

On the differential equation $yy''+2xy'=0$

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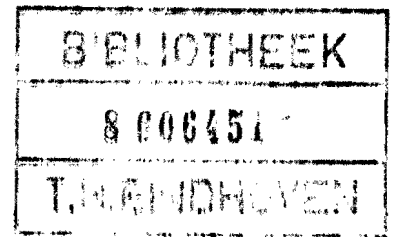
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ON THE DIFFERENTIAL EQUATION $yy'' + 2xy' = 0$

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Abstract

The asymptotic behaviour for $x \rightarrow \infty$ of solutions $y(x)$ of the initial value problem $yy'' + 2xy' = 0$, $y(0) = \alpha > 0$, $y'(0) = \beta$, and also the asymptotic behaviour for $\beta \rightarrow +\infty$ and for $\beta \rightarrow -\infty$ of $L := \lim_{x \rightarrow \infty} y(x)$ is investigated.

1. Introduction

In this paper we consider the following initial value problem

$$(1) \quad yy'' + 2xy' = 0 \quad (x > 0),$$

$$(2) \quad y(0) = \alpha > 0, \quad y'(0) = \beta.$$

Equation (1) occurs in boundary layer theory (see [1], p. 22-23) and arises from the diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z} \left(D \frac{\partial C}{\partial z} \right).$$

If $D = e^{-C}$, and if C only depends on $x := \frac{1}{2}z t^{-\frac{1}{2}}$, then $y := e^{-C}$ satisfies (1).

In boundary layer theory one is also interested in the boundary value problem (1), (3) with

$$(3) \quad y(0) = \alpha > 0 \quad , \quad y(\infty) = L > 0 .$$

A case of special interest is

$$(2') \quad y(0) = 1 \quad , \quad y'(0) = \gamma ,$$

since all solutions of (1), (2) are expressible in solutions of (1), (2') by means of the transformation (11) (see Section 2). We shall study the asymptotic behaviour of solutions $y(x, \gamma)$ of (1), (2') for $x \rightarrow \infty$. Moreover, we shall pay attention to the asymptotic behaviour of the limit

$L(\gamma) := \lim_{x \rightarrow \infty} y(x, \gamma)$ for $\gamma \rightarrow \infty$ and $\gamma \rightarrow -\infty$, since physicists seem interested in large values of $|\gamma|$.

2. Results

In Section 3 the following fundamental results are proved. There exists a unique solution $y(\cdot, \alpha, \beta)$ of (1), (2), defined on $[0, \infty)$. This solution $y(x, \alpha, \beta)$ tends monotonically to a positive limit $L(\alpha, \beta)$ if $x \rightarrow \infty$. The transformation rule

$$(11) \quad y(x, \alpha, \beta) = \alpha y(\alpha^{-\frac{1}{2}} x, \gamma) \quad , \quad L(\alpha, \beta) = \alpha L(\gamma) \quad , \quad \gamma = \alpha^{-\frac{1}{2}} \beta \quad ,$$

enables us to consider only the solution $y(\cdot, \gamma)$ of (1), (2') and its limit $L(\gamma)$. This limit is a continuous increasing bijection $L : \mathbb{R} \rightarrow (0, \infty)$. Hence the boundary value problem (1), (3) has a unique solution.

In Section 4 the asymptotic behaviour of $y(x, \gamma)$ for $x \rightarrow \infty$ is determined.

$$(20) \quad y(x, \gamma) = L - A \operatorname{erfc}(L^{-\frac{1}{2}} x) + \mathcal{O}(x^{-2} \exp[-2 L^{-1} x^2]) \quad (x \rightarrow \infty) \quad ,$$

where

$$A = \frac{1}{2} \pi^{\frac{1}{2}} L^{\frac{1}{2}} \gamma \exp \left[2 \int_0^{\infty} s(L^{-1} - (y(s, \gamma))^{-1}) ds \right] \quad .$$

In Sections 5 and 6 bounds for the limit $L(\gamma)$ are determined.

$$(36) \quad 2 L_0 \log(1 + L_0 \gamma^2) < L(\gamma) - L_0 \gamma^2 < 47 L_0 \log(1 + 3(16)^{-1} \gamma^2) + 47 L_0 \quad (\gamma > 1.8)$$

$$(56) \quad 1 - \gamma^{-2} < L(\gamma) \exp \left[\frac{1}{2} \gamma^2 + \frac{1}{2} \right] < 1 + \gamma^{-2} \quad (\gamma < -7) \quad ,$$

where $L_0 = 0.3574\dots$ denotes the limit of the special solution y of (1) with initial values $y(0) = 0$, $y'(0) = 1$.

