

Solution to Problem 93-15 : Two integrals arising from a reaction-diffusion problem

Citation for published version (APA):

Boersma, J., & Doelder, de, P. J. (1994). Solution to Problem 93-15 : Two integrals arising from a reaction-diffusion problem. *SIAM Review*, 36(3), 498-499.

Document status and date:

Published: 01/01/1994

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Two Integrals Arising from a Reaction-Diffusion Problem

Problem 93-15, by JOHN BILLINGHAM (Schlumberger Cambridge Research, Cambridge, England).

Evaluate the two integrals

$$(i) \int_0^{\infty} e^{-s^2} ds / A_n^2(s), \quad n \text{ even and } \geq 0,$$

$$(ii) \int_0^{\infty} \left\{ 1/4s^2 - e^{-s^2} / A_n^2(s) \right\} ds, \quad n \text{ odd and } \geq 1,$$

where

$$A_n(s) = \sum_{r=0}^{n/2} (2s)^{2r} (n/2)! / (2r)! (n/2 - r)!, \quad n \text{ even and } \geq 0,$$

$$A_n(s) = \sum_{r=0}^{(n-1)/2} (2s)^{2r+1} ((n-1)/2)! / (2r+1)! ((n-1)/2 - r)!, \quad n \text{ odd and } \geq 1.$$

This problem arose as the leading order approximation to the small time solution of a reaction-diffusion problem [1].

REFERENCE

- [1] J. BILLINGHAM AND D. J. NEEDHAM, *The development of travelling waves in quadratic and cubic autocatalysis with unequal diffusion rates. II. An initial value problem with an immobilized or nearly immobilized autocatalyst*, Philos. Trans. Roy. Soc. London, Ser. A, 336 (1991), pp. 497-539.

Solution by J. BOERSMA AND P. J. DE DOELDER (Eindhoven University of Technology, Eindhoven, the Netherlands).

The two integrals are evaluated by a method due to Belevitch [1, §3], for obtaining indefinite integrals involving independent solutions of a linear differential equation, by means of the Wronskian. First we express $A_n(s)$ in terms of confluent hypergeometric functions Φ [2, §6.1] and of the Hermite polynomial H_n by means of [2, form. 10.13(17), (18)], viz.

$$A_n(s) = \Phi\left(-\frac{1}{2}n, \frac{1}{2}; -s^2\right) = \frac{i^{-n} \left(\frac{1}{2}n\right)!}{n!} H_n(is), \quad n \text{ even and } \geq 0,$$

$$A_n(s) = 2s \Phi\left(-\frac{1}{2}n + \frac{1}{2}, \frac{3}{2}; -s^2\right) = \frac{i^{-n} \left(\frac{1}{2}n - \frac{1}{2}\right)!}{n!} H_n(is), \quad n \text{ odd and } \geq 1.$$

From the differential equation for Hermite polynomials (see [2, form. 10.13(12)]) we infer that $u = H_n(is)$ satisfies the equation

$$u'' + 2su' - 2nu = 0,$$

where a prime denotes differentiation with respect to s . The latter equation has two independent power-series solutions

$$u_1(s) = \Phi\left(-\frac{1}{2}n, \frac{1}{2}; -s^2\right), \quad u_2(s) = s \Phi\left(-\frac{1}{2}n + \frac{1}{2}, \frac{3}{2}; -s^2\right),$$

constructed in the standard manner. The Wronskian of these solutions is readily found to be

$$W(u_1, u_2) = u_1 u_2' - u_1' u_2 = e^{-s^2}.$$

Furthermore, we establish the expansions

$$\begin{aligned} \frac{u_1(s)}{u_2(s)} &= \frac{1}{s} + O(s) \quad (s \rightarrow 0), \\ \frac{u_1(s)}{u_2(s)} &= \frac{2\Gamma(\frac{1}{2}n + 1)}{\Gamma(\frac{1}{2}n + \frac{1}{2})} + O(s^{-2}) \quad (s \rightarrow \infty), \end{aligned}$$

where the latter expansion was obtained from the asymptotics of the Φ -function, as discussed in [1, §6.13.1].

The evaluation of the integrals is now straightforward by setting $e^{-s^2} = u_1 u_2' - u_1' u_2$, $A_n(s) = u_1$ for even $n \geq 0$, and $A_n(s) = 2u_2$ for odd $n \geq 1$. As a result we find

$$\begin{aligned} \int_0^\infty \frac{e^{-s^2}}{A_n^2(s)} ds &= \int_0^\infty \frac{u_1 u_2' - u_1' u_2}{u_1^2} ds = \left. \frac{u_2(s)}{u_1(s)} \right|_{s=0}^{s=\infty} \\ &= \frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{2\Gamma(\frac{1}{2}n + 1)}, \quad n \text{ even and } \geq 0, \\ \int_0^\infty \left(\frac{1}{4s^2} - \frac{e^{-s^2}}{A_n^2(s)} \right) ds &= \frac{1}{4} \int_0^\infty \left(\frac{1}{s^2} - \frac{u_1 u_2' - u_1' u_2}{u_2^2} \right) ds \\ &= \frac{1}{4} \left(\frac{u_1(s)}{u_2(s)} - \frac{1}{s} \right) \Big|_{s=0}^{s=\infty} = \frac{\Gamma(\frac{1}{2}n + 1)}{2\Gamma(\frac{1}{2}n + \frac{1}{2})}, \quad n \text{ odd and } \geq 1. \end{aligned}$$

Remark. The two results may be combined into the integral

$$\int_0^\infty \frac{e^{-s^2}}{H_n^2(is)} ds = \frac{2^{-n-1} \sqrt{\pi}}{n!}, \quad n = 0, 1, 2, \dots$$

where in the case of odd n , the finite part of the divergent integral should be taken.

REFERENCES

[1] V. BELEVITCH. *The Gauss hypergeometric ratio as a positive real function*. SIAM J. Math. Anal., 13 (1982), pp. 1024-1040.
 [2] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, *Higher Transcendental Functions*, Vol. I, II, McGraw-Hill, New York, 1953.

Also solved by C. C. GROSJEAN (University of Ghent, Ghent, Belgium), W. B. JORDAN (Scotia, NY), and the proposer.

ERRATA

*Problem 92-11**, by MALTE HENKEL (Université de Genève, Switzerland) and R. A. WESTON (University of Durham, Durham, UK).