Enumerative combinatorial problems concerning structures

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ENUMERATIVE COMBINATORIAL PROBLEMS CONCERNING STRUCTURES

BY

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1. *Introduction*. Before we state the main theme of this paper, we present the definition of the cycle index of a permutation group and we describe a typical special case of Pólya's famous counting theorem connected with it.

Let D be a finite set with n elements. If g is any permutation of D then g splits into cycles; let $b_j(g)$ be the number of cycles of length j (whence $\sum jb_j(g) = n$). Let G be a group of permutations of D; $|G|$ denotes the order of the group. Let $y_1, y_2, y_3, \dots$ be variables. To each element $g \in G$ we associate a product $y_1{}^{b_1(g)} y_2{}^{b_2(g)} \dots$ (we do not bother to write down the last factor in the product, for we may adopt the convention that an infinite product $y_{k+1}{}^0 y_{k+2}{}^0 \dots$ represents unity). The "*cycle index*" (or "cycle index polynomial") is the average of these products:

$$P_G(y_1, y_2, y_3, \dots) = |G|^{-1} \sum_{g \in G} y_1{}^{b_1(g)} y_2{}^{b_2(g)} y_3{}^{b_3(g)} \dots . \tag{1.1}$$

It is obviously a polynomial of finite degree in a finite number of variables. We now describe a special case of Pólya's theorem (see Pólya [7], and, for generalizations, De Bruijn [1, 2]). We consider mappings f of D into a set R. Two of these, f_1 and f_2, are called equivalent if there is a $g \in G$ such that $f_1 g = f_2$. An equivalence class of mappings is called a *mapping pattern*. Then Pólya's theorem states that the number of patterns is

$$P_G(m, m, m, \dots), \tag{1.2}$$

where m is the number of elements of R.

It seems that so far most applications of (1.2) were made with isolated cases, using only one group at a time, and that hardly any attempt was made to do something with a whole class of groups. This is connected with the fact that there are not many classes of groups for which we posses something like a catalogue of the cycle indexes. It is the object of the present paper to establish that there are cases where we have a sequence of groups $G_1, \ldots, G_N$, where the sum

$$\Sigma_{j=1}^{N} P_{G_j}(m, m, m, \ldots) \tag{1.3}$$

has a combinatorial significance, and where the sum

$$U(y_1, y_2, \ldots) = \Sigma_{j=1}^{N} P_{G_j}(y_1, y_2, \ldots) \tag{1.4}$$

can be evaluated more or less successfully. In fact $U(m, m, m, \ldots)$ has a combinatorial meaning which is similar to the one of (1.2), so U plays the same role as the cycle index of a permutation group. We shall call it the U-polynomial.

We give here a brief description of one of the typical examples. Let $\Gamma_1, \ldots, \Gamma_N$ be the different (i.e. pairwise non-isomorphic) graphs with n nodes and k edges, and let G_j be the automorphism group of Γ_j. Then, $P_{G_j}(m, m, m, \ldots)$ represents, by Pólya's theorem, the number of essentially different ways to colour the nodes of Γ_j, if the colours have to be chosen from a given set of m colours. Hence (1.3), that is $U(m, m, m, \ldots)$, is the number of different m-coloured graphs with n nodes and k edges. Similarly it will be established that $U(0, 2, 0, 2, \ldots)$ is the number of symmetrically bicoloured graphs with n nodes and k edges. These two examples may suffice to show the combinatorial significance of $U(y_1, y_2, \ldots)$. It is, of course, only a matter of orderly administration to multiply this expression by $w^n z^k$ (z and w are independent variables) and to sum for n and k. This produces a generating function $U(w, z; y_1, y_2, \ldots)$ which gives the answer to many combinatorial questions concerning the set of *all* graphs.

In these ideas, the notion of a graph can be replaced by any other class of structures on finite sets, and for a number of them (see secs. 3 and 4) we get quite simple expressions for the U-polynomial or for its generating function.

In sec. 5 we consider *pairs* of structures. As in particular a colouring of a set may be considered as a structure on that set, a coloured structure is, in fact, already a structure pair.

Notation. If D and R are sets, then the set of all mappings of D into R is denoted by R^D. If $d \in D$, $f \in R^D$, then the image of d under the mapping f is indicated sometimes by $f(d)$, sometimes by fd. If

D, R, S, T are sets, if $f \in R^D$, $g \in S^T$, and if $R \subset T$, then the composition gf is the mapping of D into S, defined as follows: $(gf)(d) = g\{f(d)\}$ for all $d \in D$. If A and B are sets, then $A \times B$ is the cartesian product, i.e. the set of all pairs $(a, b)(a \in A, b \in B)$. If D is a finite set, then $|D|$ represents the number of elements of D.

2. *Structures on a finite set.* We shall not define here explicitly what we mean by the word "structure". We shall only assume that there is a set of things which are called "structures" and which are permuted in a certain way.

Let D be a finite set, to be called the *base set.* Let G be a group of permutations of D. Let S be a finite set, the elements of which are called *structures,* or in particular, *structures on D.* Finally we assume that we have a representation σ of G as permutations of S. That is, to each $g \in G$ there corresponds a permutation $\sigma(g)$ of S, and $\sigma(g_1)\,\sigma(g_2) = \sigma(g_1 g_2)$ for all $g_1, g_2 \in G$. Throughout this section, the symbols D, G, S, σ will keep this fixed meaning.

Two elements s_1, s_2 of S are called *isomorphic* if there is a $g \in G$ such that $\sigma(g)s_1 = s_2$. Isomorphism is an equivalence relation; the equivalence classes are called *structure classes.* If $g \in G$ and $s \in S$ are such that $\sigma(g)s = s$, then g is called an *automorphism* of s. These g form a group H_s, called the automorphism group of s.

If two elements of S are isomorphic, then their automorphism groups are conjugated: if $\sigma(h)s = s$, then $\sigma(ghg^{-1}) = \sigma(g)\,\sigma(h)\,\sigma(g)^{-1}$ leaves $\sigma(g)s$ invariant, whence $H_{\sigma(g)s} = gH_s g^{-1}$. It follows that H_s and $H_{\sigma(g)s}$ have the same cycle index. So if K is a structure class, we can define

$$Z_K(y_1, y_2, \ldots) = P_H(y_1, y_2, \ldots), \tag{2.1}$$

where H is the automorphism group of any arbitrary structure of the class. And we define

$$U(y_1, y_2, \ldots) = \textstyle\sum_K Z_K(y_1, y_2, \ldots), \tag{2.2}$$

where the sum runs over all possible classes of S. Since D and S are finite, U is a polynomial; it will be referred to as the *U-polynomial* of S.