3. Preliminaries

We shall investigate existence and uniqueness properties, and prove the continuous dependence of the limit as functions on the initial values.

(4) Theorem. There exists a unique solution y of the initial value problem (1), (2) which is defined on $[0, \infty)$. This solution y is positive and monotone, and $y(x)$ tends to a positive limit L if $x \rightarrow \infty$.

Proof. According to well-known existence and uniqueness theorems (see for example Sections I.1 and I.2 of [2]), there is a positive number, say δ , such that there is a unique solution, say y , of (1), (2), defined on $[0, \delta)$. Let I be the maximal interval of the form $[0, a)$, (possibly a may be ∞) on which y is positive and satisfies (1), (2). Dividing both sides of (1) by yy' , integrating over $[0, x]$ with $x \in I$ we get

$$(5) \quad y'(x) = \beta \exp \left[-2 \int_0^x s(y(s))^{-1} ds \right] \quad (x \in I) ,$$

a result also correct if $y' = 0$. It follows that

$$(6) \quad 0 \leq y'(x) \operatorname{sgn} \beta \leq |\beta| \quad (x \in I) .$$

Let $a > 0$, $a \in I$. Dividing both sides of (1) by xy integrating from a to x , $x \in I$, $x > a$, taking exponentials, we find

$$(7) \quad y(x) = y(a) \exp \left[\frac{1}{2} a^{-1} y'(a) - \frac{1}{2} x^{-1} y'(x) - \frac{1}{2} \int_a^x s^{-2} y'(s) ds \right] .$$

From (6) and (7) we infer that

$$(8) \quad y(x) \begin{matrix} \leq \\ > \end{matrix} y(a) \exp[\frac{1}{2}a^{-1}y'(a)] \quad (a < x, x \in I) ,$$

where \leq holds if $\beta \geq 0$ and $>$ if $\beta < 0$. It follows from (5) and (8) that y is bounded, and increasing if $\beta > 0$, and that y is bounded away from zero and decreasing if $\beta < 0$, and that $I = [0, \infty)$. Clearly $\lim_{x \rightarrow \infty} y(x)$ exists and is positive.

We shall denote the solution of (1), (2) by $y(x, \alpha, \beta)$ and its limit by $L(\alpha, \beta)$. In the case of (1), (2') we write $y(x, \gamma)$ and $L(\gamma)$. Simply by inspection we can prove

(9) Theorem. For every $\lambda > 0$

$$(10) \quad y(x, \lambda^2 \alpha, \lambda \beta) = \lambda^2 y(\lambda^{-1} x, \alpha, \beta) \quad , \quad L(\lambda^2 \alpha, \lambda \beta) = \lambda^2 L(\alpha, \beta) .$$

Theorem (9) enables us to consider, without loss of generality, only the initial value problem (1), (2'). We have

$$(11) \quad y(x, \alpha, \beta) = \alpha y(\alpha^{-\frac{1}{2}} x, \gamma) \quad , \quad L(\alpha, \beta) = \alpha L(\gamma) \quad , \quad \gamma = \alpha^{-\frac{1}{2}} \beta .$$

(12) Theorem. $L(\cdot) : \mathbb{R} \rightarrow (0, \infty)$ is an increasing continuous bijection.

