the δ_i'').

We will assume thoughout that F (resp. G) is differentiable on $(0, \infty)$ with continuous and positive derivative f (resp. g), and hence continuous on $[0, \infty)$ with $F(0) = 0$ (resp. $G(0) = 0$). Moreover, we assume that H and K are continuous on $(-\infty, \infty)$ and allow $H_-(\infty)$ and $K_-(\infty)$ to be less than one. (Here $H_-(\infty)$ denotes $\lim_{x \to \infty} H(x)$; $K_-(\infty)$ is defined similarly.) In particular, $H_-(\infty) = K_-(\infty) = 0$ corresponds to the uncensored case, the censoring times U_i and V_i being then infinite with probability one. Denote by $F^{\text{inv}}(s) = \inf\{x \geq 0 \ : \ F(x) \geq s\}$ and $G^{\text{inv}}(s) = \inf\{x \geq 0 \ : \ G(x) \geq s\}$ for $0 \leq s \leq 1$ the quantile functions pertaining to F and G. Set further $T_H = \sup\{x \ : \ H(x) < 1\}$ and $T_K = \sup\{x \ : \ K(x) < 1\}$, and introduce the functions

$$\begin{aligned} h(s) &= \int_0^s (1-u)^{-2}\left(1 - H\left(F^{\text{inv}}(u)\right)\right)^{-1} du \text{ for } 0 \leq s < F(T_H) , \\ h(s) &= s \text{ for } s \leq 0 , \end{aligned} \qquad (1.3)$$

and

$$k(s) = \int_0^s (1-u)^{-2}\Big(1 - K\big(G^{\text{inv}}(u)\big)\Big)^{-1} du \text{ for } 0 \le s < G(T_K) \, , \qquad (1.4)$$
$$k(s) = s \text{ for } s \le 0 \; .$$

Consider now the empirical processes $\alpha_n'(x) = m^{1/2}\big(F_m(x) - F(x)\big)$ for $-\infty < x \le \infty$, and $\alpha_n''(x) = n^{1/2}\big(G_n(x) - G(x)\big)$ for $-\infty < x \le \infty$, and the reduced empirical processes

$$a_m'(s) = \alpha_m'\big(F^{\text{inv}}(s)\big) \; = m^{1/2}\Big(F_m\big(F^{\text{inv}}(s)\big) - s\Big) \; =: \; m^{1/2}(\Gamma_m'(s) - s)$$
$$\text{for } 0 \le s \le 1 \, , \qquad (1.5)$$

and

$$a_n''(s) = \alpha_n''\big(G^{\text{inv}}(s)\big) \; = n^{1/2}\Big(G_n\big(G^{\text{inv}}(s)\big) - s\Big) \; =: \; n^{1/2}(\Gamma_n''(s) - s)$$
$$\text{for } 0 \le s \le 1 \, . \qquad (1.6)$$

Burke, S. Csörgő and Horváth (1988) and Major and Rejtő (1988) proved that the following strong approximations of a_m' and a_n'' hold. Assuming that the original probability space is sufficiently rich, it is possible to define two independent standard two-parameter Wiener processes W' and W'' in such a way that, for any fixed $\vartheta_F \in \big(0, F(T_H)\big)$ and $\vartheta_G \in \big(0, G(T_K)\big)$, we have almost surely

$$||a_m' - m^{-1/2}(1-I)W'(h,n)||_0^{\vartheta_F} \; =: ||a_m' - \Lambda_m'||_0^{\vartheta_F} = O(m^{-1/2}\log^2 m)$$
$$\text{as } m \to \infty \, , \qquad (1.7)$$

and

$$||a_n'' - n^{-1/2}(1-I)W''(k,n)||_0^{\vartheta_G} \; =: ||a_n'' - \Lambda_n''||_0^{\vartheta_G} = O(n^{-1/2}\log^2 n)$$
$$\text{as } n \to \infty \, , \qquad (1.8)$$

where I denotes the identity function, and $||\varphi||_c^d := \sup_{c \le x \le d} |\varphi(x)|$.

In the uncensored case where $H_-(\infty) = K_-(\infty) = 0$, we have $h = k = I/(1-I)$, so that the processes $m^{-1/2}(1-I)W'(h,m)$ and $n^{-1/2}(1-I)W''(k,n)$ in (1.7)-(1.8) are Brownian bridges. In this case, (1.7) and (1.8) are valid for $\vartheta_F = \vartheta_G = 1$, being then nothing else but the Kiefer process approximations of the corresponding empirical processes due to Komlós, Major and Tusnády (1975). Moreover, F_m and G_n coincide with the classical empirical distribution functions based on $X_1, ..., X_m$ and $Y_1, ..., Y_n$ respectively. Motivated by this example, it is tempting to test the null hypothesis that $F = G$ by Kolmogorov-Smirnov-type statistics of the form

$$D_{FG;mn}^{\nu} = \Big(\frac{mn}{m+n}\Big)^{1/2} ||F_m - G_n||_0^{\nu} \, . \qquad (1.9)$$

Unfortunately, even in the case where $F = G$ and $H = K$, the limiting distribution of $D^{\nu}_{FG;mn}$ depends on the unknown values of F and H. Moreover, the plot of F_m against G_n has the inconvenience of having a poor visual interpretation. To overcome this latter drawback, one may use quantile-quantile $(Q-Q)$ and probability-probability $(P-P)$ plots, which we will now introduce.

Denote by $F_m^{\text{inv}}(s) = \inf\{x \geq 0 \, : \, F_m(x) \geq s\}$ and $G_n^{\text{inv}}(s) = \inf\{x \geq 0 \, : \, G_n(x) \geq s\}$, for $0 \leq s \leq 1$, the empirical quantile functions pertaining to F_m and G_n. The *PL* $Q-Q$ plot of F against G is then defined by

$$\Delta_{FG;mn}(x) = F_m^{\text{inv}}\Big(G_n(x)\Big) \quad \text{for } 0 \leq x \leq \infty \, , \tag{1.10}$$

while the *PL* $P-P$ plot of F against G is defined by

$$\tilde{\Delta}_{FG;mn}(s) = F_m\Big(G_n^{\text{inv}}(s)\Big) \quad \text{for } 0 \leq s \leq 1 \, . \tag{1.11}$$

Statistics such as in (1.10) and (1.11) have a great intuitive appeal since they converge to the identity function (on appropriate intervals) if $F = G$. In the uncensored case such $Q-Q$ and $P-P$ plots have received considerable attention in the recent literature. We refer to Fisher (1983) and the references therein, Aly (1986a), Aly, M. Csörgő and Horváth (1987), and Beirlant and Deheuvels (1990). In the censored situation, $Q-Q$ plots have been considered in Aly (1986b).

The aim of this paper is threefold. In Section 2, we will consider the strong approximation of empirical processes based on $P-P$ and $Q-Q$ plots by Gaussian processes. Our theorems yield the best possible rates of approximation given the construction we use, and correspond to the results obtained for $P-P$ and $Q-Q$ plots in the classical uncensored case by Beirlant and Deheuvels (1990). In Section 3, we will establish the asymptotic distribution of the censored version of a Bahadur-Kiefer type two-sample statistic (devised by Deheuvels and Mason (1990b)) under the null hypothesis that $F = G$; this statistic is a two-sample version of the Bahadur-Kiefer statistic as considered in Deheuvels and Mason (1990a) and Beirlant and Einmahl (1990), for the uncensored and censored case respectively. Finally, in Section 4, we present two-sample tests of the hypothesis that $F = G$ based upon these limiting theorems.