We first explain the combinatorial significance of U. We take a third finite set R, the elements of which we shall refer to as *colours.* And we consider pairs (s, f), where $s \in S$, $f \in R^D$ (i.e. f is a mapping of D into R, i.e. a *colouring* of D). These pairs (s, f) will be called *coloured structures* (although it is the base set D that is coloured, and not the structure).

The set of coloured structures may be considered as a new set S^* of structures, as we can introduce a representation σ^* of G as permutations of S^*. If $g \in G$, we define $\sigma^*(g)$ by

$$\sigma^*(g)\,(s, f) = (\sigma(g)s, fg^{-1}) \qquad (s \in S, f \in R^D).$$

For example, if S is the set of all possible graphs whose set of nodes is D, then the coloured structures are graphs with coloured nodes. If s is a graph, and g a permutation of D, then we define $\sigma(g)s$ as the graph in which d_1 and d_2 are connected if and only if $g^{-1}(d_1)$ and $g^{-1}(d_2)$ are connected in the original graph s. Further, if s^* is a colouring of s, then $\sigma^*(g)s$ is the colouring of $\sigma(g)s$ obtained by giving each d the colour that $g^{-1}d$ had in the original coloured graph s^*. Another point of view, which we shall not adopt here, however, is that $\sigma^*(g)$ does nothing to the graph and nothing to the colours but that it only changes the *names* of the elements of D.

Returning to our general structure, we again introduce equivalence classes, this time in the set of coloured structures. Two coloured structures s_1^*, s_2^* will be called isomorphic if there is a $g \in G$ such that $\sigma^*(g)s_1^* = s_2^*$. Again, this is a equivalence relation; for the equivalence classes we shall use the name *coloured structure patterns*.

Following Pólya's theorem, with the terminology of [3], we attach a *weight* $w(r)$ to each colour (the weights may be numbers, or variables, or, more generally, the weights may be elements of some commutative ring which is, at the same time, a vector space over the rationals). To each colouring $f \in R^D$ we attach as weight

$$W(f) = \Pi_{d \in D}\, w(f(d)), \tag{2.3}$$

and, if F is a coloured structure pattern, we may define its weight $W(F)$ by taking any arbitrary pair $(s, f) \in F$ and putting $W(F) = W(f)$ (the weight does not depend on the structure, but only on the colouring). We want to express the sum of the weights of the coloured structure patterns, and that is achieved by theorem 1.

Theorem 1. $\Sigma_F\, W(F) = U(\Sigma_{r \in R}\, w(r),\ \Sigma_{r \in R}\, (w(r))^2, \ldots)$.

Corollary. In particular, by choosing $w(r) = 1$ for all r, we obtain that the *number* of coloured structure patterns equals $U(|R|, |R|, |R|, \ldots)$ ($|R|$ is the number of colours). Again taking a special case, viz. $|R| = 1$, we infer that the total number of structure classes equals $U(1, 1, 1, \ldots)$.

Proof. If (s, f) is a coloured structure, and if s_1 is isomorphic with s, then there is an f_1 such that (s, f) and (s_1, f_1) are isomorphic. And if s_1 is not isomorphic with s, there is no such f_1. So in order to enumerate all F it suffices to select a single s from each structure class K, and to investigate the pairs (s, f). We notice that (s, f_1) and (s, f_2) are isomorphic if and only if there is a $g \in H_s$ such that $f_1 = f_2 g$. So the relation between f_1 and f_2 is exactly the equivalence relation considered in Pólya's theorem, and we observe a one-to-one correspondence between the coloured structure patterns (with fixed s) and Pólya's mapping patterns (see sec. 1). Pólya's theorem gives for the sum of the weights

$$P_{H_s}(\Sigma_{r \in R}\, w(r),\ \Sigma_{r \in R}\, (w(r))^2,\ \Sigma_{r \in R}\, (w(r))^3,\ \ldots)$$

(the special case that all weights are 1 was mentioned in (1.2)).

So far we had the contribution of a single s. We have to take one s from each class K and to carry out summation over all classes. So by (2.1) and (2.2) the theorem follows.

It is quite easy to transform U into an expression which makes no reference to equivalences, nor to automorphism groups. If g is any element of G, we define

$$V(g) = |\{s|\sigma(g)s = s\}|, \tag{2.4}$$

that is, $V(g)$ is the number of structures which are invariant under $\sigma(g)$. With this notation we can state

Theorem 2. We have

$$U(y_1, y_2, y_3, \ldots) = |G|^{-1}\, \Sigma_{g \in G}\, V(g)\, y_1^{b_1(g)} y_2^{b_2(g)} \ldots,$$

where $b_j(g)$ denotes (as in sec. 1) the number of cycles of g with length j.

Proof. We fix our attention to a special structure class K. We notice that for all $s \in K$ the automorphism group H_s has the same cycle index, viz. Z_K (see (2.1)), that all H_s ($s \in K$) have the same order, and that, for all $s \in K$, the quotient $|G|/|H_s|$ represents the number of elements in K. Therefore

$$Z_K = |H_s| \,.\, |G|^{-1}\, \Sigma_{s \in K}\, P_{H_s}.$$

Applying the definition (1.1) we find

$$Z_K = |G|^{-1}\, \Sigma_{s \in K}\, \Sigma_{g \in H_s}\, \psi(g),$$

where

$$\psi(g) = y_1^{b_1(g)} y_2^{b_2(g)} \ldots. \tag{2.5}$$

In other words, $Z_K = |G|^{-1} \sum_s \sum_g \psi(g)$, where the summation runs over all pairs (s, g) with $s \in K$, $g \in G$, $\sigma(g)s = s$. Carrying out summation with respect to K, we obtain that

$$U = \sum_K Z_K = |G|^{-1} \sum_g \sum_s \psi(g);$$

where the summation is now restricted by $s \in S$, $g \in G$, $\sigma(g)s = s$. If g is fixed, the number of possible s equals $V(g)$, so our proof is complete.

