

Denumerations of rooted trees and multisets

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DENUMERATIONS OF ROOTED TREES AND MULTISSETS

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The paper describes (i) the relation between denumeration of multisets of non-negative integers and denumeration of abstract rooted trees (topological trees), and (ii) a fast algorithm for dealing with the denumeration of multisets.

1. Notation and terminology

\mathbb{N} stands for the set of positive integers, and \mathbb{N}_0 for the set of non-negative integers.

If S is a set then $|S|$ is the number of elements of S .

If S is a set, then a *multiset* of elements of S is a mapping f of S into \mathbb{N}_0 with the property that the set of s with $f(s) > 0$ is finite.

The set of all these multisets is denoted $M(S)$.

Some of our multisets are multisets of natural numbers. Number theorists might prefer the name 'partition' for this kind of multiset. If $f \in M(\mathbb{N})$, and if $n = \sum_{j=1}^{\infty} j f(j)$, then this f provides a way to write n as a sum of positive integers, viz. the sum of $f(1)$ one's, $f(2)$ two's, $f(3)$ three's etc. In number theory this is called a partition of n .

If V is a set, then $P_2(V)$ is the set of all 2-element subsets of V , i.e., the set

$$\{\{a, b\} \mid a \in V, b \in V, a \neq b\}.$$

A rooted tree is a triple (V, E, r) where V ('the set of vertices') is a finite set, E ('the set of edges') a subset of $P_2(V)$, and r ('the root') is an element of V , with the restriction that from each $v \in V$ with $v \neq r$ there is exactly one double-point-free path to r . A double-point-free path from v_1 to v_n is a sequence v_1, v_2, \dots, v_n of elements of V , with $v_i \neq v_j$ whenever $i \neq j$, such that $\{v_i, v_{i+1}\} \in E$ ($i = 1, 2, \dots, n-1$).

Two rooted trees (V, E, r) and (V', E', r') are called isomorphic if there is a bijection ϕ of V onto V' such that $\phi(r) = r'$, and such that $\{a, b\} \in E$ if and only if $\{\phi(a), \phi(b)\} \in E'$.

The set \mathcal{T} is defined as the set of all equivalence classes of rooted trees (the equivalence is isomorphism). The elements of \mathcal{T} are called *abstract rooted trees* (some authors call them 'topological rooted trees').

A special element of \mathcal{T} is the class t_0 consisting of all rooted trees (V, E, r) with

$|V| = 1$. So these trees have the form $(\{r\}, \emptyset, r)$. We call them *singleton trees*, and the equivalence class of all singleton trees is called the *abstract singleton tree*.

Every equivalence class of rooted trees contains some (V, E, r) , where V is a finite subset of \mathbb{N} . There are countably many of such (V, E, r) , and it follows that \mathcal{T} is countable.

A *denumeration* of a countable set C is a one-to-one mapping of \mathbb{N} onto C .

2. Mapping \mathcal{T} onto the set of multisets of \mathcal{T}

Consider any rooted tree (V, E, r) . We define the set W as the subset of all $w \in V$ with the property that $\{w, r\} \in E$. For each $w \in W$ we consider the set V_w of all $v \in V$ with the property that $v \neq r$ and that the double-point-free path from v to r has the form v, \dots, w, r . Now

$$(V_w, P_2(V_w) \cap E, w) \quad (2.1)$$

is a rooted tree. It is called a *principal subtree* of (V, E, r) .

Given the tree (V, E, r) , we define the mapping f of \mathcal{T} into \mathbb{N}_0 as follows. If $t \in \mathcal{T}$, then $f(t)$ is the number of $w \in W$ with the property that the rooted tree (2.1) belongs to the equivalence class t . (In more popular terms $f(t)$ is the number of copies of t among the principal subtrees.) Since V is finite, this f is a multiset: $f \in M(\mathcal{T})$.