2. Strong Approximations of the *PL* $P-P$ and $Q-Q$ Plot Processes

Throughout the sequel, we will consider the case where the samples have equal sizes. The assumption that $m = n$ has the advantage of reducing the technicalities of our proofs to a great extent. Extensions of our results to unequal sample sizes can be achieved through lengthy additional arguments of relatively minor interest which we omit for the sake of brevity.

It will be convenient to introduce the following notation. Let

$$\Gamma_n'(s) = F_n\Big(F^{\text{inv}}(s)\Big) \, , \quad \Gamma_n''(s) = G_n\Big(F^{\text{inv}}(s)\Big) \quad \text{for } 0 \leq s \leq 1 \, , \tag{2.1}$$

and

$$\Gamma_n'(s) = F_n\Big(F^{\mathrm{inv}}(s)\Big) \ , \quad \Gamma_n''(s) = G_n\Big(F^{\mathrm{inv}}(s)\Big) \quad \text{for } 0 \le s \le 1 \ , \tag{2.1}$$

and

$$\Gamma_n'^{\,\mathrm{inv}}(s) = F\Big(F_n^{\mathrm{inv}}(s)\Big) \ , \ \Gamma_n''^{\mathrm{inv}}(s) = F\Big(G_n^{\mathrm{inv}}(s)\Big) \ \text{ for } 0 \le s \le 1 \ . \tag{2.2}$$

Define the reduced PL $P-P$ plot process of F against G by

$$\tilde{A}_n(s) = n^{1/2}\Big(\Gamma_n'\Big(\Gamma_n''^{\mathrm{inv}}(s)\Big) - s\Big) \quad \text{for } 0 \le s \le 1 \ , \tag{2.3}$$

and the reduced PL $Q-Q$ plot process of F against G by

$$A_n(s) = n^{1/2}\Big(\Gamma_n'^{\,\mathrm{inv}}\Big(\Gamma_n''(s)\Big) - s\Big) \quad \text{for } 0 \le s \le 1 \ . \tag{2.4}$$

Whenever $F = G$, the reduced PL $P-P$ plot process has a simple expression in terms of the PL $P-P$ plot $\tilde{\Delta}_{FF;nn}$. We have namely

$$\tilde{A}_n(s) = n^{1/2}\Big(F_n\Big(G_n^{\mathrm{inv}}(s)\Big) - s\Big) \ = n^{1/2}(\tilde{\Delta}_{FF;nn}(s) - s) \quad \text{for } 0 \le s \le 1 \ . \tag{2.5}$$

Such a simple relation does not exist between the reduced PL $Q-Q$ plot process and the PL $Q-Q$ plot $\Delta_{FF;nn}$. However, we will establish later on that A_n can be closely approximated when $F = G$ by the following empirical process. Let

$$\begin{aligned} \hat{A}_n(s) &= \ n^{1/2} f\Big(F^{\mathrm{inv}}(s)\Big)\Big(F_n^{\mathrm{inv}}\Big(G_n\Big(F^{\mathrm{inv}}(s)\Big)\Big) - F^{\mathrm{inv}}(s)\Big) \\ &= \ n^{1/2} f\Big(F^{\mathrm{inv}}(s)\Big)\Big(\Delta_{FF;nn}\Big(F^{\mathrm{inv}}(s)\Big) - F^{\mathrm{inv}}(s)\Big) \qquad \text{for } 0 < s < 1 \ ; \\ \hat{A}_n(0) &= 0 \ . \end{aligned} \tag{2.6}$$

The following process turns out to be a natural approximant of $\tilde{A}_n$, $-A_n$ and $-\hat{A}_n$. Let $\Theta = \min\Big(F(T_H), G(T_K)\Big)$, and set

$$M_n(s) = n^{-1/2}(1-s)\Big(W'(h(s),n) - W''(k(s),n)\Big) \ =: \Lambda_n'(s) - \Lambda_n''(s) \quad \text{for } 0 \le s < \Theta \ . \tag{2.7}$$

The main result of this section gives the exact uniform rates of approximation of $\tilde{A}_n$, $-A_n$ and $-\hat{A}_n$ by $M_n = \Lambda_n' - \Lambda_n''$. Let "$\to_P$" denote convergence in probability.

Theorem 2.1.
Assume that $F = G$. Then, for any $\vartheta \in (0, \Theta)$, we have, as $n \to \infty$,

$$n^{1/4}(\log n)^{-1/2} \, \|\tilde{A}_n - M_n\|_0^\vartheta \Big/ \Big(\Big\|\Lambda_n''\Big(\frac{1}{1-H(F^{\mathrm{inv}})} + \frac{1}{1-K(F^{\mathrm{inv}})}\Big)\Big\|_0^\vartheta\Big)^{1/2} \to_P 1 \ , \tag{2.8}$$

$$n^{1/4}(\log n)^{-1/2}\,\|A_n + M_n\|_0^\vartheta$$
$$/\Big(\Big\|(\Lambda_n' - \Lambda_n'')\Big(\frac{1}{1-H(F^{\text{inv}})}\Big)\Big\|_0^\vartheta\Big)^{1/2} \to_P 1\,. \tag{2.9}$$

Assume further that $\lim_{u\downarrow 0}\Big(u\log(1/u)\Big)^{-1/2}\sup_{0<s<F^{\text{inv}}(\vartheta);|s-t|\le u}|f(t)-f(s)| = 0$ and that f is right continuous and positive at 0. Then (2.9) holds with A_n being replaced by $\hat{A}_n$.

Let W be a standard Wiener process and for $-\infty < s < \Theta$, write $l(s) = h(s) + k(s)$.

Corollary 2.1.
As $n \to \infty$

$$n^{1/4}(\log n)^{-1/2}\,\|\tilde{A}_n - M_n\|_0^\vartheta$$
$$\to_d \Big(\Big\|(1-I)W(k)\Big(\big(1-H(F^{\text{inv}})\big)^{-1} + \big(1-K(F^{\text{inv}})\big)^{-1}\Big)\Big\|_0^\vartheta\Big)^{1/2},$$
$$n^{1/4}(\log n)^{-1/2}\,\|A_n + M_n\|_0^\vartheta \to_d \Big(\Big\|(1-I)W(l)\big(1-H(F^{\text{inv}})\big)^{-1}\Big\|_0^\vartheta\Big)^{1/2}.$$

Remark 2.1.
In the uncensored case, $H_-(\infty) = K_-(\infty) = 0$, $\Theta = 1$, and Λ_n' and Λ_n'' are Brownian bridges. In this case, (2.8) and (2.9) hold for $\vartheta = 1$. While (2.8) is then similar to Theorem 2.1 of Beirlant and Deheuvels (1990), (2.9) is new and may be restated as

$$n^{1/4}(\log n)^{-1/2}\,\|A_n + M_n\|_0^\vartheta/(\|M_n\|_0^\vartheta)^{1/2} \to_P 1 \quad \text{as } n \to \infty\,. \tag{2.10}$$

Remark 2.2.
Corollary 2.1 implies that the rate $O_P(n^{-1/4}(\log n)^{1/2})$ is the best possible for an approximation of $\tilde{A}_n$, $-A_n$ or $-\hat{A}_n$ by M_n (see Aly (1986b) for related results).

Remark 2.3.
It easily follows that (2.8) remains valid with Λ_n'' in the denominator replaced by a_n''. Similarly, we may replace in the denominator of (2.9), $\Lambda_n' - \Lambda_n''$ by A_n.