We close this section by indicating an application of the polynomial U that is not a direct consequence of Pólya's theorem. We take a set of two colours. A structure s is said to be *symmetrically bicoloured* if there is an automorphism of the structure that interchanges the colours, i.e. if there is a $g \in H_s$ such that $fg^{-1} = pf$, where p is the permutation that interchanges the two elements of R. If in a coloured structure the colouring has this symmetry, then the same is true for all coloured structures of the same coloured structure pattern, and we may call the pattern a *symmetrically bicoloured structure pattern.*

Theorem 3. The number of symmetrically bicoloured structure patterns equals $U(0, 2, 0, 2, 0, 2, \ldots)$.

Proof. As in the proof of theorem 1, we have to select a single s from each structure class K, and to count the number of symmetric bicolourings of s. According to [2] (or to sec. 4 of the present paper) the number of symmetric bicolourings of a set (symmetry being defined by a permutation group H) equals $P_H(0, 2, 0, 2, \ldots)$. In our case we have to take for the group H the automorphism group H_s, whence the number of symmetric bicolourings of the structure s equals $Z_K(0, 2, 0, 2, \ldots)$. Now taking the sum over all classes K, the theorem follows.

A generalization of theorem 3 can be obtained from a result of sec. 4 (see (4.4)). We take a set R of colours, and a permutation p of R. For $j = 1, 2, 3, \ldots$, let λ_j denote the number of elements r of R with $p^j r = r$. Then the number of coloured structure patterns which are not affected by the colour permutation p, equals $U(\lambda_1, \lambda_2, \lambda_3, \ldots)$. In the case that R contains only two colours, and p interchanges these, we have $\lambda_1 = \lambda_3 = \lambda_5 = \ldots = 0$, $\lambda_2 = \lambda_4 = \ldots = 2$, and we get theorem 3. In the case that R contains any number of colours, and p is the identity, we get the corollary of theorem 1.

3. *Examples with symmetric group G.* In each example we take for S the set of all structures of a given type, and G is always the full symmetric group of D. The number of elements of D is called n. In all examples of this section the U-polynomial will depend on n, and will be interpreted as the coefficient of w^n in a generating function

$$U(w; y_1, y_2, \ldots) = \Sigma_0^\infty w^n U_n(y_1, y_2, \ldots). \tag{3.1}$$

(i) *"Trivial" structures.* Having the trivial structure on D means that D is considered as just a set. Or, rather, the set S consists of only one element, and the representation σ can be only the trivial one. So there is only one structure class, and the automorphism group of the one element in that class is G itself. So $U(y_1, y_2, \ldots) = P_G(y_1, y_2, \ldots)$. As G is the symmetric group of degree n, we know (see [7, 2]) that (3.1) equals

$$U(w; y_1, y_2, \ldots) = \exp(y_1 w + \tfrac{1}{2}y_2 w^2 + \tfrac{1}{3}y_3 w^3 + \ldots).$$

(ii) *Mappings into a fixed set.* Let R be a fixed finite set. We take $S = R^D$, so the structures are the mappings of D into R. The representation σ is defined as follows: for each $g \in G$, $s \in S$ we have $\sigma(g)s = sg^{-1}$. So the structure classes are just the mapping patterns, or colour patterns, of sec. 1. We claim that

$$U(w; y_1, y_2, \ldots) = \exp(ry_1 w + \tfrac{1}{2}ry_2 w^2 + \tfrac{1}{3}ry_3 w^3 + \ldots), \tag{3.2}$$

where r stands for the number of elements of R. This can be obtained from theorem 2. We suppress the derivation, since (3.2) does not lead to new results. We have to bear in mind that colouring the structures with a colour set R_1 means the same thing as colouring D itself with a colour set $R \times R_1$, so that the number of coloured structures can be obtained directly from Pólya's theorem.

(iii) *Ordered couplings.* An ordered coupling of a set D is a set of pairs (d_1, d_2) of elements of D, such that each element occurs exactly once in a pair. This can only happen, of course, if n is even. An alternative definition of an ordered coupling is: a one-to-one mapping of a subset of D onto its complement. The natural definition of the representation σ is obtained by requiring that if (d_1, d_2) occurs in the coupling s, then (gd_1, gd_2) occurs in the coupling $\sigma(g)s$.

For this type of structure, the generating function contains even powers of w only. We can show that

$$U(w; y_1, y_2, \ldots) = \exp(y_1^2 w^2 + \tfrac{1}{2}y_2^2 w^4 + \tfrac{1}{3}y_3^2 w^6 + \ldots), \tag{3.3}$$

but we again suppress the proof, as (3.3) does not lead to new results. For colouring an ordered couple with colours taken from the colour set R, can also be described as colouring the first component of the couple with a colour taken from the set $R \times R$.

(iv) *Permutations.* Let S be the set of all permutations of D. If $s \in S$, $g \in G$, we define $\sigma(g)s = gsg^{-1}$ (so if s carries d into d', then $\sigma(g)s$ carries gd into gd'). We shall show that, in this case

$$U(w; y_1, y_2, \ldots) = \{(1 - y_1 w)(1 - y_2 w^2)(1 - y_3 w^3) \ldots\}^{-1}. \quad (3.4)$$

Notice that the *number* of structure classes on D is equal to the number of partitions of n, and, indeed, (3.4) produces the well-known generating function for these partition numbers by setting $y_1 = y_2 = \ldots = 1$.

We prove (3.4) by application of theorem 2. According to the definition of σ, the number $V(g)$ (see (2.4)) equals the number of s with $gs = sg$. The permutations commuting with g map cycles of g into cycles of the same length. If g has b_1 cycles of length 1, b_2 of length 2, etc., then there are $b_1!\, b_2!\, b_3! \ldots$ possibilities for mapping the set of cycles onto itself by mappings that preserve the length of the cycles. And from each one of these mappings we obtain $2^{b_2} 3^{b_3} \ldots$ possibilities for s. So

$$V(g) = b_1!\, 1^{b_1} b_2!\, 2^{b_2} b_3!\, 3^{b_3} \ldots.$$

We have $b_1 + 2b_2 + \ldots = n$, and the number of $g \in G$ corresponding to this sequence $b_1, b_2, \ldots$ equals

$$n!/\{b_1!\, 1^{b_1} b_2!\, 2^{b_2} \ldots\}. \quad (3.5)$$

Now applying theorem 2 we find that $U(y_1, y_2, \ldots)$ equals the coefficient of w^n in the product

$$\{\Sigma_{b_1} y_1{}^{b_1} w^{b_1}\} \{\Sigma_{b_2} y_2{}^{b_2} w^{2b_2}\} \ldots,$$

and (3.4) follows.

