

Graph topology and gap topology for unstable plants

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Graph topology and gap topology
for unstable plants

by

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GRAPH TOPOLOGY AND GAP TOPOLOGY FOR UNSTABLE PLANTS

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Abstract

This paper provides a reformulation of the graph topology and the gap topology in a very general setting in the frequency domain. Many essential properties and their comparison are clearly presented in the reformulation. It is shown that the gap topology is suitable for the general systems rather than square systems with unit feedback, which is the situation studied in [2,3,9]. It is also revealed that, whenever an unstable plant can be stabilized by a feedback, it is a closed operator, mapping input space to output space. Hence the gap topology can always be applied whenever the unstable plants can be stabilized. The graph topology and the gap topology are suitable for different unstable subsets, and have many similar characteristics. If one confines them to the same subset, they will be identical. Finally, the definitions of the graph metric and the gap metric are discussed.

Keywords: graph topology, gap topology, unstable plants, feedback system.

1. Introduction

In many control-system-design problems a topology (or metric) is required for measuring the distance between two systems. For example, in order to specify:

- 1) how well one system approximates another;
- 2) the uncertainties that can be tolerated in a system without destroying such characteristic as stability when feedback is applied;
- 3) the sensitivity of the interconnected system to a change in its components.

For stable systems, represented by input - output mappings, the operator norm can be used to generate a metric. However, the norm cannot be used for comparing unstable systems. Besides, the norm of the difference can be a poor measure of the distance between systems. And there are many systems remaining close together for many practical purposes, even though the norm of their difference approaches infinity. To illustrate this fact let K_1 and K_2 be two frequency response operators with

$$K_1(s) = \frac{1}{s + \varepsilon} ; \quad K_2(s) = \frac{1}{s + 0.1}$$

the closed-loop responses corresponding to K_1 and K_2 with unit feedback is relatively close and the difference is

$$\| (I + K_1)^{-1} - (I + K_2)^{-1} \| \rightarrow 0.0905 \quad (\varepsilon \rightarrow 0+)$$

but the difference between K_1 and K_2 is

$$\| K_1 - K_2 \| \rightarrow +\infty \quad (\varepsilon \rightarrow 0+)$$

where $\| \cdot \|$ is defined by $\| K \| = \sup_{s \in C_+} |K(s)|$.

Developing a topology for unstable systems should be related to a special design purpose. It may be that a topology is suitable for one control design purpose and can not match another one.

There are two topologies which are well developed for the problem of robust stabilization, the graph topology [6,7] and the gap topology [2,3,9]. The graph topology is suitable for unstable systems which have coprime factorizations whereas the gap topology for the unstable system can be regarded as a closed operator from input space to output space.

There exist different opinions on comparison between the graph topology and the gap topology. According to [2,3,9], they are equivalent. Vidyasagar's opinions [6] are:

- 1) The graph topology can conclude causality as well as boundedness whereas the gap topology concludes only boundedness;

- 2) The graph topology is carried out for non-square plants and general (not necessarily unit or even stable) feedback whereas the analysis of the gap topology in [2,3,9] is only performed for square plants under unit feedback;
- 3) The scope of the result in [2,3,9] can be extended by treating the plant-compensator combo as a square system. But the closed-loop response transfer matrix is not shown to be related to individual variations of system and compensator while this is done for the graph topology.

The graph topology and gap topology are certainly not the same topology (for one thing the subsets of the plants they are concerned with are different). But in some sense, they do produce the same convergence. It is the purpose of this paper to reveal the essential properties of these topologies and give a clear comparison.

This paper is not going to be concerned with the causality, because all the results presented here are in the frequency domain while the best framework to consider the causality is in the time domain. A detailed treatment of the causality involved in the topology aspect will be given elsewhere.

A further study is made for the gap topology in this paper. The results show:

- 1) The gap topology can be carried out for non-square plants and general (not necessarily unit or even stable) feedback;
- 2) The closed-loop response transfer matrix can be related to the individual variations of system and compensator.

This paper is organized as follows: In the preliminary section (Section 2), we first introduce the subsets of plants which are dealt with in this article. They lie in a rather abstract family of plants including distributed as well as lumped Linear Time Invariant (LTI) systems. Then, the characteristics of the topologies for unstable plants related with the robustness of a feedback system are presented. In Section 3, the graph topology is reformulated in a very general setting. Section 4 is a preparatory stage of the gap topology, and the gap topology is treated in Section 5. It is shown that the gap topology is suitable for non-square plants with general feedback. Moreover, the closed-loop response transfer matrix is related to the individual variations of both system and compensator. In Section 6, the conclusion and comparison between the graph topology and the gap topology are presented. Section 7 is concerned with the designing metric aspects of these topologies.

For the sake of convenience, we introduce some symbols which will be used. Denote by $\|\cdot\|$ the element norm or operator norm. The norm in the product space of two Banach spaces is defined as

$$\|(x, y)\| := [\|x\|^2 + \|y\|^2]^{1/2}.$$

If X and Y are two Banach spaces, $B(X, Y)$ (or $C(X, Y)$) denote the set of all bounded (or closed) linear operators.

2. Plant subsets and characteristics of the topology for unstable plants

Let H be an integral domain with identity and let $F \supset H$ be a subset of the quotient field of H . Assume X is a Banach space. To build our framework, we impose the following basic assumption:

Basic assumption

Every element $P \in F$ is a linear operator mapping X to X and the operator P is bounded iff P is in H .

We interpret H as the set of SISO stable plants, F as the universe of all SISO plants (including stable plants as a subset) and X as the input and output space. Under the basic assumption, we know that the unit set U of H is an open subset of H and the mapping: $u \rightarrow u^{-1}$ ($u \in U$) is continuous in the topology induced by the operator norm. Denote by $M(F)$ (or $M(H)$) the set of all matrices with entries in F (or H) and suppose $U(H)$ is the set of all unimodular matrices in $M(H)$. In case where it is necessary to display the order of the matrices, a notation of the form $M(F)^{n \times m}$ is used to indicate the subset of $M(F)$ consisting of all $n \times m$ matrices.