Without formal proof we mention

(i) If (V, E, r) and (V', E', r') are isomorphic, they give rise to exactly the same f . If t is the equivalence class to which (v, E, r) and (V', E', r') belong, this f will be denoted $\Omega(t)$.

(ii) To each multiset $f \in M(\mathcal{T})$ there belongs exactly one $t \in \mathcal{T}$ with the property that $f = \Omega(t)$.

Note that Ω is a bijection of \mathcal{T} onto the set of multisets of \mathcal{T} . And in particular note that Ω maps the abstract singleton tree onto the ‘empty’ multiset (i.e. the f with $f(t) = 0$ for all $t \in \mathcal{T}$).

3. Denumeration of the set of multisets of positive integers derived from a denumeration of the set of abstract rooted trees

Let γ be a denumeration of \mathcal{T} (so γ is a bijection of \mathbb{N} onto \mathcal{T}). We shall associate with this γ a denumeration of $M(\mathbb{N})$ (the set of all multisets of positive integers).

If $n \in \mathbb{N}$, then $\gamma(n) \in \mathcal{T}$, and $\Omega(\gamma(n)) \in M(\mathcal{T})$.

If $f \in M(\mathcal{T})$, then f maps \mathcal{T} into \mathbb{N}_0 , the composite mapping $f\gamma$ maps \mathbb{N} into \mathbb{N}_0 , and we have $f\gamma \in M(\mathbb{N})$. Actually the mapping that sends f to the corresponding $f\gamma$ is a bijection of $M(\mathcal{T})$ onto $M(\mathbb{N})$. This mapping sends $\Omega(\gamma(n))$ to a multiset we shall call $\Gamma(n)$. So for all $n \in \mathbb{N}$ we have $\Gamma(n) \in M(\mathbb{N})$, and

$$\Gamma(n)(k) = (\Omega(\gamma(n)))(\gamma(k)) \quad (3.1)$$

for all $n, k \in \mathbb{N}$. In popular terms, $\Gamma(n)(k)$ is the number of copies of $\gamma(k)$ among

the principal subtrees of $\gamma(n)$.

Since Ω is a bijection of \mathcal{T} onto $M(\mathcal{T})$, we conclude that Γ (i.e. the mapping that sends n to $\Gamma(n)$) is a denumeration of $M(\mathbb{N})$.

If to a second denumeration γ' of \mathcal{T} there corresponds, in the same way as above, the denumeration Γ' of $M(\mathbb{N})$, and if $\Gamma = \Gamma'$ then $\gamma = \gamma'$. In order to see this we assume $\Gamma = \Gamma'$ and that there is a smallest t (smallest in the sense of the number of vertices) for which $n, m \in \mathbb{N}$ exist with $\gamma(n) = \gamma'(m) = t$ and $n \neq m$. Let us abbreviate $\Omega(t)$ to μ . This μ is a multiset (mapping \mathcal{T} into \mathbb{N}_0), and whenever $s \in \mathcal{T}$, $\mu(s) \neq 0$ we infer that s is smaller than t (since principal subtrees of a tree are smaller than the tree itself). So if k satisfies $\gamma(k) = s$ we have $\gamma'(k) = s$ too (by the minimality of t). Interchanging γ and γ' we find $\gamma(k) = s \Leftrightarrow \gamma'(k) = s$. From this we infer that the composite mappings $\mu\gamma$ and $\mu\gamma'$ are equal. By (3.1) $\mu\gamma = \Gamma(n)$, $\mu\gamma' = \Gamma'(m)$. Since $\Gamma = \Gamma'$, and Γ is one-to-one, we get $n = m$.

Let us denote by Θ the mapping that sends γ to Γ as expressed in (3.1):

$$\Theta(\gamma) = \Gamma. \tag{3.2}$$

What we just proved can be expressed as

Theorem 3.1. *Θ is an injection of the set of all denumerations of \mathcal{T} into the set of all denumerations of $M(\mathbb{N})$.*

4. Denumeration of the set of abstract rooted trees derived from a denumeration of the set of multisets of positive integers

We shall see that not every denumeration of $M(\mathbb{N})$ has the form $\Theta(\gamma)$ where γ is a denumeration of \mathcal{T} . And we shall characterize the set of all $\Theta(\gamma)$'s.