(v) *Cyclic arrangements.* A cyclic arrangement of D is the same thing as a cyclic permutation of D. For this type of structure we get, using the same σ as in the previous example,

$$U(w; y_1, y_2, \ldots) = \Sigma_{j=1}^{\infty} \frac{\varphi(j)}{j} \log \frac{1}{1 - y_j w^j},$$

where φ denotes Euler's function. This can be obtained by application of theorem 2, but also directly from the definition (1.4): notice that D has only one class of cyclic arrangements, and that the automorphism group of D is just the cyclic group of order n, whose cycle index is

$$n^{-1} \sum_{j|n} \varphi(j) y_j^{n/j}.$$

(vi) *Nestings.* A *nesting* of D is a finite decreasing sequence of non-empty subsets of D, starting with D itself. ("decreasing" means that of any two consecutive sets in the sequence the latter is a proper subset of the former). If $g \in G$ and if s is a nesting, then $\sigma(g)s$ is defined as the nesting whose subsets are obtained from those of s upon transformation by g.

A nesting can also be described by looking at the differences of the consecutive nested sets. If we number these differences 1, 2, 3, ..., we get what can be called a "labeled partition", or a "preferential arrangement" (see [4]).

We want to determine $V(g)$, i.e. the number of nestings invariant under $\sigma(g)$. A nesting is invariant under $\sigma(g)$ if and only if for each cycle of g and for each set of the nesting sequence it is true that either the cycle is part of the set or the cycle and the set are disjoint. It follows that the number of invariant nestings equals the total number of nestings of a set with m elements, where m is the number of cycles of g. Obviously $m = b_1(g) + b_2(g) + b_3(g) + \dots$. Denoting this total number of nestings by q_m, and putting $q_0 = 1$, it is easy to derive the recurrence relation

$$q_\nu = \sum_{0 \leq \mu < \nu} \binom{\nu}{\mu} q_\mu, \qquad (\nu > 0),$$

from which we obtain the generating function

$$\sum_{\nu=0}^{\infty} q_\nu x^\nu/\nu! = (2 - e^x)^{-1} = 1 + x + \tfrac{3}{2}x^2 + \tfrac{13}{6}x^3 + \dots.$$

Hence the number of nestings invariant under $\sigma(g)$ can be represented as the value of

$$\left(\frac{d}{dx}\right)^{b_1(g)} \left(\frac{d}{dx}\right)^{b_2(g)} \dots (2 - e^x)^{-1}$$

at $x = 0$. Applying (3.5) and theorem 2, we obtain that $U(y_1, y_2, \dots)$ equals the coefficient of w^n of

$$\sum_{b_1=0}^{\infty} \frac{1}{b_1!} \left(\frac{y_1 w}{1} \frac{d}{dx}\right)^{b_1} \sum_{b_2=0}^{\infty} \frac{1}{b_2!} \left(\frac{y_2 w^2}{2} \frac{d}{dx}\right)^{b_2} \dots (2 - e^x)^{-1}$$

at $x = 0$. Now Taylor's formula

$$\{\exp (a\, d/dx)\} f(x) = f(x + a)$$

leads to the final result

$$U(w; y_1, y_2, \ldots) = \left\{2 - \exp\left(\frac{y_1 w}{1} + \frac{y_2 w^2}{2} + \frac{y_3 w^3}{3} + \ldots\right)\right\}^{-1}.$$

We can also split this expansion according to the length of the sequence of sets involved in the nesting. Let $q_{\nu\lambda}$ denote the number of nestings of length λ, constructed in a set of ν elements, and put $Q_\nu(z) = \sum_\lambda q_{\nu\lambda} z^\lambda$, $(\nu \geq 1)$, $Q_0(z) = 1$. Then it is not difficult to derive a recurrence for $Q_\nu(z)$, viz.

$$Q_\nu(z) = z \sum_{0 \leq \mu < \nu} \binom{\nu}{\mu} Q_\mu(z),$$

and we derive

$$\sum_{\nu=0}^{\infty} Q_\nu(z) x^\nu / \nu! = \{1 - z(e^x - 1)\}^{-1}.$$

As above, we can substitute $x = y_1 w + \frac{1}{2} y_2 w^2 + \ldots$ in order to get the corresponding result for $U(w; y_1, y_2, \ldots)$.

(vii) *Graphs*. We consider graphs without loops and without double connections, with D as the set of nodes. And σ is defined in the obvious way: if s is one of the graphs, and if $g \in G$, then $\sigma(g)s$ is the graph in which d_1, d_2 are connected if and only if $g^{-1}d_1$, $g^{-1}d_2$ are connected in s.

We split our set of structures according to the number of nodes as well as to the number of edges. Let S_{nj} be the set of all graphs with n nodes and j edges. Its U-polynomial is denoted by $U_{nj}(y_1, y_2, \ldots)$. Accordingly, $V_{nj}(g)$ denotes the number of graphs with n nodes and j edges which are invariant under $\sigma(g)$. As G is the symmetric group, $V_{nj}(g)$ only depends on the cycle structure of g, i.e. on $b_1, b_2, b_3, \ldots$ (where $b_i = b_i(g)$). We may write $V_{nj}(g) = V_j(b_1, b_2, b_3, \ldots)$; we need not indicate n on the right-hand side, since n depends on the b's: $n = b_1 + 2b_2 + 3b_3 + \ldots$. The number of g having these b's is expressed by (3.5).

We build the generating function

$$U(w, z; y_1, y_2, \ldots) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} w^n z^j U_{nj}(y_1, y_2, \ldots). \qquad (3.6)$$

From theorem 2 we deduce the following result: $U(w, z; y_1, y_2, \ldots)$ is obtained by expanding

$$\exp(y_1 w + \tfrac{1}{2} y_2 w^2 + \tfrac{1}{3} y_3 w^3 + \ldots)$$

in terms of powers of $y_1, y_2, y_3, \ldots$, and multiplying each term $y_1^{b_1} y_2^{b_2} \ldots$ by the factor

$$V(z; b_1, b_2, \ldots) = \sum_{j=0}^{\infty} z^j V_j(b_1, b_2, b_3, \ldots). \qquad (3.7)$$

(needless to say, $V_j = 0$ from a certain j onward, as the number of edges cannot exceed $\frac{1}{2}(n^2 - n)$).