The framework built above is a very general set-up including both lumped and distributed, both continuous-time and discrete-time LTI systems' situations. The following examples are presented to show that the basic assumption is reasonable and includes many important situations.

Example 1

Assume that H is the set of all proper rational functions without poles in the closed right half plane (RHP) and F is the set of all rational functions. Let X be the Hardy space $H^2(\mathcal{C}_+)$ (on the RHP). Then in this case the basic assumption is satisfied. This case indicates the continuous time lumped LTI systems.

Example 2

Assume that H is the set of all proper rational functions without poles in the closed unit disc and F is the set of all rational functions. Let X be the Hardy space $H^2(D)$ (on the unit disc). Then, the basic assumption is satisfied in this case, which stands for the discrete time lumped LTI systems.

Example 3

Let H be $\hat{A}_-(0)$, which denotes the well-known \mathcal{C} -algebra of transfer functions studied by Callier and Desoer, and F be $\hat{B}(0)$. Assume X is the Hardy space $H^2(\mathcal{C}_+)$. This situation is a good model of continuous time distributed LTI systems, which can be put into our framework, i.e. the basic assumption is satisfied.

Now let us define the subsets of plants which we are going to work on.

Let

$$C(F)^{n \times m} := \{P \in M(F)^{n \times m} : P \in C(X^m, X^n)\}$$

and

$$C(F) := \bigcup_{m,n} C(F)^{n \times m}.$$

Denote by $R(F)$ the subset of $M(F)$ consisting of all plants which have both right coprime fractions (r.c.f.) and left coprime fractions (l.c.f.) over $M(H)$ [7]. $R(F)^{n \times m}$ is defined in an obvious way.

Below we introduce the concept of uncertainty of a feedback system and the characteristics of the topologies for unstable plants discussed in this article.

Let us consider the standard feedback system shown in Figure 2.1 and suppose P is the plant and C is the compensator. In this paper we consider the case when both P and C are in the same plant subset $R(F)$ (or $C(F)$).

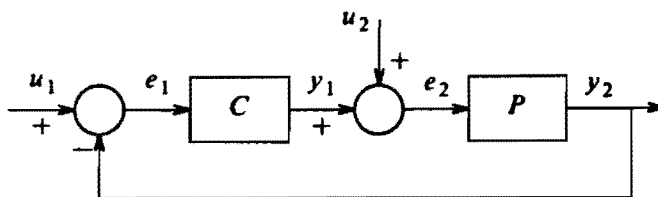


Figure 2.1. Feedback system

Now the closed-loop transfer matrix is given by

$$H(P, C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix}$$

where the well-posed condition $|I+CP| \neq 0$ is always supposed. The compensator of C stabilizes P if $H(P, C) \in M(H)$.

The topologies for unstable plants studied here are concerned with robustness of feedback stability in the presence of plant and/or compensator perturbations.

Suppose further that the plant and/or compensator uncertainties can be modeled by considering the family of plants P_λ and/or the family of compensators C_λ respectively, where the uncertainty parameter λ is assumed to be in a T_2 -topology space Λ .

Here we present two characteristics of the topology for unstable plants which are needed essentially in most of the robustness problems of control system designs. Assume T is a topology over the plant subset $M \subset M(F)$. We say T has Property 1 (or Property 2), if it satisfies the following Condition 1 (or Condition 2).

- 1) In the topology T , the stable plants in M forms an open subset, i.e. $M(H) \cap M$ is open in M ;
- 2) The following convergences are equivalent:

- (i) $P_\lambda \xrightarrow{T} P_{\lambda_0} \quad (\lambda \rightarrow \lambda_0);$
 $C_\lambda \xrightarrow{T} C_{\lambda_0} \quad (\lambda \rightarrow \lambda_0). \quad (\text{simultaneously})$
- (ii) $H(P_\lambda, C_\lambda) \xrightarrow{T} H(P_{\lambda_0}, C_{\lambda_0}) \quad (\lambda \rightarrow \lambda_0).$

where $\{P_\lambda\}$, $\{C_\lambda\}$ and $H(P_\lambda, C_\lambda)$ are in M and " \xrightarrow{T} " means convergence in the topology T .

Let us conclude this section by providing some useful facts about the subsets $R(F)$ and $C(F)$ of the universe of unstable plants.

Lemma 2.1.

Assume $P, C \in M(F)$ such that $H(P, C) \in M(H)$. Then $P \in R(F)$ iff $C \in R(F)$.

This proof can be found in [7].

Lemma 2.2.

Assume $P, C \in M(F)$. If $H(P, C)$ is stable, then $P, C \in C(F)$.

This proof is based on the following lemma.

Lemma 2.3.

Assume X, Y, Z are Banach spaces. Let $T \in C(Y, Z)$, $S \in C(X, Y)$ and $T^{-1} \in B(Z, Y)$. Then $T \cdot S \in C(X, Z)$.

The proof of this lemma can easily be obtained from the definition of a closed operator.

Proof of Lemma 2.2. Since $H(P, C)$ can be written as:

$$H(P, C) = \begin{bmatrix} (I+PC)^{-1} & -(I+PC)^{-1}P \\ (I+CP)^{-1}C & (I+CP)^{-1} \end{bmatrix}$$

we know that $(I+PC)^{-1}$ and $(I+CP)^{-1}$ are bounded. Further, $(I+PC)$ and $(I+CP)$ are closed.