If $t \in \mathcal{T}$, then $\Omega(t)$ is a multiset of abstract trees. The trees s which actually occur in this multiset, i.e. the trees s with $\Omega(t)(s) > 0$, are principal subtrees of t and therefore smaller than t . (Again we take 'smaller' in the sense of 'fewer vertices', and we extend notions like 'subtrees', 'number of vertices' from trees to abstract trees in the obvious way).

Now assume that $\Gamma = \Theta(\gamma)$, i.e., assume (3.1). If $\Gamma(n)(k) > 0$ we infer that $\gamma(k)$ is smaller than $\gamma(n)$. So if $\Gamma(n_1)(n_2) > 0$, $\Gamma(n_2)(n_3) > 0$, ... then $\gamma(n_2)$ is smaller than $\gamma(n_1)$, $\gamma(n_3)$ is smaller than $\gamma(n_2)$, etc. It cannot go on forever. This gives a necessary condition for Γ to be of the form $\Theta(\gamma)$, and that condition can also be shown to be sufficient. Both will be expressed in Theorem 4.1.

For an example of a denumeration of $M(\mathbb{N})$ that does not satisfy that condition we refer to the end of Section 5.

Theorem 4.1. *A denumeration Γ of $M(\mathbb{N})$ has the form $\Theta(\gamma)$ (where γ is some denumeration of \mathcal{T}) if and only if for every sequence n_1, n_2, \dots (with $n_i \in \mathbb{N}$ for all*

$i \in \mathbb{N}$) there exists some $j \in \mathbb{N}$ with

$$\Gamma(n_j)(n_{j+1}) = 0.$$

Proof. The ‘only if’ part was shown above. In order to do the ‘if’ part we start from a denumeration Γ of $M(\mathbb{N})$ and we define a special directed graph on \mathbb{N} . The graph is (\mathbb{N}, D) , where D is the set of all ordered pairs (n, m) that satisfy $\Gamma(n)(m) > 0$.

Since Γ is a denumeration of $M(\mathbb{N})$, there is just one n_0 such that $\Gamma(n_0)$ is the zero multiset (the one for which $\Gamma(n_0)(m) = 0$ for all m). This means that (\mathbb{N}, D) has just a single endpoint n_0 .

The condition stated in the theorem says that (\mathbb{N}, D) has no cycles and that there is no point from which an infinite directed path starts. Moreover, every point has finite outdegree (since for the multiset $\Gamma(n)$ there are only finitely many m with $\Gamma(n)(m) > 0$). By Koenig’s lemma there is, for every vertex, an upper bound to the length of the paths starting at that point. Let P_k be the set of vertices for which this upper bound equals k . Hence we can split \mathbb{N} into disjoint paths as

$$\mathbb{N} = P_0 \cup P_1 \cup P_2 \cup \dots$$

such that $P_0 = \{n_0\}$, and such that for each $i > 0$ and for each $n \in P_i$ we have $\Gamma(n)(m) > 0$ for at least one $m \in P_{i-1}$ and for no $m \in P_i \cup P_{i+1} \cup \dots$.

Applying this partition we shall give a construction that uses induction with respect to i . For each $n \in \mathbb{N}$ we shall construct an abstract rooted tree t_n ($t_n \in \mathcal{T}$).

If $n = n_0$ we take for t_n the singleton tree.

If $i > 0$ and $n \in P_i$, we assume that all t_m with $m \in P_0 \cup \dots \cup P_{i-1}$ have been constructed already. Now we take the multiset $\Gamma(n)$ (it is a mapping of \mathbb{N} into \mathbb{N}_0) and use it for the construction of an element ϕ of $M(\mathcal{T})$. If t is some t_m with $m \in P_0 \cup \dots \cup P_{i-1}$, we take $\phi(t) = \Gamma(n)(m)$. If t is not such a t_m we take $\phi(t) = 0$.