Proof. Let $\gamma_2 > \gamma_1$. Then there is a $\delta > 0$ such that $y(x, \gamma_2) > y(x, \gamma_1)$ on $[0, \delta)$. Suppose that there exists a positive x_0 such that $y(x_0, \gamma_2) = y(x_0, \gamma_1)$ and $y(x, \gamma_2) > y(x, \gamma_1)$ for $0 < x < x_0$. Clearly $y'(x_0, \gamma_2) < y'(x_0, \gamma_1)$.

Dividing (1) by y and integrating from 0 to x we find

$$(13) \quad y'(x, \gamma) - \gamma + 2x \log y(x, \gamma) = 2 \int_0^x s \log y(s, \gamma) ds .$$

Applying (13) twice with $\gamma = \gamma_2$ and $\gamma = \gamma_1$ and subtracting both results we get for $x = x_0$

$$y'(x_0, \gamma_2) - y'(x_0, \gamma_1) - \gamma_2 + \gamma_1 = 2 \int_0^{x_0} s \log(y(s, \gamma_2)/y(s, \gamma_1)) ds ,$$

a contradiction since both sides have different signs. It follows that $y(x, \gamma_2) > y(x, \gamma_1)$ for all $x > 0$.

In the sequel of the proof we need the formula

$$(14) \quad \log L = \frac{1}{2} \int_0^{\infty} (\gamma - y'(s)) s^{-2} ds ,$$

where $y = y(\cdot, \gamma)$, $L = L(\gamma)$, and which is obtained by integration of $y'y^{-1} = -\frac{1}{2} x^{-1} y''$. The integral in (14) exists since $y''(s) = -2\gamma s + \sigma(s)$ ($s > 0$) and $\gamma - y'(s) = \gamma s^2 + \sigma(s^2)$ ($s > 0$). Applying (14) to $y_2 := y(\cdot, \gamma_2)$ and $y_1 := y(\cdot, \gamma_1)$ respectively, and subtracting the results we get

$$(15) \quad \log L_2 L_1^{-1} = \frac{1}{2} \int_0^{\infty} [\gamma_2 - \gamma_1 + y_1'(s) - y_2'(s)] s^{-2} ds ,$$

where $L_2 := L(\gamma_2)$ and $L_1 := L(\gamma_1)$.

If $\gamma_1 > 0$ then by (5), $y_2' > \gamma_2 \gamma_1^{-1} y_1'$ which implies directly $L_2 > 1 + \gamma_2 \gamma_1^{-1} (L_1 - 1)$, and, by (15) and (14)

$$\log L_2 L_1^{-1} < (\gamma_2 - \gamma_1) \gamma_1^{-1} \log L_1 < (\gamma_2 - \gamma_1) \gamma_1^{-1} \log L_2 .$$

If $\gamma_1 = 0$ then by (5), $y_2' < \gamma_2 \exp[-x^2 L_2]$ which upon integration gives $L_2 < (1 + \frac{1}{2} \pi^{\frac{1}{2}} \gamma_2)^2$.

If $\gamma_2 = 0$ then by (5) $y_1' > \gamma_1 \exp[-x^2]$ which leads to $L_1 > 1 + \frac{1}{2} \pi^{\frac{1}{2}} \gamma_1$.

If $\gamma_2 < 0$ then by (5) $y_2' < \gamma_2 \gamma_1^{-1} y_1'$ which implies

$$L_2 - L_1 < (1 - \gamma_2 \gamma_1^{-1})(1 - L_1) < (1 - \gamma_2 \gamma_1^{-1}),$$

and by (15) and (14) $\log L_2 L_1^{-1} > (1 - \gamma_1 \gamma_2^{-1}) \log L_2$.

It follows that $L(\cdot)$ is continuous and strictly increasing on \mathbb{R} , and, moreover $L(\gamma) \rightarrow \infty$ if $\gamma \rightarrow \infty$, $L(\gamma) \downarrow 0$ if $\gamma \rightarrow -\infty$.