If we take $T = (I+PC)$ and $S = -(I+PC)^{-1}P$, by using Lemma 2.3 we obtain that P is closed. On the other hand, if we take $T = (I+CP)$ and $S = (I+CP)^{-1}C$, we obtain that C is closed. \square

3. Graph topology

The organization of this section shows directly that the graph topology has both Property 1 and Property 2 over $R(F)$.

The graph topology is proposed by Vidyasagar and thoroughly studied in his monograph [7]. Compared with [7], there are two distinguishing features in the reformulation of this paper.

- 1) The definition and theorems are carried out in a very general setting.
- 2) In [7], the result of the spectral factorization of the rational matrix has been used in the proof of "diagonal product property" (as will become clear in the sequel). But, the spectral factorization problem over a general matrix ring has not been solved. So, we have to provide an alternative proof.

The only proof we give in the section is the diagonal product property. The proofs of all other results are some simple translations of [7], we omit them. Reader can consult the excellent treatment of the details of the graph topology in [7].

Recall that $R(F)$ consists of all matrices in $M(F)$ which have both r.c.f. and l.c.f. over $M(H)$. Let us define the graph topology over $R(F)^{n \times m}$. If $P_0 \in R(F)^{n \times m}$, then for any r.c.f. (N_0, D_0) of P_0 , we know that $N_0 \in B(X^m, X^n)$ and $D_0 \in B(X^m)$. Consequently (N_0, D_0) can be interpreted as a bounded linear operator from X^m to the product space $X^n \times X^m$.

Lemma 3.1.

Let (N_0, D_0) be an r.c.f. of P_0 . There exists a constant μ such that: if a pair $(N, D) \in M(H)$ satisfies

$$\|(N, D) - (N_0, D_0)\| < \mu .$$

Then (N, D) is also an r.c. pair and $\det D \neq 0$.

The basic neighbourhood of P_0 is defined as follows: Let (N_0, D_0) be any r.c.f. of P_0 and let ϵ be any positive number less than $\mu(N_0, D_0)$, then

$$N(N_0, D_0, \epsilon) := \{P = ND^{-1} : \|(N, D) - (N_0, D_0)\| < \epsilon\} \quad (3.1)$$

is a basic neighbourhood of P_0 .

Now by varying ϵ over all $(0, \mu(N_0, D_0))$, by varying (N_0, D_0) over all r.c.f.'s of P_0 and by varying P_0 over $R(F)^{n \times m}$, we obtain a collection of the basic neighbourhoods.

Lemma 3.2.

The collection of the basic neighbourhoods forms a topology on $R(F)^{n \times m}$.

And we call it the graph topology. In it two plants P_1 and P_2 are "close", if they have r.c.f.'s $(N_1, D_1), (N_2, D_2)$ such that $\|(N_1, D_1) - (N_2, D_2)\|$ is small.

For the family $\{P_\lambda\}$ of plants, we say P_λ converges to P_{λ_0} as $\lambda \rightarrow \lambda_0$ in the graph topology, if for every r.c.f. $(N_{\lambda_0}, D_{\lambda_0})$ of P_{λ_0} there exist an r.c.f. (N_λ, D_λ) of P_λ such that

$$\| (N_\lambda, D_\lambda) - (N_{\lambda_0}, D_{\lambda_0}) \| \rightarrow 0 \quad (\lambda \rightarrow \lambda_0) .$$

Theorem 3.1.

On $M(H)^{n \times m}$, viewed as a subset of $R(F)^{n \times m}$, the norm topology and the graph topology are the same. Moreover, $M(H)^{n \times m}$ is an open subset of $R(F)^{n \times m}$ in the graph topology, i.e. the graph topology possesses Property 1 over $R(F)^{n \times m}$.

By varying n and m over the positive integer set $\{1,2,\dots\}$, we obtain the graph topology for $R(F)$, and $M(H)$ is an open set in $R(F)$.

One of the important features of the graph topology is that it possesses the "diagonal product property" in the meaning of the following theorem. It is this feature that guarantees that the graph topology has Property 2.

Theorem 3.2.

Assume $P_\lambda \in R(F)^{n \times m}$ is block diagonal of the form

$$P_\lambda := \begin{bmatrix} P_\lambda^1 & 0 \\ 0 & P_\lambda^2 \end{bmatrix}$$

where $P_\lambda^i \in R(F)^{n_i \times m_i}$ ($i = 1,2$) and

$$n_1 + n_2 = n \quad , \quad m_1 + m_2 = m .$$

Then P_λ is continuous at $\lambda_0 \in \Lambda$ in the graph topology iff P_λ^i is continuous at $\lambda_0 \in \Lambda$ in the graph topology ($i = 1,2$).

Proof.

" \Leftarrow :" Assume $(N_{\lambda_0}^i, D_{\lambda_0}^i)$ is the r.c.f. of $P_{\lambda_0}^i$ ($i = 1,2$). Since $\{P_\lambda^i\}$ converges to $P_{\lambda_0}^i$, there are r.c.f.'s $(N_\lambda^i, D_\lambda^i)$ of P_λ^i such that

$$(N_\lambda^i, D_\lambda^i) \rightarrow (N_{\lambda_0}^i, D_{\lambda_0}^i) \quad (\lambda \rightarrow \lambda_0), \quad i = 1,2 .$$

Let

$$N_\lambda = \begin{bmatrix} N_\lambda^1 & 0 \\ 0 & N_\lambda^2 \end{bmatrix}, \quad D_\lambda = \begin{bmatrix} D_\lambda^1 & 0 \\ 0 & D_\lambda^2 \end{bmatrix}; \quad (*)$$

$$N_{\lambda_0} = \begin{bmatrix} N_{\lambda_0}^1 & 0 \\ 0 & N_{\lambda_0}^2 \end{bmatrix}, \quad D_{\lambda_0} = \begin{bmatrix} D_{\lambda_0}^1 & 0 \\ 0 & D_{\lambda_0}^2 \end{bmatrix} .$$

Then clearly (N_λ, D_λ) is an r.c.f. of P_λ , $(N_{\lambda_0}, D_{\lambda_0})$ is an r.c.f. of P_{λ_0} and

$$(N_\lambda, D_\lambda) \rightarrow (N_{\lambda_0}, D_{\lambda_0}) \quad (\lambda \rightarrow \lambda_0)$$

because the norm topology is a product topology. Therefore, $\{P_\lambda\}$ converges to P_{λ_0} in the

graph topology.