We know (Section 3) that Ω bijects \mathcal{T} onto $M(\mathcal{T})$, so there is exactly one element in \mathcal{T} that is mapped onto ϕ . We now define t_n as that special element, i.e., $\Omega(t_n) = \phi$.

Thus we have constructed, for all $n \in \mathbb{N}$, an element t_n of \mathcal{T} .

For all $n, m \in \mathbb{N}$ we have

$$\Omega(t_n)(t_m) = \Gamma(n)(m) \quad (4.1)$$

(note that, with the above notation, if $m \in P_i \cup P_{i+1} \cup \dots$, we have $\Omega(t_n)(t_m) = 0$ by the definition of ϕ , and $\Gamma(n)(m) = 0$ by the construction of the P ’s). And if $s \in \mathcal{T}$ is such that there is no $m \in \mathbb{N}$ with $s = t_m$, then

$$\Omega(t_n)(s) = 0. \quad (4.2)$$

We next show that t_1, t_2, \dots represent a denumeration of \mathcal{T} . Assume that some t is the smallest tree that does not occur exactly once in the sequence. The singleton tree occurs exactly once, so t is not the singleton tree. Now consider $\Omega(t)$. If $s \in \mathcal{T}$ and $\Omega(t)(s) > 0$ then s is smaller than t (principal subtrees of a tree are smaller than the tree itself) whence there is a unique k with $s = t_k$. So with $\Omega(t)$ we can associate

an element δ of $M(\mathbb{N})$ by requiring $\delta(m) = \Omega(t)(t_m)$ for all $m \in \mathbb{N}$.

Since Γ is a denumeration of $M(\mathbb{N})$, there is exactly one $n \in \mathbb{N}$ such that $\Gamma(n) = \delta$. Now $\Gamma(n)(m) = \delta(m) = \Omega(t)(t_m)$ for all m . So by (4.1) we have $\Omega(t_n)(t_m) = \Omega(t)(t_m)$ for all $m \in \mathbb{N}$. As we marked above we infer from $s \in \mathcal{T}$, $\Omega(t)(s) > 0$ that s equals some t_k , so if there is no m with $s = t_m$ we have $\Omega(t)(s) = 0$. By (4.2) we see that $\Omega(t_n)(s) = \Omega(t)(s)$ in these cases. So $\Omega(t_n)(s) = \Omega(t)(s)$ for all $s \in \mathcal{T}$, and we infer that $\Omega(t_n) = \Omega(t)$. As Ω is bijective (Section 2) we have $t_n = t$.

Thus we have shown that t occurs in the sequence t_1, t_2, t_3, \dots . It remains to prove that it does not occur more than once. This is easy. If $t_n = t_k$ we apply (4.1) and get that $\Gamma(n)(m) = \Gamma(k)(m)$ for all m . Hence $\Gamma(n) = \Gamma(k)$. Since Γ is a denumeration of $M(\mathbb{N})$ this implies $n = k$.

This finishes the proof of Theorem 4.1.

5. Göbel's denumeration of the set of abstract rooted trees

In [1] F. Göbel presented a denumeration of \mathcal{T} depending on the use of factorization of natural numbers. In our notation his construction is actually $\Theta(\Gamma)$ where Γ is a special denumeration of $M(\mathbb{N})$.

For Γ we take the following multiset denumeration. Let p_1, p_2, p_3, \dots be the primes in ascending order ($p_1 = 2, p_2 = 3, \dots$). Then to each $n \in \mathbb{N}$ we attach a multiset $\Gamma(n)$ (mapping \mathbb{N} into \mathbb{N}_0). This $\Gamma(n)$ is given by the condition that, for each $m \in \mathbb{N}$, the number of times that p_m occurs in the prime decomposition of m is $\Gamma(n)(m)$:

$$n = \prod_{m=1}^{\infty} (p_m)^{\Gamma(n)(m)}. \tag{5.1}$$

If p_m is a prime factor of m we have $m < n$. Therefore $\Gamma(n)(m) > 0$ implies $n > m$, and it follows that Γ satisfies the condition of Theorem 4.1.