Using the notation (k, m) for the greatest common divisor of k and m, we shall show that

$$V(z; b_1, b_2, \ldots) = \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 + z^{km/(k,m)})^{\frac{1}{2}(k,m)b_k b_m} \times \times \prod_{k=1}^{\infty} (1 + z^k)^{-\frac{1}{2}b_k} \prod_{r=1}^{\infty} \left\{ \frac{(1 + z^r)^2}{1 + z^{2r}} \right\}^{\frac{1}{2}b_{2r}}. \tag{3.8}$$

In order to prove this, we take a special permutation g of G, with $b_1(g) = b_1$, $b_2(g) = b_2$, etc. We want to count the number of graphs (with D as the set of nodes) which are invariant under $\sigma(g)$, or, rather, we want to have the sum of the *weights* of those graphs, where each graph has the weight w_j, if j is the number of edges. Such graphs can be found by superposition of a number of possibly simpler graphs which we call *primitive*: a primitive graph is obtained by choosing a single edge and constructing the graph whose edges are just all edges obtained from this special one by repeated application of g. We want to evaluate the primitive graph inventory, i.e. the sum of the weights of all primitive graphs (we still keep g fixed).

First, we have primitive graphs arising from a connection between two different cycles, one of length k and one of length m (k and m are not necessarily different). Once these cycles have been fixed, it is easy to see that there are still (k, m) primitive graphs possible, and that each one of these has $km/(k, m)$ edges. Their contribution to the inventory is $(k, m)z^{km/(k,m)}$. This was for one single pair of cycles; if we take the sum over all pairs we obtain

$$\tfrac{1}{2} \textstyle\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} b_k b_m (k, m) z^{km/(k,m)} - \tfrac{1}{2} \sum_{k=1}^{\infty} b_k k z^k \tag{3.9}$$

(the subtracted sum arises from the fact that connections within a cycle were excluded for the moment).

Secondly, we consider the remaining primitive graphs, e.g. those arising from a connection between two different points of one and the same cycle. If the cycle has length k, and if k is odd, then there are $\frac{1}{2}(k - 1)$ such primitive graphs, and they all have k edges; if k is even there are $\frac{1}{2}k - 1$ primitive graphs with k edges, and a single primitive graph with $\frac{1}{2}k$ edges. So the contribution to the inventory is $\frac{1}{2}(k - 1)z^k$ if k is odd, and $(\frac{1}{2}k - 1)z^k + z^{\frac{1}{2}k}$ if k is even. Taking the sum over all cycles, we find

$$\textstyle\sum_{k=1}^{\infty} \frac{1}{2}(k - 1) b_k z^k + \sum_{r=1}^{\infty} b_{2r}(z^r - \frac{1}{2}z^{2r}). \tag{3.10}$$

Adding (3.9) and (3.10) we find the primitive graph inventory

$$\tfrac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} b_k b_m (k, m) z^{km/(k,m)} - \tfrac{1}{2}\sum_{k=1}^{\infty} b_k z^k + \\ + \sum_{r=1}^{\infty} b_{2r}(z^r - \tfrac{1}{2}z^{2r}). \qquad (3.11)$$

Notice that (3.11) is a power series in z with non-negative coefficients. If we wish, we can write it as

$$\sum_{h=1}^{\infty} c_h z^h, \qquad (3.12)$$

where c_h is the number of primitive graphs with h edges.

Returning to the question of *all* graphs invariant under g, we notice that an arbitrary graph of that type is obtained by taking any subset of the set of primitive graphs, and constructing the superposition of the primitive graphs belonging to that subset. It follows that the inventory of the set of all graphs invariant under g (this inventory is $V(z; b_1, b_2, \ldots)$, by definition) equals

$$\prod_{h=1}^{\infty} (1 + z^h)^{c_h}. \qquad (3.13)$$

Finally remarking that (3.12) is an abbreviation for (3.11), we can rewrite (3.13) as the right-hand side of (3.8).

There are some modifications of our problem which can be dealt with in the same fashion. For example, we might admit loops in our graphs (at most one loop at each node). This would have added the sum $\sum_{k=1}^{\infty} b_k z^k$ to the primitive graph inventory, and that would have resulted in replacing, in (3.8), the second factor

$$\prod_{k=1}^{\infty} (1 + z^k)^{-\frac{1}{2}b_k} \text{ by } \prod_{k=1}^{\infty} (1 + z^k)^{\frac{1}{2}b_k}.$$

Another modification is obtained by admitting multiple connections between nodes. In that case, we have to replace (3.13) by

$$\prod_{h=1}^{\infty} (1 + z^h + z^{2h} + \ldots)^{c_h},$$

and than the expression for $V(z; b_1, b_2, \ldots)$ is altered accordingly.

Our method can also be applied, for instance, to oriented graphs (digraphs), or to trees.

As pointed out in the introduction, $U_{nj}(1, 1, 1, \ldots)$ represents the total number of "different" graphs with n nodes and j edges. This number was also determined by HARARY [5]. And $U_{nj}(2, 2, 2, \ldots)$ is the number of bicoloured graphs (i.e. bicoloured structure patterns), which was also evaluated in a different way by HARARY [6]. $U_{nj}(0, 2, 0, 2, \ldots)$ is the number of symmetrically bicoloured graphs, etc.

4. *Examples with a general group G.* We again take a finite set D, and a permutation group G of D, which is, in contrast to the previous section, not necessarily the symmetric group.

(viii) *TH-structures.* Let T be another finite set, and let H be a group of permutations of T. If both f_1 and f_2 are mappings of T into D, then they are called equivalent if and only if $f_1 = f_2 h$ for some $h \in H$. The equivalence classes defined by this equivalence will be called TH-structures. If $f \in D^T$, then the TH-structure to which f belongs can be denoted by fH; if f_1 and f_2 are equivalent we have $f_1 H = f_2 H$.

The representation σ will be defined as follows: if $g \in G$, then $\sigma(g)$ maps the class fH onto gfH.