" \Rightarrow :" Suppose $(N_{\lambda_0}^i, D_{\lambda_0}^i)$ is an r.c.f. of $P_{\lambda_0}^i$ ($i = 1, 2$), then N_{λ_0} and D_{λ_0} defined by (*) are an r.c.f. of P_{λ_0} . In the same way, taking $(N_{\lambda}^i, D_{\lambda}^i)$ as an r.c.f. of P_{λ}^i ($i = 1, 2$), we can obtain an r.c.f. $(N_{\lambda}, D_{\lambda})$ of P_{λ} by (*). Since $\{P_{\lambda}\}$ converges to P_{λ_0} , there exists a family $\{U_{\lambda}\}$ of unimodular matrices such that

$$\begin{bmatrix} D_{\lambda}^1 & 0 \\ 0 & D_{\lambda}^2 \\ N_{\lambda}^1 & 0 \\ 0 & N_{\lambda}^2 \end{bmatrix} \begin{bmatrix} U_{1\lambda} & U_{3\lambda} \\ U_{4\lambda} & U_{2\lambda} \end{bmatrix} \rightarrow \begin{bmatrix} D_{\lambda_0}^1 & 0 \\ 0 & D_{\lambda_0}^2 \\ N_{\lambda_0}^1 & 0 \\ 0 & N_{\lambda_0}^2 \end{bmatrix}$$

where U_{λ} is partitioned in the obvious way. Hence

$$\begin{bmatrix} D_{\lambda}^i \\ N_{\lambda}^i \end{bmatrix} U_{i\lambda} \rightarrow \begin{bmatrix} D_{\lambda_0}^i \\ N_{\lambda_0}^i \end{bmatrix} \quad (i = 1, 2).$$

Since $(N_{\lambda_0}^i, D_{\lambda_0}^i)$ is an r.c.f. of $P_{\lambda_0}^i$, there exist $A^i \in M(H)$ and $B^i \in M(H)$ such that

$$[A^i, B^i] \begin{bmatrix} D_{\lambda_0}^i \\ N_{\lambda_0}^i \end{bmatrix} = I \quad i = 1, 2.$$

Thus,

$$[A^i, B^i] \begin{bmatrix} D_{\lambda}^i \\ N_{\lambda}^i \end{bmatrix} U_{i\lambda} \rightarrow I \quad (i = 1, 2).$$

Hence when λ is sufficiently close to λ_0 , $\Delta_{\lambda}^i U_{i\lambda}$ is a unimodular matrix, where

$$\Delta_{\lambda}^i := [A^i, B^i] \begin{bmatrix} D_{\lambda}^i \\ N_{\lambda}^i \end{bmatrix} \quad i = 1, 2.$$

Consequently, $U_{i\lambda}$ has to be a unimodular matrix. As a result, P_{λ}^i converges to $P_{\lambda_0}^i$ in the graph topology ($i = 1, 2$). \square

Theorem 3.3.

Consider the feedback system in Figure 2.1 with P and C replaced by P_{λ} and C_{λ} respectively. Then the following statements are equivalent:

- i) $P_{\lambda} \xrightarrow{\text{Gr}} P_{\lambda_0}$ ($\lambda \rightarrow \lambda_0$) and
- $C_{\lambda} \xrightarrow{\text{Gr}} C_{\lambda_0}$ ($\lambda \rightarrow \lambda_0$);

$$\text{ii) } H(\mathbf{P}_\lambda, \mathbf{C}_\lambda) \xrightarrow{\text{Gr}} H(\mathbf{P}_{\lambda_0}, \mathbf{C}_{\lambda_0}) \quad (\lambda \rightarrow \lambda_0).$$

Let us conclude this section by illustrating that the graph topology of [7] can be put into the framework we discussed above.

Here H is the ring S of the proper stable (no poles on closed RHP) rational matrices, and we take F to be the set of all rational matrices. In this case $R(F) = M(F)$. Let H_2 denote Hardy spaces in RHP . If we take $X = H_2$, then the framework built in this section will produce the same theory as [6,7].

4. The gap between closed subspaces

Let X be any Banach space and let ϕ and ψ be any closed subspaces in X . The directed gap from Φ to ψ is defined as

$$\vec{\delta}(\Phi, \psi) = \sup_{x \in S_\Phi} \inf_{y \in \psi} \|x - y\| \quad (4.1)$$

and the gap between ϕ and ψ as

$$\delta(\Phi, \psi) = \max(\vec{\delta}(\Phi, \psi), \vec{\delta}(\psi, \Phi))$$

where

$$S_\Phi := \{x \in \Phi : \|x\| = 1\}.$$

Moreover, define

$$\vec{\delta}(0, \psi) = 0$$

$$\vec{\delta}(\Phi, 0) = 1 \quad (\text{if } \Phi \neq 0).$$

The following relations follow directly from the definition

$$\vec{\delta}(\Phi, \psi) = 0 \iff \Phi \subset \psi$$

$$\delta(\Phi, \psi) = 0 \iff \Phi = \psi$$

$$\delta(\Phi, \psi) = \delta(\psi, \Phi) \quad 1 \leq \delta(\Phi, \psi) \leq 1.$$

Usually, $\delta(\cdot, \cdot)$ is not a metric because $\delta(\cdot, \cdot)$ does not satisfy in general the triangle inequality required of a metric function. But the function $d(\cdot, \cdot)$ defined by

$$\vec{d}(\Phi, \psi) = \sup_{x \in S_\Phi} \inf_{y \in S_\psi} \|x - y\|$$

$$d(\Phi, \psi) = \max[\vec{d}(\Phi, \psi), \vec{d}(\psi, \Phi)]$$

$$d(0, \psi) = 0 ; \quad d(\Phi, 0) = 2 \quad (\text{if } \Phi \neq 0)$$

is a metric function on the set of all closed linear subspaces of X and it generates the same topology as the gap function.