In the remainder of this section, we prove Theorem 2.1. We will make use of the following decomposition. Throughout, we assume that $F = G$.

$$\begin{aligned}\tilde{A}_n(s) &= n^{1/2}\Big(\Gamma_n'\Big(\Gamma_n''^{\text{inv}}(s)\Big) - s\Big)\\ &= n^{1/2}\Big(\Gamma_n'\Big(\Gamma_n''^{\text{inv}}(s)\Big) - \Gamma_n''^{\text{inv}}(s)\Big) + n^{1/2}(\Gamma_n''^{\text{inv}}(s) - s\Big)\\ &=: a_n'\Big(\Gamma_n''^{\text{inv}}(s)\Big) + b_n''(s)\end{aligned}$$

$$
\begin{aligned}
&=: a_n'\Big(\Gamma_n''^{\rm inv}(s)\Big) - a_n''\Big(\Gamma_n''^{\rm inv}(s)\Big) + R_{1,n}(s)\\
&=: \Lambda_n'\Big(\Gamma_n''^{\rm inv}(s)\Big) - \Lambda_n''\Big(\Gamma_n''^{\rm inv}(s)\Big) + R_{1,n}(s) + R_{2,n}(s)\\
&= \ M_n\Big(s + n^{-1/2}b_n''(s)\Big) + R_{1,n}(s) + R_{2,n}(s)\\
&=: M_n\Big(s - n^{-1/2}\Lambda_n''(s)\Big) + R_{1,n}(s) + R_{2,n}(s) + R_{3,n}(s)\\
&=: M_n(s) + R_{1,n}(s) + R_{2,n}(s) + R_{3,n}(s) + R_{4,n}(s)\ , \qquad (2.11)
\end{aligned}
$$

where we have used (1.5) and (2.1)-(2.7). We will evaluate successively $||R_{j,n}||_0^\vartheta$ for $j = 1,2,3$ and 4, and show that these remainder terms are negligible for $j = 1,2$ and 3, the main contribution being obtained for $j = 4$ and yielding (2.8). These results are captured in the following lemmas.

Lemma 2.1 (Sander (1975), see also Aly, M. Csörgő and Horváth (1985, pp. 193–194)). We have for all $\vartheta \in (0,\Theta)$ almost surely

$$||R_{1,n}||_0^\vartheta = O(n^{-1/2}) \quad \text{as } n \to \infty\ . \qquad (2.12)$$

Lemma 2.2.
We have for all $\vartheta \in (0,\Theta)$ almost surely

$$||R_{2,n}||_0^\vartheta = O(n^{-1/2}\log^2 n) \quad \text{as } n \to \infty\ . \qquad (2.13)$$

Proof. From the obvious inequality

$$||R_{2,n}||_0^\vartheta \le ||a_n' - \Lambda_n'||_0^{\Gamma_n''^{\rm inv}(\vartheta)} + ||a_n'' - \Lambda_n''||_0^{\Gamma_n''^{\rm inv}(\vartheta)}\ ,$$

it suffices to use the Glivenko-Cantelli property for the PL estimators, combined with (1.7) and (1.8) to obtain (2.13). □

Having established the easy parts of our proof, we turn to the more difficult treatment of $R_{3,n}$ and $R_{4,n}$. First, we assume without loss of generality that the Wiener processes W' and W'' which are used in (1.7)-(1.8) are defined on $(-\infty,\infty) \times (0,\infty)$. This is obtained by extending each of these processes on $(-\infty,0] \times [0,\infty)$ by using independent standard two-parameter Wiener processes $\tilde{W}'$ and $\tilde{W}''$ on $[0,\infty) \times [0,\infty)$, independent of W' and W'', and by setting $W'(s,t) = \tilde{W}'(-s,t)$ and $W''(s,t) = \tilde{W}''(-s,t)$ for $s < 0$ and $t > 0$. Using the corresponding definition of M_n as given in (2.7), we see that $M_n\Big(s - n^{-1/2}\Lambda_n''(s)\Big)$ as given in (2.11) is properly defined (note that $s - n^{-1/2}\Lambda_n''(s)$ may be negative).

Lemma 2.3.
We have for all $\vartheta \in (0,\Theta)$

$$\|R_{3,n}\|_0^\vartheta = O_P(n^{-3/8}(\log n)^{3/4}) \quad \text{as } n \to \infty \ . \tag{2.14}$$

Proof. Define a sequence of standard two-sided Wiener processes on $\Big(-\infty, l^{\mathrm{inv}}(\Theta)\Big)$ through the identity

$$(1-I)W_n(l) = M_n \ . \tag{2.15}$$

From (2.7) and (2.11) it easily follows that

$$\begin{aligned} \|R_{3,n}\|_0^\vartheta \ &\le n^{-1/2}\Big\|b_n''W_n\Big(l(I+n^{-1/2}b_n'')\Big)\Big\|_0^\vartheta \\ &+ n^{-1/2}\Big\|\Lambda_n''W_n\Big(l(I-n^{-1/2}\Lambda_n'')\Big)\Big\|_0^\vartheta \\ &+ \Big\|W_n\Big(l(I+n^{-1/2}b_n'')\Big) - W\Big(l(I-n^{-1/2}\Lambda_n'')\Big)\Big\|_0^\vartheta \ . \end{aligned} \tag{2.16}$$

It is straightforward to show that the first two terms in the right hand side of (2.16) are $O_P(n^{-1/2})$ as $n \to \infty$, so we focus on the third one.

By Theorem 3 and Corollary 1 in Beirlant and Einmahl (1990), we see that as $n \to \infty$

$$\|b_n'' + a_n''\|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2}) \ . \tag{2.17}$$

This, in conjunction with (1.8), yields as $n \to \infty$

$$\|(I+n^{-1/2}b_n'') - (I-n^{-1/2}\Lambda_n'')\|_0^\vartheta = O_P(n^{-3/4}(\log n)^{1/2}) \ . \tag{2.18}$$

Note also that for any fixed $0 < \varepsilon < \Theta - \vartheta$, as $n \to \infty$,

$$\begin{aligned} &P\Big(s + n^{-1/2}b_n''(s) \in [0, \vartheta+\varepsilon] \text{ and } s - n^{-1/2}\Lambda_n''(s) \in [-\varepsilon, \vartheta+\varepsilon] \ , \\ &\quad \text{for all } 0 \le s \le \vartheta\Big) \to 1 \ . \end{aligned} \tag{2.19}$$

Moreover, on $[0, \vartheta+\varepsilon]$, we have $2 \le l' \le C$, for some $C \in (0,\infty)$, where l' denotes the derivative of l. Note that $l'(s) = 2$ for $s < 0$.

Combining the above results, we see that it suffices to show, to complete the proof of this lemma, that for arbitrary $C' \in (0,\infty)$, as $n \to \infty$,

$$\sup_{\substack{-2\varepsilon \le s,t \le C \\ |s-t| \le C' n^{-3/4}(\log n)^{1/2}}} |W_n(s) - W_n(t)| = O_P(n^{-3/8}(\log n)^{3/4}) \ , \tag{2.20}$$

which readily follows from the oscillation modulus results for a standard Wiener process (see (2.25) below). □

We now turn to the study of $R_{4,n}$. Recall by (2.11) that

$$R_{4,n}(s) = M_n\Big(s - n^{-1/2}\Lambda_n''(s)\Big) - M_n(s) \quad \text{for } 0 \le s < \Theta . \tag{2.21}$$

Denote by $\{W(t), -\infty < t < \infty\}$ a standard Wiener process extended to the real line (i.e. such that $\{W(t), t \ge 0\}$ and $\{W(-t), t \ge 0\}$ are two independent standard Wiener processes). In view of (2.21) and (2.15), the following proposition, in the spirit of Proposition 4 of Deheuvels and Mason (1990a), will give the proper evaluation of $||R_{4,n}||_0^\vartheta$.