By a small modification of this example we can get a multiset denumeration that does not satisfy that condition of Theorem 4.1. We just interchange p_1 and p_2 : $p_1 = 3, p_2 = 2, p_3 = 5, p_4 = 7, \dots$. If we again define Γ according to (5.1), we get $\Gamma(2)(2) = 1$, and the condition fails.

6. An alternative denumeration with fast algorithms for ranking and unranking

We first define $w(f)$ (the weight of f) for an arbitrary element of $M(\mathbb{N})$, by

$$w(f) = \sum_{j=1}^{\infty} jf(j). \tag{6.1}$$

As it was remarked in Section 1, the multiset f defines a number-theoretic partition of $w(f)$. As an example we give the multiset $(0, 3, 1, 0, 1)$, with a notation that is short for $f(1) = 0, f(2) = 3, f(3) = 1, f(4) = 0, f(5) = 1, f(j) = 0$ for all $j > 5$. This has weight 14, and corresponds to the partition $14 = 2 + 2 + 2 + 3 + 5$.

There is an obvious lexicographic ordering of the set of number-theoretical partitions of a fixed number. As an example we list the partitions of 5, along with the corresponding multisets.

$1 + 1 + 1 + 1 + 1$	(5)
$1 + 1 + 1 + 2$	(3, 1)
$1 + 1 + 3$	(2, 0, 1)
$1 + 2 + 2$	(1, 2)
$1 + 4$	(1, 0, 0, 1)
$2 + 3$	(0, 1, 1)
5	(0, 0, 0, 0, 1)

The partitions are in what we might call increasing order, but we prefer to look at the multisets. The multisets are in decreasing lexicographic order. If f and g are in $M(\mathbb{N})$, we say that $f < g$ is in the lexicographic order if there is some $j \in \mathbb{N}$ such that $f(j) < g(j)$ and $f(i) = g(i)$ for all i with $1 \leq i < j$.

The denumeration of $M(\mathbb{N})$ to be described in this section is obtained as follows. First take the zero multiset, i.e. the f with $w(f) = 0$, then the one with $w(f) = 1$, then those with $w(f) = 2$, etc. And for each n , the f 's with $w(f) = n$ will be arranged in decreasing lexicographic order (like in the example above).

We first define, for $f \in M(\mathbb{N})$, the number $p(f)$ by

$$p(f) = \min\{i \in \mathbb{N} \mid f(i) > 0\}. \quad (6.2)$$

If f is the zero multiset, $p(f)$ is not defined by this. We then just put $p(f) = 0$. In the language of number-theoretical partitions, $p(f)$ is the smallest term in the partition.

If $f \in M(\mathbb{N})$ and $f \neq 0$, we define its *reduction* f^* by

$$f^*(j) = f(j) - 1 \quad \text{if } j = m(f), \quad f^*(j) = f(j) \quad \text{otherwise.}$$

Note that $w(f^*) = w(f) - m(f)$, $p(f^*) \geq p(f)$, unless $f^* = 0$.

We can now describe the organization of the list. Put the zero multiset as the first item on the list. Next assume for some pair (n_0, k_0) (with $1 \leq k_0 \leq n_0$) that we have listed all f with $(w(f), p(f)) < (n_0, k_0)$. (Note that $(n, k) < (n', k')$ means that either $n < n'$ or both $n = n'$ and $k < k'$). Then we put all f with $(w(f), p(f)) = (n_0, k_0)$ on the list, in the order in which their reductions are on the list already.

If a multiset is the m -th entry of the list, this m is called the *rank* of the multiset. Finding m when the multiset is given, is called *ranking*, finding the multiset when m is given is called *unranking*.