Although the gap $\delta(\cdot, \cdot)$ is not a metric function, it is more convenient than the proper metric function $d(\cdot, \cdot)$ for applications, because its definition is slightly simpler.

If X is a Hilbert space, the gap function is a proper metric and has a simple representation

$$\delta(\Phi, \Psi) = \|P_{\Phi} - P_{\Psi}\|$$

where P_{Φ} is defined by

$$P_{\Phi}x = \text{solution of } \min_{y \in \Phi} \|x - y\|$$

i.e. P_{Φ} is the projection on Φ .

Finally, we point out that more information about the gap function can be found in [4].

5. Gap topology

In this section we will develop the gap topology for the plant set $C(F)$. Recall that $C(F)^{n \times m}$ consists of all plants in $M(F)^{n \times m}$ which can be interpreted as closed operators mapping X^m to X^n . Since $P_0 \in C(F)^{n \times m}$ is a closed operator, it has a closed graph $G(P_0)$, i.e.

$$G(P_0) := \{(x, P_0x) : x \in \text{Dom}(P_0) \subset X^m, P_0x \in X^n\} \quad (5.1)$$

is a closed subspace in $X^m \times X^n$.

The gap between two plants in $C(F)^{n \times m}$ is defined as the gap between their graphs, i.e.

$$\vec{\delta}(P_1, P_2) = \vec{\delta}(G(P_1), G(P_2)) \quad (5.2.1)$$

$$\delta(P_1, P_2) = \delta(G(P_1), G(P_2)). \quad (5.2.2)$$

It is easy to see

$$\delta(P_1, P_2) = 0 \iff P_1 = P_2.$$

The basic neighbourhood of $P_0 \in C(F)^{n \times m}$ is defined as

$$N(P_0, \varepsilon) = \{P \in C(F)^{n \times m} : \delta(P, P_0) < \varepsilon\}. \quad (5.3)$$

Now by varying ε over $(0,1]$ and by varying P_0 over $C(F)^{n \times m}$, we obtain a collection of basic neighbourhoods and this collection forms a base for a topology on $C(F)^{n \times m}$ which is called the gap topology.

The following properties, Theorem 5.1-5.3, are quoted from Kato [4].

Theorem 5.1.

Let $P_0 \in M(H)^{n \times m}$ and let $P_1 \in C(F)^{n \times m}$ satisfy

$$\delta(P_0, P_1) < (1 + \|P_0\|^2)^{\frac{1}{2}} \quad (5.4)$$

then $P_1 \in M(H)^{n \times m}$. Moreover, the gap topology is identical with the operator norm topology on $M(H)^{n \times m}$.

The first part of this theorem says that in the gap topology the stable plant set $M(H)^{n \times m}$ viewed as subset of $C(F)^{n \times m}$ is open, i.e. the gap topology possesses Property 1.

By varying n and m , we obtain a topology on $C(F)$. And $M(H)$ is an open subset of $C(F)$ under the gap topology.

Theorem 5.2.

Let $P_i \in C(F)^{n \times m}$ ($i = 1, 2$) and $P_0 \in M(H)^{n \times m}$. Then

$$\delta(P_0 + P_1, P_0 + P_2) \leq 2(1 + \|P_0\|^2) \delta(P_1, P_2) \quad (5.5)$$

From (5.5), one can easily prove:

$$\delta(P_1, P_2) \leq 2(1 + \|P_0\|^2) \delta(P_0 + P_1, P_0 + P_2). \quad (5.6)$$

Theorem 5.3.

If $P_i \in C(F)^{n \times m}$ ($i = 1, 2$) considered as operators mapping X^m to X^n are invertible, then

$$\delta(P_1^{-1}, P_2^{-1}) = \delta(P_1, P_2). \quad (5.7)$$

As in the case of the graph topology the "diagonal product property" plays an important role in the process of proving that the gap topology has Property 2.

Let $P_\lambda \in C(F)^{n \times m}$ have following form

$$P_\lambda = \begin{bmatrix} P_\lambda^1 & 0 \\ 0 & P_\lambda^2 \end{bmatrix} \quad (5.8.1)$$

where $P_\lambda^i \in C(F)^{n_i \times m_i} \subset C(X^{m_i}, X^{n_i})$ ($i = 1, 2$). It is obvious that $P_\lambda \in C(F)^{n \times m} \subset C(X^m, X^n)$

$$n_1 + n_2 = n, \quad m_1 + m_2 = m.$$

Theorem 5.4.