Proposition 2.1.
Let $-\infty < a < c < d < b < \infty$ be fixed, and let $\{\Phi(t), a \le t \le b\}$ and $\{\Psi(t), a \le t \le b\}$ be functions such that

(i) Φ is positive and has a derivative φ which is continuous on $[a,b]$;

(ii) Ψ has a derivative ψ which is positive and continuous on $[a,b]$.

Then, there exists an event E of probability one such that, on E, we have for all continuous functions η on $[a,b]$

$$\lim_{u\downarrow 0} \sup_{c\le t\le d} \Big(2u\log(1/u)\Big)^{-1/2} \left| \frac{W\Big(\Psi\big(t + u\eta(t)\big)\Big)}{\Phi\big(t + u\eta(t)\big)} - \frac{W\big(\Psi(t)\big)}{\Phi(t)} \right|$$
$$= \| \, |\psi\eta|^{1/2}/\Phi\|_c^d . \tag{2.22}$$

Proof. Let $C_1 > 0$ be a lower bound for Φ and $C_2 < \infty$ be an upper bound for $|\varphi|$ on $[a,b]$. On the event of probability one that W is bounded over any compact subset of $(-\infty,\infty)$, we have, as $u \downarrow 0$,

$$\sup_{c\le t\le d} \Big(2u\log(1/u)\Big)^{-1/2} \left| W\Big(\Psi(t + u\eta(t)\Big) \Big(\frac{1}{\Phi\big(t+u\eta(t)\big)} - \frac{1}{\Phi(t)}\Big)\right|$$
$$\le (C_2/C_1^2)\Big(u/\big(2\log(1/u)\big)\Big)^{1/2} ||\eta||_c^d \, ||W(\Psi)||_a^b \to 0 . \tag{2.23}$$

Thus, for (2.24), it suffices to show that

$$\lim_{u\downarrow 0} \sup_{c\le t\le d} \Big(2u\log(1/u)\Big)^{-1/2} \left| \frac{W\Big(\Psi(t+u\eta(t)\Big) - W\big(\Psi(t)\big)}{\Phi(t)} \right|$$
$$= \sup_{c\le t\le d} \frac{|\psi(t)\eta(t)|^{1/2}}{|\Phi(t)|} . \tag{2.24}$$

Observe, by changing u into λu, $\lambda > 0$, that if (2.24) holds, then it also holds when η is replaced by $\lambda\eta$. Thus, excluding the trivial case where $\| \, |\psi\eta|^{1/2}/\Phi\|_c^d = 0$, we may limit ourselves to proving that (2.24) holds almost surely for all continuous functions η on $[a,b]$ such that $\| \, |\psi\eta|^{1/2}/\Phi\|_c^d = 1$. We will now make use of the following property of the Wiener process (see e.g. M. Csörgő and Révész (1981), pp. 26–29). There exists an event E' of probability one such that on E', we have for all $-\infty < C < D < \infty$

$$\lim_{u\downarrow 0} \sup_{\substack{C\le t\le D\\ |s-t|\le u}} \frac{|W(s)-W(t)|}{\bigl(2u\log(1/u)\bigr)^{1/2}}$$

$$= \lim_{u\downarrow 0} \sup_{C\le t\le D} \frac{|W(t\pm u)-W(t)|}{\bigl(2u\log(1/u)\bigr)^{1/2}} = 1 \,. \qquad (2.25)$$

In the remainder of our proof, we will assume that E' holds, and show that (2.24) is true on E' for all continuous functions η on $[a,b]$ satisfying $\| \, |\psi\eta|^{1/2}/\Phi\|_c^d = 1$. Let η be such a function. By continuity of ψ and η, and using the assumption that ψ is positive, for any $0 < \varepsilon < 1$, there exists a $u_1 > 0$ such that for all $0 < u \le u_1$, we have

$$\Bigl|\Bigl(\Psi\bigl(t+u\eta(t)\bigr) - \Psi(t)\Bigr) - u\psi(t)\eta(t)\Bigr| \le \varepsilon u \min(1/2, \psi(t)|\eta(t)|)$$
$$\text{for all } c\le t\le d \,. \qquad (2.26)$$

Fix any $c \le \gamma < \delta \le d$, and set $C = \Psi(\gamma) < D = \Psi(\delta)$, and $M = (1+\varepsilon)\|\psi\eta\|_\gamma^\delta$. We have by (2.26) for all $u > 0$ sufficiently small

$$\sup_{\gamma\le t\le\delta} \bigl(2u\log(1/u)\bigr)^{-1/2} \Bigl|\frac{W\Bigl(\Psi\bigl(t+u\eta(t)\bigr)\Bigr) - W\bigl(\Psi(t)\bigr)}{\Phi(t)}\Bigr|$$

$$\le \|1/\Phi\|_\gamma^\delta \sup_{\substack{C\le t\le D\\ |s-t|\le Mu}} \frac{|W(s)-W(t)|}{\bigl(2u\log(1/u)\bigr)^{1/2}} \,, \qquad (2.27)$$

which in turn tends to $(1+\varepsilon)^{1/2}(\|\psi\eta\|_\gamma^\delta)^{1/2}\,\|1/\Phi\|_\gamma^\delta$ as $u \downarrow 0$. Since there exists an $N_1 \ge 1$ such that for all $c \le \gamma < \delta \le d$ with $|\gamma-\delta| \le 1/N_1$, we have

$$(1+\varepsilon)^{1/2}(\|\psi\eta\|_\gamma^\delta)^{1/2}\,\|1/\Phi\|_\gamma^\delta \le (1+\varepsilon)\,\|\,|\psi\eta|^{1/2}/\Phi\|_\gamma^\delta \le 1+\varepsilon \,, \qquad (2.28)$$

by covering $[c,d]$ with a finite number of intervals $[\gamma,\delta]$ of span less than $1/N_1$, we conclude from (2.27) and (2.28) that

$$\limsup_{u\downarrow 0} \sup_{c\le t\le d} \bigl(2u\log(1/u)\bigr)^{-1/2} \Bigl|\frac{W\Bigl(\Psi\bigl(t+u\eta(t)\bigr)\Bigr) - W\bigl(\Psi(t)\bigr)}{\Phi(t)}\Bigr|$$

$$\le 1+\varepsilon \,. \qquad (2.29)$$

Let $c \le \tau \le d$ be such that $|\psi(\tau)\eta(\tau)|^{1/2}/|\Phi(\tau)| = 1$, and set $m = \psi(\tau)\eta(\tau)$; let $0 < \varepsilon < |m|/2$. There exists a subinterval $[\gamma,\delta] \subset [c,d]$ with $\gamma \le \tau \le \delta$, such that

$$|\psi(t)\eta(t) - m| \le \frac{\varepsilon}{2} \text{ and } 1/\Phi(t) \ge (1-\varepsilon)/\Phi(\tau) \quad \text{for all } \gamma\le t\le\delta \,. \qquad (2.30)$$