We shall show that for the set of all TH-structures the U-polynomial $U(y_1, y_2, y_3, \ldots)$ is equal to

$$P_H\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots\right) P_G(y_1 e^{z_1+z_2+z_3+\cdots}, y_2 e^{2(z_2+z_4+z_6+\ldots)}, y_3 e^{3(z_3+z_6+z_9+\ldots)}, \ldots) \qquad (4.1)$$

(evaluated at $z_1 = z_2 = \ldots = 0$). In order to prove this, we apply theorem 2, so for any $g \in G$ we have to determine $V(g)$, i.e. the number of TH-structures invariant under $\sigma(g)$. Obviously $V(g)$ is equal to the number of equivalence classes which entirely consist of functions f satisfying $gf \in fH$.

For determining the number of equivalence classes, we first use a device that was applied in Burnside's lemma ([3], sec. 145, theorem viii): the number of classes is $\sum_f^* \{Q(f)\}^{-1}$, where the asterisk indicates that we only take those f for which $gf \in fH$, and $Q(f)$ is the number of elements in the equivalence class to which f belongs. We have $Q(f) = |H|/|H_f|$, where H_f is the group of all h with $fh = f$, since the left cosets of H_f can be brought into 1–1-correspondence with the elements of the equivalence class of f. So

$$V(g) = \textstyle\sum_f^* |H_f|/|H|.$$

If f has the property that $gf \in fH$, there are exactly $|H_f|$ elements $h \in H$ with $gf = fh$; if f does not have the property there are none. It follows that

$$V(g) = |H|^{-1} \textstyle\sum_{h \in H} (\text{number of } f \in D^T \text{ with } gf = fh). \qquad (4.2)$$

This number of f with $gf = fh$ only depends on the cycle representation of g and h. Denoting $b_i(g)$ by b_i, $b_i(h)$ by c_i, the number of

f with $gf = fh$ can be shown to be (see [2])

$$b_1^{c_1}(b_1 + 2b_2)^{c_2}(b_1 + 3b_3)^{c_3}(b_1 + 2b_2 + 4b_4)^{c_4} \dots =$$

$$= \left(\frac{\partial}{\partial z_1}\right)^{c_1}\left(\frac{\partial}{\partial z_2}\right)^{c_2} \dots \exp\{\textstyle\sum_{j=1}^{\infty} jb_j(z_j + z_{2j} + z_{3j} + \dots)\}, \quad (4.3)$$

evaluated at $z_1 = z_2 = \dots = 0$. Taking the sum over all $h \in H$, and dividing by $|H|$, we get $V(g)$. In order to get $U(y_1, y_2, \dots)$ we apply theorem 2, and that produces (4.1).

The special case $y_1 = y_2 = \dots = 1$ in (4.2) reproduces a result of [1, 2]. For, $U(1, 1, 1, \dots)$ is nothing but the number of structure classes, and these structure classes can also be interpreted as follows: consider in D^T the stronger equivalence defined by: $f_1 \sim f_2$ if and only if $f_1 \in Gf_2H$. For the number of these classes we obtained ([1, 2]) the expression (4.1) with $y_1 = y_2 = y_3 = \dots = 1$.

We remark that our above computation of $V(g)$ gives a direct proof for the result on symmetric bicolourings that we used in the proof of theorem 3. For, according to the left-hand side of (4.3), we have:

$$V(g) = P_H(\textstyle\sum_{t|1} tb_t, \sum_{t|2} tb_t, \sum_{t|3} tb_t, \dots). \quad (4.4)$$

The expression on the right occurred in [1], without the present interpretation, however. The result $P_H(0, 2, 0, 2, \dots)$ for symmetric bicolourings arises from (4.4) by taking for D a set of two elements, and for g the permutation that interchanges those two; in that case we have $b_1 = 0$, $b_2 = 1$, $b_3 = b_4 = \dots = 0$.

Notice that the entries $\sum_{t|j} tc_t$ admit a direct interpretation: for each j it is the number of elements of D which are invariant under g^j.

As a further application of (4.4), we ask for the number of ways to colour the six faces of a cube with colours red, white and blue, in such a way that the cyclic permutation red → white, white → blue, blue → red, does not alter the colour scheme essentially ("essentially" means: but for rotations of the cube). Now g is the cyclic permutation of 3 elements, so $b_3 = 1$, $b_1 = b_2 = b_4 = b_5 = \dots = 0$, and the required number is

$$P_H(0, 0, 3, 0, 0, 3, \dots).$$

As $P_H = (y_1^6 + 6y_1^2y_4 + 3y_1^2y_2^2 + 6y_2^3 + 8y_3^2)/24$, our result is $8 . 3^2/24 = 3$.

A further question is, in how many ways we can colour the six faces with red, white and blue in such a way that red and blue can be interchanged without altering the colour scheme essentially. In that case we have $b_1 = 1$, $b_2 = 1$, $b_3 = b_4 = \dots = 0$, whence the required number is $P_H(1, 3, 1, 3, \dots) = 9$. These nine solutions are easily obtained experimentally: one is all-white; two have one red and one blue face, four have two red and two blue faces, and two have three red and three blue faces.

We next show that our notion of TH-structures is sufficiently general for the simulation of all possible (D, G, S, σ) situations of sec. 2, if we exclude the case that D has only one element. To be precise: let D and G be given (G is any group of permutations of D), $|D| > 1$, let S_0 be a set of structures, and let σ_0 be some representation of G by permutations of S_0. We shall construct an exact copy (D, G, S_1, σ_1) of the situation (D, G, S_0, σ_0), where S_1 is a subset of the set of all TH-structures (with a suitable set T and a suitable group H of permutations of T), and σ_1 is essentially the standard representation of G by permutations of S_1. Needless to say, S_1 has to have the property that if it contains a class fH then it contains the class gfH, for each $g \in G$; and σ_1 is defined by $\sigma_1(g)(fH) = gfH$. By saying that (D, G, S_1, σ_1) is an exact copy of (D, G, S_0, σ_0), we mean that there is a 1–1 mapping ψ of S_1 onto S_0 such that $\psi\sigma_1(g) = \sigma_0(g)\psi$ for all $g \in G$.