Assume P_λ is defined by (5.8.1). Then

$$\delta(P_\lambda, P_{\lambda_0}) \rightarrow 0 \quad (\lambda \rightarrow \lambda_0)$$

if and only if

$$\delta(P_\lambda^i, P_{\lambda_0}^i) \rightarrow 0 \quad (\lambda \rightarrow \lambda_0) \quad (i = 1, 2).$$

This result follows from following two lemmas and in these two lemmas we only consider two arbitrary plants P_{λ_1} and P_{λ_2} in $\{P_\lambda\}$. For the sake of simplicity, we denote

$$P_i := P_{\lambda_i} = \begin{bmatrix} P_{\lambda_i}^1 & 0 \\ 0 & P_{\lambda_i}^2 \end{bmatrix} =: \begin{bmatrix} P_i^1 & 0 \\ 0 & P_i^2 \end{bmatrix} \quad i = 1, 2. \quad (5.8.2)$$

Lemma 5.1.

$$\delta(P_1, P_2) \geq \max [\delta(P_1^1, P_2^1), \delta(P_1^2, P_2^2)] \quad (5.9)$$

Proof. By definition

$$\vec{\delta}(P_1, P_2) = \sup_{x \in S_G(P_1)} \inf_{\tilde{x} \in G(P_2)} \|x - \tilde{x}\|$$

where x and \tilde{x} in $[X^{m_1} \times X^{m_2}] \times [X^{n_1} \times X^{n_2}]$.

$$x = (x^1, x^2, P_1^1 x^1, P_1^2 x^2)$$

$$\tilde{x} = (\tilde{x}^1, \tilde{x}^2, P_2^1 \tilde{x}^1, P_2^2 \tilde{x}^2).$$

hence

$$\begin{aligned} \vec{\delta}(P_1, P_2) &= \sup_{x \in S_{G(P_1)}} \inf_{\tilde{x} \in G(P_2)} [\|x^1 - \tilde{x}^1\|^2 + \\ &+ \|P_1^1 x^1 - P_2^1 \tilde{x}^1\|^2 + \|x^2 - \tilde{x}^2\|^2 + \|P_1^2 x^2 - P_2^2 \tilde{x}^2\|^2]^{\frac{1}{2}} \end{aligned} \quad (5.10)$$

$$\begin{aligned} &\geq \sup_{\substack{(x^1, P_1^1 x^1) \in S_{G(P_1)} \\ x^2 = P_1^2 x^2 = 0}} \inf_{\tilde{x} \in G(P_2)} [\|x^1 - \tilde{x}^1\|^2 + \\ &+ \|P_1^1 x^1 - P_2^1 \tilde{x}^1\|^2 + \|0 - \tilde{x}^2\|^2 + \|0 - P_2^2 \tilde{x}^2\|^2]^{\frac{1}{2}} \end{aligned} \quad (5.11)$$

$$= \sup_{\substack{y \in S_{G(P_1)} \\ y = (x^1, P_1^1 x^1)}} \inf_{\substack{z \in G(P_2) \\ z = (\tilde{x}^1, P_2^1 \tilde{x}^1) \\ \tilde{x}^2 = P_2^2 \tilde{x}^2 = 0}} \|y - z\|$$

$$= \sup_{y \in S_{G(P_1)}} \inf_{z \in G(P_2)} \|y - z\|$$

$$= \vec{\delta}(P_1^1, P_2^1)$$

i.e.

$$\vec{\delta}(P_1, P_2) \geq \vec{\delta}(P_1^1, P_2^1). \quad (5.12)$$

From (5.10) to (5.11), if we take $x^1 = P_1^1 x^1 = 0$ instead of $x^2 = P_1^2 x^2 = 0$, we can get

$$\vec{\delta}(P_1, P_2) \geq \vec{\delta}(P_1^2, P_2^2). \quad (5.13)$$

Combining (5.12) and (5.13), one knows

$$\vec{\delta}(P_1, P_2) \geq \max \{ \vec{\delta}(P_1^1, P_2^1), \vec{\delta}(P_1^2, P_2^2) \}.$$

By symmetry,

$$\vec{\delta}(P_2, P_1) \geq \max \{ \vec{\delta}(P_2^1, P_1^1), \vec{\delta}(P_2^2, P_1^2) \} .$$

Consequently, (5.9) is proved. □

Lemma 5.2.

$$\delta(P_1, P_2) \leq \delta(P_1^1, P_2^1) + \delta(P_1^2, P_2^2) . \quad (5.14)$$

Proof. By applying

$$(\alpha + \beta)^{\frac{1}{2}} \leq \alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}} \quad \forall \alpha, \beta \geq 0$$

to (5.10), we get

$$\begin{aligned} \vec{\delta}(P_1, P_2) &\leq \sup_{x \in S_G(P_1)} \inf_{\bar{x} \in G(P_2)} [(\|x^1 - \bar{x}^1\|^2 \\ &\quad + \|P_1^1 x^1 - P_2^1 \bar{x}^1\|^2)^{\frac{1}{2}} + (\|x^2 - \bar{x}^2\|^2 + \|P_1^2 x^2 - P_2^2 \bar{x}^2\|^2)^{\frac{1}{2}}] \\ &= \sup_{x \in S_G(P_1)} [\inf_{(\bar{x}^1, P_2^1 \bar{x}^1) \in G(P_2)} (\|x^1 - \bar{x}^1\|^2 + \|P_1^1 x^1 - P_2^1 \bar{x}^1\|^2)^{\frac{1}{2}} \\ &\quad + \inf_{(\bar{x}^2, P_2^2 \bar{x}^2) \in G(P_2)} (\|x^2 - \bar{x}^2\|^2 + \|P_1^2 x^2 - P_2^2 \bar{x}^2\|^2)^{\frac{1}{2}}] \\ &\leq \sup_{x \in S_G(P_1)} \inf_{(\bar{x}^1, P_2^1 \bar{x}^1) \in G(P_2)} (\|x^1 - \bar{x}^1\|^2 + \|P_1^1 x^1 - P_2^1 \bar{x}^1\|^2)^{\frac{1}{2}} \\ &\quad + \sup_{x \in S} \inf_{(\bar{x}^2, P_2^2 \bar{x}^2) \in G(P_2)} (\|x^2 - \bar{x}^2\|^2 + \|P_1^2 x^2 - P_2^2 \bar{x}^2\|^2)^{\frac{1}{2}} \\ &= \sup_{\substack{\|x\| \leq 1 \\ x \in G(P_1^1)}} \inf_{y \in G(P_2^1)} \|x - y\| + \sup_{\substack{\|x\| \leq 1 \\ x \in G(P_1^2)}} \inf_{y \in G(P_2^2)} \|x - y\| . \end{aligned}$$