Thus, combining (2.26) and (2.30), we obtain that for all $u > 0$ sufficiently small,

$$\sup_{c\le t\le d}\ \Big(2u\log(1/u)\Big)^{-1/2}\Big|\frac{W\Big(\Psi\big(t+u\eta(t)\big)\Big)-W\Big(\Psi(t)\Big)}{\Phi(t)}\Big|$$

$$\ge\ \sup_{\gamma\le t\le\delta}\ \Big(2u\log(1/u)\Big)^{-1/2}\Big|\frac{W\Big(\Psi\big(t+u\eta(t)\big)\Big)-W\Big(\Psi(t)\Big)}{\Phi(t)}\Big|$$

$$\ge\frac{1-\varepsilon}{\Phi(\tau)}\Big(\sup_{C\le t\le D}\ \frac{|W(t+mu)-W(t)|}{\Big(2u\log(1/u)\Big)^{1/2}}$$

$$-\sup_{\substack{C-\varepsilon\le t\le D+\varepsilon\\ |s-t|\le\varepsilon u}}\ \frac{|W(s)-W(t)|}{\Big(2u\log(1/u)\Big)^{1/2}}\Big)\ , \tag{2.31}$$

which, by (2.25) tends to

$$(1-\varepsilon)\Big(\frac{|\psi(\tau)\eta(\tau)|^{1/2}}{\Phi(\tau)}-\frac{\varepsilon^{1/2}}{\Phi(\tau)}\Big)=(1-\varepsilon)\Big(1-\frac{\varepsilon^{1/2}}{\Phi(\tau)}\Big)\quad \text{as } u\downarrow 0\ . \tag{2.32}$$

We conclude from (2.31) and (2.32) that

$$\liminf_{u\downarrow 0}\ \sup_{c\le t\le d}\ \Big(2u\log(1/u)\Big)^{-1/2}\Big|\frac{W\Big(\Psi\big(t+u\eta(t)\big)\Big)-W\Big(\Psi(t)\Big)}{\Phi(t)}\Big|$$

$$\ge\ (1-\varepsilon)\Big(1-\frac{\varepsilon^{1/2}}{\Phi(\tau)}\Big)\ . \tag{2.33}$$

Since $0<\varepsilon<\min(1,|m|/2)$ in (2.29) and (2.33) is arbitrary, we have (2.24) as sought. This completes the proof of Proposition 2.1. □

We now return to the proof of Theorem 2.1. By choosing $\Phi=1/(1-I)$ and $\Psi=l$, recalling that $l=h+k$, we see from (1.3) and (1.4) that Φ and Ψ satisfy the assumptions of Proposition 2.1 for any choice of $-1<a<c=0<d=\vartheta<b<\Theta$. Consider now two independent standard Wiener processes W' and W'' extended to the real line, and define (similar to 2.15), a third Wiener process W through the identity

$$M\ :=\ (1-I)W(l)=(1-I)\Big(W'(h)+W''(k)\Big)\ =:\ \Lambda'+\Lambda''\ . \tag{2.34}$$

It is obvious from (2.7) and (2.34) that, for every $n\ge 1$,

$$\{M_n,\Lambda_n',\Lambda_n''\}\ =_d\ \{M,\Lambda',\Lambda''\}\ . \tag{2.35}$$

Since (2.22), when applied for $\eta=-\Lambda''$, implies that almost surely

$$\lim_{u\downarrow 0}\ \Big(2u\log(1/u)\Big)^{-1/2}\ \|M(I-u\Lambda'')-M\|_0^{\vartheta}/\|\ |l'\Lambda''|^{1/2}(1-I)\|_0^{\vartheta}=1\ ,$$

it follows from (2.21), (2.35), and the fact that almost sure convergence implies convergence in probability, that as $n \to \infty$

$$n^{1/4}(\log n)^{-1/2}\,\|R_{4,n}\|_0^{\vartheta}/\|\,|l'\Lambda_n''|^{1/2}(1-I)\|_0^{\vartheta} \to_P 1\ . \tag{2.36}$$

Recall, by (1.3) and (1.4), that

$$l' = h' + k' = (1-I)^{-2}\Big(\big(1 - H(F^{\text{inv}})\big)^{-1} + \big(1 - K(F^{\text{inv}})\big)^{-1}\Big)\ . \tag{2.37}$$

Thus, combining (2.11) and (2.36)–(2.37) with Lemmas 2.1, 2.2 and 2.3, we obtain readily (2.8).

The proof of the second statement of Theorem 2.1 makes use of the following decomposition, in the spirit of that given in (2.11).

$$\begin{aligned}
A_n(s) &= n^{1/2}\Big(\Gamma_n'^{\,\text{inv}}\big(\Gamma_n''(s)\big) - s\Big)\\
&= n^{1/2}\Big(\Gamma_n'^{\,\text{inv}}\big(\Gamma_n''(s)\big) - \Gamma_n''(s)\Big) + n^{1/2}(\Gamma_n''(s) - s)\\
&=: b_n'\big(\Gamma_n''(s)\big) + a_n''(s)\\
&=: -a_n'\Big(\Gamma_n'^{\,\text{inv}}\big(\Gamma_n''(s)\big)\Big) + a_n''(s) + R_{1,n}'(s)\\
&=: -\Lambda_n'\Big(\Gamma_n'^{\,\text{inv}}\big(\Gamma_n''(s)\big)\Big) + \Lambda_n''(s) + R_{1,n}'(s) + R_{2,n}'(s)\\
&=: -\Lambda_n'\big(s - n^{-1/2}M_n(s)\big) + \Lambda_n''(s) + R_{1,n}'(s) + R_{2,n}'(s) + R_{3,n}'(s)\\
&=: -M_n(s) + R_{1,n}'(s) + R_{2,n}'(s) + R_{3,n}'(s) + R_{4,n}'(s)\ .
\end{aligned} \tag{2.38}$$

Similarly as in (2.11)–(2.37), we evaluate $\|R_{j,n}'\|_0^{\vartheta}$ for $j = 1,2,3,4$. In the first place, we apply Lemma 2.1 with the formal replacements of a_n'' and b_n'' by a_n' and b_n'. We so obtain that for all $\lambda \in (0,\Theta)$,

$$\|b_n' + a_n'(\Gamma_n'^{\,\text{inv}})\|_0^{\lambda} = O(n^{-1/2}) \quad \text{a.s. as } n \to \infty\ . \tag{2.39}$$

By using the Glivenko-Cantelli property for PL estimators, it follows readily from (2.38) and (2.39) that, for any $\vartheta \in (0,\Theta)$,

$$\|R_{1,n}'\|_0^{\vartheta} = O(n^{-1/2}) \quad \text{a.s. as } n \to \infty\ . \tag{2.40}$$

Next, we see from (1.7) and (1.8) that, for any $\vartheta \in (0,\Theta)$,

$$\|R_{2,n}'\|_0^{\vartheta} = O(n^{-1/2}\log^2 n) \quad \text{a.s. as } n \to \infty\ . \tag{2.41}$$

For the term $R_{3,n}'$ some more work is needed. What we first need is a rough bound, compared to (2.9) itself, for

$$\Gamma_n'^{\,\mathrm{inv}}\Big(\Gamma_n''(s)\Big) - s + n^{-1/2}M_n(s) = n^{-1/2}\Big(A_n(s) + M_n(s)\Big)\ . \tag{2.42}$$

Note that

$$\begin{aligned} A_n(s) + M_n(s) = {} & b_n'\Big(\Gamma_n''(s)\Big) + a_n''(s) + M_n(s) \\ =: {} & -\Lambda_n'\Big(\Gamma_n''(s)\Big) + \Lambda_n''(s) + M_n(s) + T_{1,n}(s) \\ =: {} & T_{1,n}(s) + T_{2,n}(s)\ . \end{aligned} \tag{2.43}$$

Similar to (2.18), and using (1.8), we have as $n \to \infty$

$$\|T_{1,n}\|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2})\ . \tag{2.44}$$

Using essentially the same techniques as in the proof of Lemma 2.3, we easily see that $(n \to \infty)$

$$\|T_{2,n}\|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2})\ . \tag{2.45}$$

Thus we have shown that the left hand side of (2.42) is $O_P(n^{-3/4}(\log n)^{1/2})$ as $n \to \infty$, uniformly in $0 \le s \le \vartheta$.