We shall describe fast algorithms for ranking and unranking. These algorithms require numbers $r(n, k)$, defined as follows. If $1 \leq k \leq n$, and also if $n = k = 0$, this $r(n, k)$ is the number of multisets with

$$(w(f), p(f)) \leq (n, k).$$

The first item is $r(0, 0) = 1$, and we refer to Table 1 for values with $1 \leq k \leq n \leq 9$. Note that $r(n, k)$ is constant in the range $\frac{1}{2}(n-2) < k < n$ (if n is fixed, $n > 2$). This is a consequence of the fact that for $\frac{1}{2}n < k < n$ there are no partitions of n with smallest term k .

If $1 \leq k \leq n$ we denote by $r'(n, k)$ the value of r in the last pair preceding (n, k) . So $r'(n, k) = r(n-1, n-1)$ if $k=1$ and $=r(n, k-1)$ if $1 < k \leq n$. We can now express how the $r(n, k)$ can be obtained by recursion.

$$r(n, k) = r'(n, k) + \begin{cases} 1 & \text{if } 0 < k = n, \\ 0 & \text{if } 0 < n - k < k, \\ r(n - k, n - k) - r'(n - k, k) & \text{if } n - k \geq k \geq 0. \end{cases}$$

The idea behind this is that, as long as $0 < k < n$, the f^* with $(w(f), p(f)) = (n, k)$ have their reduction f^* in the range

$$(n - k, k) \leq (w(f^*), p(f^*)) \leq (n - k, n - k), \tag{6.3}$$

and that every multiset in that range is obtained exactly once as an f^* .

As a preparation to both the ranking and the unranking algorithm, we take any integer N and we prepare a list of the $r(n, k)$ up to $n = k = N$. This list will enable us to rank all multisets f with $w(f) \leq N$. Conversely, the unranking algorithm will enable us to find the m th multiset for any given m with $1 \leq m \leq r(N, N)$.

The amount of work and storage space required for this preparation is of the order of N^2 , but it can be used for quite a large range of m . Note that $r(N, N) = 1 + p(1) + \dots + p(N)$ (where $p(k)$ stands for the number of partitions of k) and that $p(k) \sim (4k)^{-1} 3^{-1/2} \exp(\pi(2k/3)^{1/2})$ (see [3]).

We first describe the ranking algorithm. Let f_0 be a given multiset with $w(f_0) \leq N$. We want to find the number m such that f_0 is the m th multiset of the list. We put $w(f_0) = n$, $p(f_0) = k$ and denote by f_0^* the reduction of f_0 . Assuming that f_0^* has rank m^* , we have

$$m = r(n, k) - r(n - k, n - k) + m^*. \tag{6.4}$$

Table 1
Values of $r(n, k)$

$n \backslash k$	1	2	3	4	5	6	7	8	9
1	2								
2	3	4							
3	6	6	7						
4	10	11	11	12					
5	17	18	18	18	19				
6	26	28	29	29	29	30			
7	41	43	44	44	44	44	45		
8	60	64	65	66	66	66	66	67	
9	89	93	95	96	96	96	96	96	97

This can be applied recursively, using $f_0(1)+f_0(2)+f_0(3)+\dots$ steps in total.

The motivation for (6.4) is that the sublist of f 's with $(w(f), p(f))=(n, k)$ appears in the same order as the list of their f^* 's in the range (6.3). The distance from f_0 to the last entry of the sublist is $r(n, k)-m-1$, and the distance of f_0^* to the last entry of the list of f^* 's is $r(n-k, n-k)-m^*-1$.

As an example we take the partition $1+2+2+3$ (with $w(f)=8$, $p(f)=1$). Its position on the list is $(r(0, 1)-r(7, 7))+(r(7, 2)-r(5, 5))+(r(5, 2)-r(3, 3))+r(3, 3)-r(0, 0)+1=(60-45)+(43-19)+(18-7)+(7-1)+1=57$.