Now one can easily prove:

$$\vec{\delta}(P_1^1, P_2^1) := \sup_{\substack{\|x\| \leq 1 \\ x \in G(P_1^1)}} \inf_{y \in G(P_2^1)} \|x - y\| = \sup_{\substack{\|x\| \leq 1 \\ x \in G(P_1^1)}} \inf_{y \in G(P_2^1)} \|x - y\| .$$

Thus

$$\vec{\delta}(P_1, P_2) \leq \vec{\delta}(P_1^1, P_2^1) + \vec{\delta}(P_1^2, P_2^2) .$$

By symmetry

$$\vec{\delta}(P_2, P_1) \leq \vec{\delta}(P_2^1, P_1^1) + \vec{\delta}(P_2^2, P_1^2).$$

Consequently, (5.14) is proved. □

Remark 5.1.

In a completely analogous way, one can prove that (5.9) and (5.14) are still true if P_i is defined by

$$P_i := \begin{bmatrix} 0 & P_i^1 \\ P_i^2 & 0 \end{bmatrix} \quad i = 1, 2 \quad (5.15)$$

instead of (5.8.2). Moreover, one can easily prove

$$\delta \left[\begin{bmatrix} 0 & P_1^1 \\ P_1^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & P_2^1 \\ P_2^2 & 0 \end{bmatrix} \right] = \delta \left[\begin{bmatrix} P_1^1 & 0 \\ 0 & P_1^2 \end{bmatrix}, \begin{bmatrix} P_2^1 & 0 \\ 0 & P_2^2 \end{bmatrix} \right]$$

and

$$\delta(-P_1^1, -P_2^1) = \delta(P_1^1, P_2^1).$$

Now, we are in a position to prove that the gap topology has Property 2.

We can rewrite the closed-loop transfer matrix $H(P, C)$ in Figure 2.1 as the following form

$$H(P, C) = [I + FG]^{-1}$$

where

$$G = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix} \quad F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Let $\{P_\lambda\}$ and $\{C_\lambda\}$ in $C(F)$ be families of plants and compensators respectively. The corresponding closed transfer matrix family is

$$H_\lambda := H(P_\lambda, C_\lambda) = (I + FG_\lambda)^{-1}$$

where

$$G_\lambda = \begin{bmatrix} C_\lambda & 0 \\ 0 & P_\lambda \end{bmatrix}.$$

Theorem 5.5.

Assume $\{P_\lambda\}$, $\{C_\lambda\}$ and $\{H_\lambda\}$ are defined as above. Then

$$\delta(H_\lambda, H_{\lambda_0}) \rightarrow 0 \quad (\lambda \rightarrow \lambda_0)$$

if and only if

$$\delta(P_\lambda, P_{\lambda_0}) \rightarrow 0 \quad (\lambda \rightarrow \lambda_0)$$

and

$$\delta(C_\lambda, C_{\lambda_0}) \rightarrow 0 \quad (\lambda \rightarrow \lambda_0).$$

Proof. By (5.7), we know

$$\begin{aligned} \delta(H_\lambda, H_{\lambda_0}) &= \delta((I + FG_\lambda)^{-1}, (I + FG_{\lambda_0})^{-1}) = \\ &= \delta(I + FG_\lambda, I + FG_{\lambda_0}). \end{aligned}$$

By (5.5) and (5.6),

$$\begin{aligned} \delta(H_\lambda, H_{\lambda_0}) &\leq 4\delta(FG_\lambda, FG_{\lambda_0}) \\ \delta(H_\lambda, H_{\lambda_0}) &\geq \frac{1}{4} \delta(FG_\lambda, FG_{\lambda_0}). \end{aligned}$$

By Remark 5.1, one can easily show

$$\delta(FG_\lambda, FG_{\lambda_0}) = \delta(G_\lambda, G_{\lambda_0}).$$

By (5.9) and (5.14), we have

$$\begin{aligned} \delta(H_\lambda, H_{\lambda_0}) &\leq 4(\delta(P_\lambda, P_{\lambda_0}) + \delta(C_\lambda, C_{\lambda_0})) \\ \delta(H_\lambda, H_{\lambda_0}) &\geq \frac{1}{4} \max [\delta(P_\lambda, P_{\lambda_0}), \delta(C_\lambda, C_{\lambda_0})]. \end{aligned}$$

As a result, the conclusion is true. □

6. Conclusion and comparison

We present our conclusion in the following items:

1. According to Theorem 3.1 and Theorem 5.1, the graph topology and the gap topology are identical with the operator norm topology on $M(H)$. Consequently, they are the same topology on $M(H)$.
2. Both the graph topology and the gap topology possess both Property 1 and Property 2 on different subsets of unstable plants, and both of them can be carried out for non-square plants with general feedbacks.
3. The gap topology is suitable for more general plants than the graph topology. We illustrate this point by:

Lemma 6.1.

Assume $R(F)$ and $C(F)$ are defined in Section 2. Then

$$R(F) \subset C(F) .$$

Proof. According to [7], each $P \in R(F)$ has a stabilizing compensator $C \in R(F)$, i.e. $H(P, C)$ is stable. By Lemma 2.2, we know $P, C \in C(F)$. Hence $R(F) \subset C(F)$. \square

4. Here we claim that the graph topology and the gap topology are identical on $R(F)$.

Theorem 6.1.

Let us consider the feedback system in Figure 2.1. Assume we have a family $\{P_\lambda\}$ of plants and a family $\{C_\lambda\}$ of compensators and both of them are in $R(F)$. Further, suppose $H(P_{\lambda_0}, C_{\lambda_0})$ is stable. Then, if $\{P_\lambda\}$ and $\{C_\lambda\}$ converge to P_{λ_0} and C_{λ_0} respectively in the graph topology then $\{P_\lambda\}$ and $\{C_\lambda\}$ converge to P_{λ_0} and C_{λ_0} respectively in the gap topology, and vice versa.