Now we can again use similar arguments as in the proof of Lemma 2.3 to show that this implies, as $n \to \infty$,

$$\|R_{3,n}'\|_0^\vartheta = O_P(n^{-3/8}(\log n)^{3/4})\ . \tag{2.46}$$

For brevity's sake these straightforward arguments are omitted.

Finally, we have

$$R_{4,n}'(s) = \Lambda_n'(s) - \Lambda_n'\Big(s - n^{-1/2}M_n(s)\Big) =_d \Lambda'(s) - \Lambda'\Big(s - n^{-1/2}M(s)\Big)\ , \tag{2.47}$$

where Λ' and M are as in (2.34). By applying Proposition 2.1 with $\Psi = h$, $\Phi = 1/(1-I)$, and $\eta = -M$, we see that almost surely

$$n^{1/4}(\log n)^{-1/2}\,\|\Lambda' - \Lambda'(I - n^{-1/2}M)\|_0^\vartheta / \|\,|h'M|^{1/2}(1-I)\|_0^\vartheta \to 1 \quad \text{as } n \to \infty\ . \tag{2.48}$$

Recalling that $h' = (1-I)^{-2}\Big(1 - H(F^{\mathrm{inv}})\Big)^{-1}$, by (2.47) and (2.48) and the fact that almost sure convergence implies convergence in probability, we obtain

$$n^{1/4}(\log n)^{-1/2}\,||R'_{4,n}||_0^\vartheta/\Big(\Big\|M_n/\big(1-H(F^{\rm inv})\big)\Big\|_0^\vartheta\Big)^{1/2}\ \to_P\ 1$$

$$\text{as } n\to\infty\ . \qquad (2.49)$$

By combining (2.38) with (2.40), (2.41), (2.46) and (2.49), we obtain readily (2.9).

To complete the proof of Theorem 2.1 it suffices to show that under the additional conditions on f

$$n^{1/4}(\log n)^{-1/2}\,||\hat{A}_n - A_n||_0^\vartheta \to_P 0 \quad \text{as } n\to\infty\ . \qquad (2.50)$$

From the mean value theorem we have

$$\begin{aligned}\hat{A}_n(s) &= n^{1/2} f\big(F^{\rm inv}(s)\big)\Big(F^{\rm inv}\big(s+n^{-1/2}A_n(s)\big)-F^{\rm inv}(s)\Big)\\ &= \frac{f\big(F^{\rm inv}(s)\big)}{f\big(F^{\rm inv}(\mu_{s,n})\big)}\,A_n(s)\ ,\end{aligned} \qquad (2.51)$$

with $|\mu_{s,n}-s|\le n^{-1/2}\,|A_n(s)|$. Hence

$$||\hat{A}_n - A_n||_0^\vartheta \le ||A_n||_0^\vartheta\,\Big\|\frac{f(F^{\rm inv})}{f\big(F^{\rm inv}(\mu_{I,n})\big)}-1\Big\|_0^\vartheta\ . \qquad (2.52)$$

Because of (2.9) and the conditions on f we see from (2.52) that for a proof of (2.50) it remains to show that

$$||F^{\rm inv}-F^{\rm inv}(\mu_{I,n})||_0^\vartheta = O_P(n^{-1/2}) \quad \text{as } n\to\infty\ . \qquad (2.53)$$

But for some $C\in(0,\infty)$

$$\begin{aligned}&||F^{\rm inv}-F^{\rm inv}(\mu_{I,n})||_0^\vartheta \le C\,||I-\mu_{I,n}||_0^\vartheta\\ &\le\ C\,n^{-1/2}\,||A_n||_0^\vartheta = O_P(n^{-1/2}) \quad \text{as } n\to\infty\ .\end{aligned} \qquad (2.54)$$

Thus we have shown (2.50) and hence the proof of Theorem 2.1 is completed. □

Remark 2.4.
We conjecture that, by following the lines of Deheuvels and Mason (1990a), it is possible to obtain strong versions of (2.8) and (2.9), in the spirit of Theorem 2.2 of Beirlant and Deheuvels (1990) in the uncensored case. We omit the corresponding arguments which require much more than minor modifications of our proofs. By doing so, one replaces the optimal rate $O_P(n^{-1/4}(\log n)^{1/2})$ mentioned in Remark 2.2 by an optimal almost sure rate of $O(n^{-1/4}(\log n)^{1/2}(\text{loglog}\, n^{1/4})$. As in the uncensored case, it is likely that some improvements may be brought to these rates by using different sequences of approximating processes. To our best knowledge, this problem is open at present.

3. A Two-Sample Bahadur-Kiefer Type Test of Fit for $P-P$ Plots

In this section, we consider the statistic defined by

$$\delta_n^\vartheta = n^{1/2}\,\|F_n(G_n^{\text{inv}}) + G_n(F_n^{\text{inv}}) - 2I\|_0^\vartheta \quad \text{for } 0 < \vartheta < \Theta \ . \tag{3.1}$$

Whenever $F = G$, by (2.1), (2.2) and (3.1), we have

$$\delta_n^\vartheta = n^{1/2}\,\|\Gamma_n'(\Gamma_n''^{\text{inv}}) + \Gamma_n''(\Gamma_n'^{\text{ inv}}) - 2I\|_0^\vartheta \ . \tag{3.2}$$

Recall, by (2.3), that $\tilde{A}_n(s) = n^{1/2}\Big(\Gamma_n'\Big(\Gamma_n''^{\text{inv}}(s)\Big) - s\Big)$ for $0 \le s \le 1$. Define likewise $\tilde{A}_n^*(s) = n^{1/2}\Big(\Gamma_n''\Big(\Gamma_n'^{\text{ inv}}(s)\Big) - s\Big)$ for $0 \le s \le 1$. An application of Theorem 2.1 shows that $\|\tilde{A}_n - M_n\|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2})$ as $n \to \infty$. Likewise, the same theorem when used with the first and second sample reversed shows that $\|\tilde{A}_n^* + M_n\|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2})$ as $n \to \infty$. Bearing in mind that (3.2) may be rewritten as $\delta_n^\vartheta = \|\tilde{A}_n + \tilde{A}_n^*\|_0^\vartheta$, it follows that, under the null hypothesis that $F = G$, we have

$$\delta_n^\vartheta = O_P(n^{-1/4}(\log n)^{1/2}) \to_P 0 \quad \text{as } n \to \infty \ . \tag{3.3}$$

By (3.3), we see that, whenever $F = G$, $n^{1/4}(\log n)^{-1/2}\delta_n^\vartheta = O_P(1)$, while the same expression tends to infinity with a rate of $n^{3/4}(\log n)^{-1/2}$ in general when $F \not\equiv G$. This motivates the study of the limiting distributional behavior of $n^{1/4}(\log n)^{-1/2}\delta_n^\vartheta$ when $F = G$. This problem has been solved in the uncensored case by Deheuvels and Mason (1990b). The following theorem extends their results to the general censored model considered here.