The unranking algorithm works on the same principle. Let m be given, $1 \leq m \leq r(N, N)$. Let (n, k) be the lowest pair such that $r(n, k) \geq m$. This gives us the value n for $w(f)$ and k for $p(f)$. Next take $m^*=m-r(n, k)+r(n-k, n-k)$ and find the m^* th multiset of the list. This is the reduction of the f we are looking for. Since we know $p(f)$, this determines f uniquely.

As an example we determine the 86th multiset. We have $r(8, 8)=67$, $r(9, 1)=89$, so $n=9$, $k=1$. We next look for the 64th multiset, because $86-89+r(9-1, 9-1)=64$. Since $r(8, 1)=60$, $r(8, 2)=64$ we get the value 2 for the next k , and we look for the 30th multiset, because $64-64+r(8-2, 8-2)=30$. Since $r(6, 5)=29$, $r(6, 6)=30$ our next k is 6, and we look for multiset number 1, because $30-30+r(6-6, 6-6)=1$. We are left with the zero multiset, and the algorithm stops. Therefore the 86th multiset is the one corresponding to the partition $9=1+2+6$.

We quote a few computer-produced cases with larger numbers:

$$374225: 3+3+4+4+5+11+13,$$

$$10^7: 1+1+1+3+3+3+4+4+4+4+4+4+4+4+5+5+9,$$

$$2 \cdot 10^7: 1+1+1+1+1+1+1+1+1+1+1+2+3+3+4+4+4+9+10+19,$$

$$3 \cdot 10^7: 3+4+6+6+8+19+24.$$

For these cases the value $N=70$ suffices. The preparation requires the evaluation of about $\frac{1}{4}N^2$ of the $r(n, k)$. The number of steps required for finding the m th partition is the number of terms in the partition, and that can never be more than N .

If we apply our multiset denumeration to tree denumeration we get principal subtrees with relatively low rank, since these numbers correspond to the terms in the partition. In the example of the 374225th tree (see Fig. 1) we get as principal subtrees

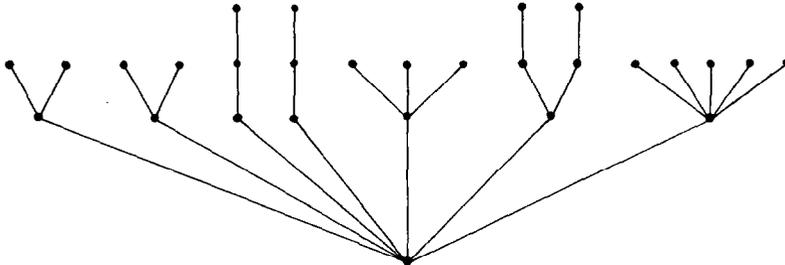


Fig. 1. Tree nr. 374225.

twice tree nr. 3, twice tree nr. 4, tree nr. 5, tree nr. 11 and tree nr. 13. Partition nr. 3 is $1+1$, nr. 4 is 2, nr. 5 is $1+1+1$, nr. 11 is $2+2$ and nr. 13 is $1+1+1+1+1$.

Tree nr. 1 is the singleton tree, and tree nr. 2 the tree with two vertices. So we can draw the picture as in Fig. 1. An easy code for this tree, establishing the structure as well as the list-positions of the subtrees is

$$374225 (3(1, 1), 3(1, 1), 4(2(1)), 4(2(1)), 4(2(1)), \\ 5(1, 1, 1), 11(2(1), 2(1)), 13(1, 1, 1, 1, 1)).$$

Similarly, the code for tree nr. 30000000 is

$$30000000 (3(1, 1), 4(2(1)), 6(1, 2(1)), 6(1, 2(1)), 8(1, 1, 1, 1), \\ 19(5(1, 1, 1)), 24(1, 1, 4(2(1))))).$$

In contrast to this method, Göbel's denumeration gives quite some tall trees ranking low on the list. And Göbel's method can be time-consuming. In order to find his m th tree we need the prime decomposition of m . And if m happens to be prime, we have to find t such that m is the t th prime.

References

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