The construction is as follows. We start from our equivalence in S_0 (s and s' are equivalent if and only if there is a $g \in G$ with $\sigma_0(g)s = s'$. Let $\Sigma_1, \dots, \Sigma_m$ be the equivalence classes (i.e. the structure classes), and select an element s_i from each Σ_i. The automorphism group of s_i (i.e. the group of all $h \in G$ with $\sigma_0(h)s_i = s_i$) is denoted by H_i. We now define T and H by

$$T = D \times \dots \times D \ (m \text{ factors}), \quad H = H_1 \times \dots \times H_m,$$

the elements of H acting on T as indicated by

$$(h_1, \dots, h_m)(d_1, \dots, d_m) = (h_1 d_1, \dots, h_m d_m) \qquad (h_i \in H_i,\ d_i \in D).$$

We next consider the mapping f_1 of T onto D, defined by

$$f_1(d_1, \dots, d_m) = d_1,$$

and similarly we define $f_2, \dots, f_m$. And we consider the set S_1 of all those TH-structures which can be written in the form gf_jH ($g \in G$, $j = 1, \dots, m$). This set has obviously the property that if it contains a class fH, then it contains the class gfH, for each $g \in G$. We shall define the mapping ψ of S_1 onto S by

$$\psi(gf_jH) = \sigma_0(g)s_j, \tag{4.5}$$

but we have to show first that this definition is unambiguous. Assume that gf_jH represents the same TH-structure as $g'f_iH$. Then there is an element $h \in H$ (we put $h = (h_1, \ldots, h_m)$) such that $gf_j = g'f_ih$. An arbitrary element $(d_1, \ldots, d_m)$ of T is mapped by gf_j onto gd_j, and by $g'f_ih$ onto

$$g'f_i(h_1d_1, \ldots, h_md_m) = g'h_id_i.$$

So g, g', h_i, i and j are such that $gd_j = g'h_id_i$ for all possible $(d_1, \ldots, d_m)$. This implies that d_j is uniquely determined by d_i; since $|D| > 1$ it follows that $i = j$. Moreover, $g = g'h_i$; as $h_i \in H$, we have $\sigma_0(h_i)s_i = s_i$. Therefore

$$\sigma_0(g)s_j = \sigma_0(g)s_i = \sigma_0(g'h_i)s_i = \sigma_0(g')\sigma_0(h_i)s_i = \sigma_0(g')s_i.$$

This means that (4.5) gives for $\psi(gf_jH)$ the same result as for $\psi(g'f_iH)$, so ψ has been defined unambiguously.

Next we show that the mapping ψ is one-to-one. Assume that g, g', i, j are such that $\sigma_0(g)s_j = \sigma_0(g')s_i$. Then s_i and s_j are equivalent, whence $i = j$. And we infer that $\sigma_0(g'^{-1}g)s_i = s_i$, whence $g'^{-1}g \in H_i$. We define $h = (h_1, \ldots, h_m)$ by taking h_k to be the identy if $k \neq i$, and $h_i = g'^{-1}g$. Now we have, for all $(d_1, \ldots, d_m)$,

$$g'f_ih(d_1, \ldots, d_m) = g'h_id_i,$$
$$gf_j(d_1, \ldots, d_m) = gd_j,$$

and as $g'h_id_i = gd_j$, we infer that gf_ih and gf_j represent the same mapping. So ψ is one-to-one.

It is obvious that ψ maps S_1 *onto* S_0, for each $s \in S_0$ belongs to one of the Σ_j, whence it has the form $\sigma_0(g)s_j$, i.e. the form of the right-hand side of (4.5).

Finally it is easy to show that $\psi\sigma_1(g) = \sigma_0(g)\psi$ for all $g \in G$. For, the definitions of σ_1 (by $\sigma_1(g)(fH) = gfH$) and ψ (by (4.5)) imply, for all $g \in G$ and all $g_1 \in G$,

$$\psi\sigma_1(g)(g_1f'_jH) = \psi(gg_1f_jH) = \sigma_0(gg_1)s_j = \sigma_0(g)\sigma_0(g_1)s_j = \sigma_0(g)\psi(g_1f_jH).$$

The above construction fails in the trivial case that D has only one element. Indeed, if $|D| = 1$, $|S_0| > 1$, simulation by TH-structures is impossible: if T and H are given, there is only one mapping of T into D, in this case, so there is only one TH-structure, and we cannot simulate S_0.

(ix) *Colourings*. We take a set R of colours, and the structures to be considered in this example are just the colourings of D with colours from R, i.e. $S = R^D$. And, for $g \in G$, we define $\sigma(g)$ by $\sigma(g)f = fg^{-1}$ $(f \in R^D)$. This is the same situation as in example ii (sec. 3), this time without restriction to the symmetric group and without summation with respect to n. It is not difficult to obtain (for example by theorem 2) that in this case we have

$$V_2(g) = |R|^{b_1(g)+b_2(g)+\dots},$$
$$U(y_1, y_2, \dots) = P_G(|R|y_1, |R|y_2, \dots).$$

In particular we can take for R a set of two elements, and then the $f \in R^D$ can be interpreted as *subsets* of D.

(x) *Colourings with colour indifference*. We again take a colour set R, but this time we also take a permutation group of R; this group will be denoted by L. Two colourings f_1, f_2 will be called equivalent if there is an $l \in L$ with $f_1 = lf_2$. The equivalence classes defined by this equivalence will be our structures; they can be represented as left classes Lf.

The representation σ will be defined as follows: if $g \in G$, then $\sigma(g)$ maps the class Lf into Lfg^{-1}. These definitions of S and σ are very similar to those involved in the TH-patterns of example viii.

For the U-polynomial belonging to our present structures we shall derive the expression

$$P_G\left(y_1 \frac{\partial}{\partial z_1}, y_2 \frac{\partial}{\partial z_2}, \dots\right) P_L(e^{z_1+z_2+\dots}, e^{2(z_2+z_4+\dots)}, e^{3(z_3+z_6+\dots)}, \dots), \quad (4.6)$$

again evaluated at the point $z_1 = z_2 = \dots = 0$. This result strongly resembles (4.1), and the derivation will follow the same lines.