Theorem 3.1.
Assume that $F = G$. Then, for any $\vartheta \in (0, \Theta)$, we have, as $n \to \infty$,

$$n^{1/4}(\log n)^{-1/2}\delta_n^\vartheta \Big/ \Big(\Big\|(\Lambda_n' - \Lambda_n'')\Big(\frac{1}{1 - H(F^{\text{inv}})} + \frac{1}{1 - K(F^{\text{inv}})}\Big)\Big\|_0^\vartheta\Big)^{1/2} \to_P 1 \ . \tag{3.4}$$

Corollary 3.1.
As $n \to \infty$,

$$\delta_n^\vartheta \Big/ \Big(\log n \Big\|\Big(F_n(F^{\text{inv}}) - G_n(F^{\text{inv}})\Big)\Big(\frac{1}{1 - H(F^{\text{inv}})} + \frac{1}{1 - K(F^{\text{inv}})}\Big)\Big\|_0^\vartheta\Big)^{1/2} \to_P 1 \ . \tag{3.5}$$

Remark 3.1.
It follows from Theorem 2.1 in Deheuvels and Mason (1990b), that in the uncensored case we can take $\vartheta = 1$ in Theorem 3.1. This implies that $n^{1/4}(\log n)^{-1/2}\delta_n^\vartheta$ is asymptotically equivalent (in probability) to the square root of the usual Kolmogorov-Smirnov

statistic. From Corollary 3.1 we see that in the censored model, a similar assertion (but with $\vartheta < \Theta$) holds true with a weighted version of this statistic.

In the remainder of this section, we prove Theorem 3.1. Throughout the sequel, we will assume that $F = G$, so that (3.2) holds. We will make use of the decomposition:

$$\begin{aligned} R_n(s) &:= n^{1/2}\Big(\Gamma_n'\Big(\Gamma_n''^{\text{inv}}(s)\Big) + \Gamma_n''\Big(\Gamma_n'^{\text{inv}}(s)\Big) - 2s\Big) = \tilde{A}_n(s) + \tilde{A}_n^*(s) \\ &= M_n\Big(s - n^{-1/2}\Lambda_n''(s)\Big) - M_n\Big(s - n^{-1/2}\Lambda_n'(s)\Big) + R_n''(s) \ . \qquad (3.6) \end{aligned}$$

Lemma 3.1.
We have for all $\vartheta \in (0, \Theta)$

$$\|R_n''\|_0^\vartheta = O_P(n^{-3/8}(\log n)^{3/4}) \quad \text{as } n \to \infty \ . \qquad (3.7)$$

Proof. By (2.11) and Lemmas 2.1, 2.2 and 2.3, we have

$$\begin{aligned} &\|M_n(I - n^{-1/2}\Lambda'') - \tilde{A}_n\|_0^\vartheta \le \|R_{1,n}\|_0^\vartheta + \|R_{2,n}\|_0^\vartheta + \|R_{3,n}\|_0^\vartheta \\ &= O_P(n^{-3/8}(\log n)^{3/4}) \quad \text{as } n \to \infty \ . \qquad (3.8) \end{aligned}$$

We now repeat the same argument by interchanging the two samples. This amounts to changing Λ_n' (resp. Λ_n'') into Λ_n'' (resp. Λ_n'), $M_n = \Lambda_n' - \Lambda_n''$ into $-M_n = \Lambda_n'' - \Lambda_n'$, and $\tilde{A}_n$ into $\tilde{A}_n^*$ in (3.8). Combining both versions of (3.8) yields (3.7) by the triangle inequality. □

By (2.35), we have

$$\begin{aligned} &M_n(I - n^{-1/2}\Lambda_n'') - M_n(I - n^{-1/2}\Lambda_n') \\ &=_d M(I - n^{-1/2}\Lambda'') - M(I - n^{-1/2}\Lambda') \ . \qquad (3.9) \end{aligned}$$

Thus, by (3.6)-(3.9), the proof of (3.4) boils down to showing that (with the notation of (2.34)-(2.35))

$$\begin{aligned} &\Big(2u\log(1/u)\Big)^{-1/2} \|M(I - u\Lambda') - M(I - u\Lambda'')\|_0^\vartheta \\ &/ \left(\left\|(\Lambda' - \Lambda'')\Big(\frac{1}{1 - H(F^{\text{inv}})} + \frac{1}{1 - K(F^{\text{inv}})}\Big)\right\|_0^\vartheta\right)^{1/2} \to_P 1 \quad (u \downarrow 0) \ . \qquad (3.10) \end{aligned}$$

To prove (3.10), we will make use of the following refinement of Proposition 2.1 (see e.g. Proposition 2.1 in Deheuvels and Mason (1990b).

Proposition 3.1.
Under the assumptions of Proposition 2.1, there exists an event E'' of probability one such that, on E'', we have for all pairs η' and η'' of continuous functions on $[a, b]$

$$\lim_{u\downarrow 0} \sup_{c\le t\le d} \Big(2u\log(1/u)\Big)^{-1/2} \left|\frac{W\Big(\Psi\big(t+u\eta'(t)\big)\Big)}{\Phi\big(t+u\eta'(t)\big)} - \frac{W\Big(\Psi\big(t+u\eta''(t)\big)\Big)}{\Phi\big(t+u\eta''(t)\big)}\right|$$

$$= \| \, |\psi(\eta'-\eta'')|^{1/2}/\Phi \|_c^d \, . \tag{3.11}$$

Proof. The proof is very similar to that of Proposition 2.1. Therefore, we omit details.□

We now apply Proposition 3.1 with $\eta' = -\Lambda'$, $\eta'' = -\Lambda''$, $\Psi = l = h+k$, $\Phi = 1/(1-I)$, $c = 0$, $d = \vartheta$, while a and b are chosen in such a way that $-1 < a < 0 < \vartheta < b < \Theta$. Recalling, by (2.34), that $M = (1-I)W(l)$, we obtain readily (3.10) from (3.11). Since the latter statement implies (3.4), the proof of Theorem 3.1 is completed. □

Remark 3.2.
A simple modification of the just given proof of Theorem 3.1 yields the following statement. Whenever $\{\rho(t), 0 \le t < \Theta\}$ is continuous and positive on $[0,\vartheta]$, we have, for $F = G$ and as $n \to \infty$

$$n^{3/4}(\log n)^{-1/2} \, \|(F_n(G_n^{\text{inv}}) + G_n(F_n^{\text{inv}}) - 2I)\rho\|_0^\vartheta$$

$$\to_d \left(\Big\|\rho^2(1-I)W(l)\Big(\big(1-H(F^{\text{inv}})\big)^{-1} + \Big(1-K\big(F^{\text{inv}}\big)\Big)^{-1}\Big)\Big\|_0^\vartheta\right)^{1/2} . \tag{3.12}$$

4. Statistical Applications

We now return to the practical problem of testing the null hypothesis that $F = G$ given the data $\{(Z_i', \delta_i'), 1 \le i \le n\}$ and $\{(Z_i'', \delta_i''), 1 \le i \le n\}$. We assume that $F = G$ is unknown, together with H and K. By Theorems 2.1 and 3.1, and Remark 3.2, we may use statistics of the form

$$\begin{aligned} S_{1,n}'(\rho,\vartheta) &= \|(F_n(G_n^{\text{inv}}) - I)\rho\|_0^\vartheta \, , \\ S_{1,n}''(\rho,\vartheta) &= \|(G_n(F_n^{\text{inv}}) - I)\rho\|_0^\vartheta \, , \end{aligned} \tag{4.1}$$

and

$$S_{2,n}(\rho,\vartheta) = \|(F_n(G_n^{\text{inv}}) + G_n(F_n^{\text{inv}}) - 2I)\rho\|_0^\vartheta \, , \tag{4.2}$$

where $\{\rho(t), 0 \le t < \Theta\}$ is a suitable function.