Proof. The proof is based upon the fact that both the graph topology and the gap topology possess both Property 1 and Property 2. Assume

$$P_\lambda \xrightarrow{\text{Gr}} P_{\lambda_0} \quad (\lambda \rightarrow \lambda_0)$$

and

$$C_\lambda \xrightarrow{\text{Gr}} C_{\lambda_0} \quad (\lambda \rightarrow \lambda_0) .$$

By Property 2,

$$H(P_\lambda, C_\lambda) \xrightarrow{\text{Gr}} H(P_{\lambda_0}, C_{\lambda_0}) \quad (\lambda \rightarrow \lambda_0) .$$

By Property 1, if λ is sufficiently close to λ_0 , $H(P_\lambda, C_\lambda)$ is stable, i.e. $H(P_\lambda, C_\lambda) \in M(H)$. But on $M(H)$ the graph topology and the gap topology are identical. Therefore

$$\delta(H(\mathbf{P}_\lambda, \mathbf{C}_\lambda), H(\mathbf{P}_{\lambda_0}, \mathbf{C}_{\lambda_0})) \rightarrow 0 \quad (\lambda \rightarrow \lambda_0) .$$

Again, by Property 2, we have

$$\delta(\mathbf{P}_\lambda, \mathbf{P}_{\lambda_0}) \rightarrow 0 \quad (\lambda \rightarrow \lambda_0)$$

and

$$\delta(\mathbf{C}_\lambda, \mathbf{C}_{\lambda_0}) \rightarrow 0 \quad (\lambda \rightarrow \lambda_0) .$$

The reverse implication follows by reversing the above steps.

□

7. Metric design for unstable plants

In the last several sections we investigated the essential qualitative aspects of the topologies for unstable plants. For much more information on the topologies, see the excellent expositions in [2,3,9] and [6,7].

When one wants to apply this kind of topology to practical problems, a quantitative description is needed. This is the problem of metrizing these topologies.

Let us first recall some backgrounds of designing a metric for unstable plants. In the case of linear lumped system, a graph topology is metrized into a graph metric by [6,7]. For the SISO linear distributed systems, a graph metric is designed in [8], and [10] offers a design procedure of graph metric for a class of MIMO linear distributed systems. In the general setting, the main obstacle to design the graph metric is the spectral factorization problem of the plants under consideration.

Compared with the case of designing the graph metric, the situation of establishing the gap metric is better. The rest of this section is devoted to the design of a gap metric. Once one confined the gap metric to the plant subset $R(F)$, the gap metric is also a graph metric.

In general, the gap function $\delta(\cdot, \cdot)$ is not a metric. Now let us define a metric on $C(F)^{n \times m}$. Assume $P_i \in C(F)^{n \times m}$ ($i = 1, 2$), define

$$\vec{d}(P_1, P_2) = \sup_{x \in S(G(P_1))} \inf_{y \in S(G(P_2))} \|x - y\|$$

and

$$d(P_1, P_2) = \max \{ \vec{d}(P_1, P_2), \vec{d}(P_2, P_1) \} .$$

It is shown by Kato [4] that $d(\cdot, \cdot)$ is a metric function and the topology generated by $d(\cdot, \cdot)$ is the gap topology. In fact, we have

$$\delta(P_1, P_2) \leq d(P_1, P_2) \leq 2\delta(P_1, P_2) \quad \forall P_1, P_2 \in C(F)^{n \times m}$$

when X is a Hilbert space, $\delta(\cdot, \cdot)$ is a metric function and has a simple form

$$\delta(P_1, P_2) = \|\Pi_{P_1} - \Pi_{P_2}\| \quad \forall P_1, P_2 \in C(F)^{n \times m} \quad (7.1)$$

where Π_P is the projection to the graph of P .

Moreover, for every plant P in $C(F)$ we can find a representation of Π_P [1]:

$$\Pi_P = \begin{bmatrix} R_P & (PR_P)^* \\ PR_P & 1 - R_{P^*} \end{bmatrix} \quad (7.2)$$

where $R_P := (I + P^*P)^{-1}$ and P^* is the dual operator.

When restricting the gap metric to $R(F)$ we obtain a graph metric and (7.2) becomes

$$\Pi_{\mathbf{P}} = \begin{bmatrix} D S^{-1} D^* & D S^{-1} N^* \\ N S^{-1} D^* & N S^{-1} N^* \end{bmatrix} \quad (7.3)$$

where (N, D) is an r.c.f. of \mathbf{P} and $S = N^*N + D^*D$. It is easy to show that every block of (7.3) is independent of the special factorization (N, D) of \mathbf{P} .

In practice, instead of using (7.1), one can use

$$m(\mathbf{P}_1, \mathbf{P}_2) = [\|R_{\mathbf{P}_1} - R_{\mathbf{P}_2}\|^2 + \|R_{\mathbf{P}_1^*} - R_{\mathbf{P}_2^*}\|^2 + 2\|\mathbf{P}_1 R_{\mathbf{P}_1} - \mathbf{P}_2 R_{\mathbf{P}_2}\|^2]^{\frac{1}{2}}. \quad (7.4)$$

It is shown in [1] that

$$\delta(\mathbf{P}_1, \mathbf{P}_2) \leq \sqrt{2} m(\mathbf{P}_1, \mathbf{P}_2) \leq 2\delta(\mathbf{P}_1, \mathbf{P}_2). \quad (7.5)$$

Finally, we point out that the metric defined by (7.1) or (7.4) may have computational advantages over the graph metric defined by [5,6,8,10].

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