For each $g \in G$, we have to determine $V(g)$, i.e. the number of classes Lf satisfying $Lf = Lfg^{-1}$. This number of classes equals $\sum_f^* \{Q(f)\}^{-1}$, where the asterisk indicates that we only take those f for which $fg^{-1} \in Lf$, and $Q(f)$ is the number of elements in the equivalence class to which f belongs. We have $Q(f) = |L|/|L_f|$, where L_f is the group of all l with $lf = f$. So

$$V(g) = \textstyle\sum_f^* |L_f|/|L|.$$

If f satisfies $fg^{-1} \in Lf$, then there are exactly $|L_f|$ elements $l \in L$ with $fg^{-1} = lf$; if f does not have that property there are none. Therefore

$$V(g) = |L|^{-1} \textstyle\sum_{l \in L} (\text{number of } f \in R^D \text{ with } fg^{-1} = lf).$$

Denoting $b_i(g)$ by b_i, $b_i(l)$ by c_i, the number of f with $fg^{-1} = lf$ becomes (cf. (4.3))

$$\left(\frac{\partial}{\partial z_1}\right)^{b_1}\left(\frac{\partial}{\partial z_2}\right)^{b_2} \dots \exp\{\textstyle\sum_{j=1}^{\infty} jc_j(z_j + z_{2j} + z_{3j} + \dots)\},$$

evaluated at $z_1 = z_2 = \dots = 0$. In order to get $U(y_1, y_2, \dots)$, we have to take the average over all $l \in L$, to multiply by $y_1^{b_1}y_2^{b_2}\dots$, and to take the average over all $g \in G$. That leads to (4.6).

If the group L consists of the identity only, $P_L(x_1, x_2, \dots)$ reduces to $|R|$, and (4.6) becomes $P_G(|R|y_1, |R|y_2, \dots)$; this special case is nothing but example (ix).

5. *Pairs of structures.* We consider a set D, with a permutation group G, and *two* sets of structures, S_1 and S_2. And we have representations σ_1 and σ_2 of G, as permutations of S_1 and S_2, respectively. We now form the set $S_3 = S_1 \times S_2$ of all pairs (s_1, s_2) $(s_1 \in S_1, s_2 \in S_2)$, and the representation σ_3, defined by

$$\sigma_3(s_1, s_2) = (\sigma_1 s_1, \sigma_2 s_2) \qquad (s_1 \in S_1, s_2 \in S_2).$$

For each one of the three situations, S_1, S_2, S_3 we form the polynomial $U(y_1, y_2, \dots)$ (see (1.4)); we indicate these by U_1, U_2, U_3. And we denote by $V_i(g)$ $(i = 1, 2, 3)$ the number of $s \in S_i$ which are invariant under $\sigma_i(g)$.

According to the definition of σ_3, we observe that a pair (s_1, s_2) is invariant under $\sigma_3(g)$ if and only if both $\sigma_1(g)s_1 = s_1$ and $\sigma_2(g)s_2 = s_2$. Therefore

$$V_3(g) = V_1(g)V_2(g). \tag{5.1}$$

The coloured structures of sec. 2 are obviously a special case of these structure pairs. We get them by taking $S_1 = S$, $S_2 = R^D$ (see sec. 4, example (ix), $S_3 = S \times R^D$. The number of coloured structure patterns now means the same thing as the number of structure classes in S_3, and that is just $U_3(1, 1, 1, \dots)$. We have (see example (ix)),

$$V_2(g) = |R|^{b_1(g)+b_2(g)+\dots}.$$

Now (4.1) reveals, in combination with theorem 2, that $U_3(1, 1, 1, \dots)$, is obtained by substitution of $y_1 = |R|$, $y_2 = |R|$, ... in $U_1(y_1, y_2, \dots)$, so $U_3(1, 1, 1, \dots) = U_1(|R|, |R|, \dots)$; and this is again the result expressed in the corollary of theorem 1.

Similar simple results can always be obtained in situations where $V_2(g)$ only depends on $b_1(g)$, $b_2(g)$, ... For example, if S_2 is the set of nestings (see sec. 3, example vi), we obtain

$$U_3(1, 1, 1, \ldots) = \left\{ U_1\left(\frac{d}{dx}, \frac{d}{dx}, \ldots\right) (2 - e^x)^{-1} \right\}_{x=0}.$$

And if S_2 is the set of permutations (see sec. 3, example iv), it turns out that

$$U_3(1, 1, 1, \ldots) = U_1\left(\frac{d}{dx_1}, \frac{d}{dx_2}, \ldots\right) \cdot (1 - x_1)^{-1}(1 - 2x_2)^{-1}(1 - 3x_3)^{-1}\ldots,$$

where the right-hand side has to be taken at $x_1 = x_2 = \ldots = 0$.

If G is the symmetric group of all permutations of D, then $V(g)$ (defined by (2.4)) depends on $b_1(g)$, $b_2(g)$, ... only. For, if $g \in G$, $g' \in G$, $b_i(g) = b_i(g')$ $(i = 1, 2, 3, \ldots)$, then there exists a permutation h with $hgh^{-1} = g'$. As G is the symmetric group, we have $h \in G$. So if $\sigma(g)$ leaves s invariant, then $\sigma(g')$ leaves hs invariant, and vice versa. It follows that $V(g) = V(g')$. Putting

$$V(g) = \omega(b_1, b_2, \ldots),$$

we have for the U-polynomial:

$$U(y_1, y_2, \ldots) = (n!)^{-1} \textstyle\sum_{g \in G} V(g) y_1^{b_1(g)} y_2^{b_2(g)} \ldots =$$
$$= \textstyle\sum \omega(b_1, b_2, \ldots)(b_1!\, 1^{b_1} b_2!\, 2^{b_2} \ldots)^{-1} y_1^{b_1} y_2^{b_2} \ldots,$$

where the summation runs over all sets b_1, b_2, ... with

$$b_1 + 2b_2 + \ldots = |D|.$$

Now returning to our structure pairs we notice that if G is the symmetric group, then the U-polynomial for the structure pairs can be obtained from the U-polynomials for the component structure sets. If U_1, U_2, U_3 are the U-polynomials for S_1, S_2, $S_1 \times S_2$, respectively, and if

$$U_i(y_1, y_2, \ldots) = \textstyle\sum \alpha_i(b_1, b_2, \ldots) y_1^{b_1} y_2^{b_2} \ldots, \qquad (i = 1, 2)$$

then for U_3 we obtain a similar polynomial, with coefficients

$$\alpha_3(b_1, b_2, \ldots) = \alpha_1(b_1, b_2, \ldots)\alpha_2(b_1, b_2, \ldots)(b_1!\, 1^{b_1} b_2!\, 2^{b_2} \ldots).$$

Similar results can be derived for structure triples, quadruples, etc, instead of pairs. This generalizes the Redfield-Read superposition theorem (see [8, 9]). That theorem is obtained from the above result by taking each structure set to consist of a single structure class only (whence the U's become cycle indexes), and putting $y_1 = y_2 = \ldots = 1$ in the final result.

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