The question to be answered is to find for each of the tests statistics given above the appropriate critical values at level $\alpha \in (0,1)$. Following the usual large sample approximations, we may use approximate critical values by rejecting the assumption that $F = G$ whenever $S_{1,n}'(\rho,\vartheta) \ge c_\rho'(\alpha,\vartheta)n^{-1/2}$, $S_{1,n}''(\rho,\vartheta) \ge c_\rho''(\alpha,\vartheta)n^{-1/2}$, or $S_{2,n}(\rho,\vartheta) \ge c_\rho(\alpha,\vartheta)n^{-3/4}(\log n)^{1/2}$ respectively, where $c_\rho'(\alpha,\vartheta) = c_\rho''(\alpha,\vartheta)$ and $c_\rho(\alpha,\vartheta)$ are given by

$$P\Big(\|\rho W(l)(1-I)\|_0^\vartheta \ge c_\rho'(\alpha,\vartheta)\Big) = \alpha \, , \tag{4.3}$$

and

$$P\Big(\Big(\Big\|\rho^2 W(l)(1-l)\Big(\frac{1}{1-H(F^{\rm inv})}+\frac{1}{1-K(F^{\rm inv})}\Big)\Big\|_0^{\vartheta}\Big)^{1/2} \geq c_\rho(\alpha,\vartheta)\Big) = \alpha \, . \tag{4.4}$$

Unfortunately, the problem is not solved yet, since the expressions in (4.3) and (4.4) depend upon the unknown values of l, H, K and F. To overcome this difficulty, we need to introduce estimators of the unknown factors in (4.3)–(4.4). Toward this aim, we set for $-\infty < s < \infty$

$$\begin{aligned} J_n'(s) &= n^{-1}\#\{1 \leq i \leq n \ : \ Z_i' < s\} \, , \\ J_n''(s) &= n^{-1}\#\{1 \leq i \leq n \ : \ Z_i'' < s\} \, , \\ \tilde{J}_n'(s) &= n^{-1}\#\{1 \leq i \leq n \ : \ Z_i' \leq s, \delta_i' = 1\} \, , \\ \tilde{J}_n''(s) &= n^{-1}\#\{1 \leq i \leq n \ : \ Z_i'' \leq s, \delta_i'' = 1\} \, , \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} h_n(s) &= \int_0^{F_n^{\rm inv}(s)} \Big(1 - J_n'(t)\Big)^{-2} d\tilde{J}_n'(t) \, , \\ k_n(s) &= \int_0^{G_n^{\rm inv}(s)} \Big(1 - J_n''(t)\Big)^{-2} d\tilde{J}_n''(t) \, , \end{aligned} \tag{4.6}$$

for $0 \leq s < \mu_n \ := \ \min\Big(\max\{F_n(Z_i') \ : \ 1 \leq i \leq n, \delta_i' = 1\}, \max\{G_n(Z_i'') \ : \ 1 \leq i \leq n, \delta_i'' = 1\}\Big)$.

It is noteworthy (see e.g. Lemma 6.2 in Burke, S. Csörgő and Horváth (1981)) that h_n and k_n are strongly uniformly consistent estimators of h and k respectively, on any interval $[0,\vartheta]$, $0 < \vartheta < \Theta$.

By all this, we may present our first choice of ρ ($= \rho_n$ now), appropriate to (4.3). Let

$$\rho_n(s) = (1-s)^{-1}\Big(l_n(\vartheta)\Big)^{-1/2} \, , \quad \text{where } l_n \ := \ h_n + k_n \, , \ \ \text{for } 0 \leq s < \mu_n \, . \tag{4.7}$$

It is now straightforward from the preceding arguments that (for $F = G$)

$$\lim_{n\to\infty} \ P(n^{1/2} S_{1,n}'(\rho_n,\vartheta) \geq c) = \lim_{n\to\infty} \ P(n^{1/2} S_{1,n}''(\rho_n,\vartheta) \geq c)$$

$$= \ P(\|W\|_0^1 \geq c) = 1 - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\exp\Big(-\frac{\pi^2(2j+1)^2}{8c^2}\Big) \ \text{ for } c > 0 \tag{4.8}$$

(see e.g. (1.5.2) in M. Csörgő and Révész (1981)). Thus, by (4.8), we may choose $c'_{\rho_n}(\alpha,\vartheta) = c''_{\rho_n}(\alpha,\vartheta)$ to be equal to the value of c which renders the right hand side of (4.8) equal to α. We so obtain a first solution to our two-sample testing problem.

A similar argument can be used for $S_{2,n}$. The appropriate choice of ρ is here given by

$$\rho_n^*(s) = (1-s)^{-1/2}\Big(l_n(\vartheta)\Big)^{-1/4}\Big(\frac{1}{1-H_n\big(F_n^{\text{inv}}(s)\big)} + \frac{1}{1-K_n\big(G_n^{\text{inv}}(s)\big)}\Big)^{-1/2}, \tag{4.9}$$

where H_n and K_n denote strongly uniformly consistent estimators of H and K. A simple choice for H_n and K_n is obtained from the remark that J'_n and J''_n are strongly consistent estimators of $1-(1-F)(1-H)$ and of $1-(1-F)(1-K)$ respectively. Therefore, one may set

$$\begin{aligned} 1-H_n(s) &= \big(1-J'_n(s)\big)/\big(1-F_n(s)\big)\ , \\ 1-K_n(s) &= \big(1-J''_n(s)\big)/\big(1-G_n(s)\big)\ . \end{aligned} \tag{4.10}$$

In view of (4.9) and (4.10), we may simplify our choice of ρ by the observation that $1-F_n\big(F_n^{\text{inv}}(s)\big)$ and $1-G_n\big(G_n^{\text{inv}}(s)\big)$ can be replaced by $1-s$. This leads to

$$\rho_n(s) = (1-s)^{-1}\Big(l_n(\vartheta)\Big)^{-1/4}\Big(\frac{1}{1-J'_n\big(F_n^{\text{inv}}(s)\big)} + \frac{1}{1-J''_n\big(G_n^{\text{inv}}(s)\big)}\Big)^{-1/2}. \tag{4.11}$$

By all this, we obtain readily that (for $F = G$)

$$\begin{aligned} &\lim_{n\to\infty} P(n^{3/4}(\log n)^{-1/2} S_{2,n}(\rho_n,\vartheta) \geq c) \\ &= P(\|W\|_0^1 \geq c^2) = 1 - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\exp\Big(-\frac{\pi^2(2j+1)^2}{8c^4}\Big) \\ &\qquad \text{for } c > 0\ . \end{aligned} \tag{4.12}$$

Finally, we choose $c_{\rho_n}(\alpha,\vartheta)$ to be equal to the value of c which renders the right hand side of (4.12) equal to α.

The preceding tests are examples of how the theorems of Sections 2 and 3 may be applied. Other choices of ρ may be used in a similar way, leading to analogue evaluations of critical levels. We do not discuss this question further.

The complicated form of the limiting distributions in (4.8) and (4.12) suggests that the problem of finding the exact distributions of our tests for finite values of n is a very complicated task. Our methods have though the advantage of being relatively simple and valid in an asymptotical sense when the sizes of the samples to be compared become sufficiently large. Moreover, the graphical plots of $F_n(G_n^{\text{inv}})$, $G_n(F_n^{\text{inv}})$ or their sum enable a visual interpretation of the fit between both lifetime distributions.

Similar methods can be presented for the $Q-Q$ plots. However, in that case, one needs also estimators of f, which renders the statistics more delicate to handle because of the necessity of using a smoothing factor for this sake. We will not present such estimators here for the sake of brevity, noting that our methods may be applied as in the $P-P$ plot case.

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