4. Asymptotic behaviour

First we investigate the asymptotic behaviour of solutions $y := y(\cdot, \gamma)$ with $\gamma \geq 0$. Using the inequality $y(x) \leq L := L(\gamma)$ in (5) we get

$$y'(x) \leq \gamma \exp[-x^2/L] ,$$

from which it follows by integration over $[x, \infty)$ that

$$y(x) \geq L - \frac{1}{2} \pi^{\frac{1}{2}} L^{\frac{1}{2}} \operatorname{erfc}(x L^{-\frac{1}{2}}) .$$

Hence

$$(16) \quad y(x) = L + \mathcal{O}(x^{-1} \exp[-x^2/L]) \quad (x \rightarrow \infty) .$$

It follows that

$$(17) \quad Q := 2 \int_0^{\infty} s((y(s))^{-1} - L^{-1}) ds$$

exists and that

$$(18) \quad 2 \int_x^{\infty} s((y(s))^{-1} - L^{-1}) ds = \mathcal{O}(x^{-1} \exp[-x^2/L]) \quad (x \rightarrow \infty) .$$

Using (17) and (18) in (5) we get

$$(19) \quad y'(x) = \gamma \exp[-Q - x^2/L] (1 + \mathcal{O}(x^{-1} \exp[-x^2/L])) \quad (x \rightarrow \infty) ,$$

which upon integration from x to ∞ gives

$$(20) \quad y(x) = L - \frac{1}{2} \pi^{\frac{1}{2}} L^{\frac{1}{2}} \gamma e^{-Q} \operatorname{erfc}(x L^{-\frac{1}{2}}) + \mathcal{O}(x^{-2} \exp[-2x^2/L]) \quad (x \rightarrow \infty) .$$

Secondly we consider the case $\gamma < 0$. Then using $L < y(x) < 1$ in (5) we get $y'(x) \geq \gamma \exp[-x^2]$. By integration over $[x, \infty)$ it follows that

$$y(x) = L + O(x^{-1} \exp[-x^2]) \quad (x \rightarrow \infty) .$$

Repeating the same kind of arguments (after (16)) as in the case $\gamma \geq 0$ we arrive at the same Formula (20) except for the order term which is now $O(x^{-2} \exp[-x^2(L^{-1} + 1)])$. Hence Formula (16) holds. Again by repeating the arguments after (16) we get (20).

5. Bounds for the limit if $\gamma > 0$

By (10) we have

$$(21) \quad L(1, \gamma) = \gamma^2 L(\gamma^{-2}, 1) .$$

We shall prove that there are positive constants C_1, C_2 such that for $\alpha > 0$ sufficiently small

$$(22) \quad C_1 \alpha |\log \alpha| < L(\alpha, 1) - L(0, 1) < C_2 \alpha |\log \alpha| .$$

For simplicity we write y, L, y_0, L_0 instead of $y(\cdot; \alpha, 1), L(\alpha, 1), y(\cdot; 0, 1), L(0, 1)$ respectively. It is easily seen that

$$(23) \quad y_0(x) = x - x^2 + O(x^3) \quad (x \downarrow 0) .$$

Clearly, Formula (5) with $\beta = 1$ holds for y_0 . It follows by (5) that initially $y'(x) > y_0'(x)$, since initially $y(x) > y_0(x)$. By standard reasoning we can conclude that

$$(24) \quad y'(x) > y_0'(x) \quad , \quad y(x) > y_0(x) + \alpha \quad (x > 0) .$$

For later use we shall prove the following facts:

$$(25) \quad y_0(x) > x - x^2 + \frac{1}{3} x^3 > x - x^2 \quad (0 < x \leq 1)$$

$$(26) \quad y_0(x) < \frac{1}{2}(1 - e^{-2x}) \quad (x > 0)$$

$$(27) \quad L_0 = 0.35742210059\dots$$

Proof of (25), (26) and (27). From $y_0'' > 0$ and $y_0(0) = 0$ it follows that

$y_0(x) > xy_0'(x)$ ($x > 0$). From (1) we deduce $y_0'' > -2$ which leads to

$y_0 > x - x^2$. Using this last result in (5) we find $y_0' > (1 - x)^2$

($0 < x \leq 1$) whence $y_0 > x - x^2 + \frac{1}{3}x^3$ ($0 < x \leq 1$). Clearly $L_0 > y_0(1) > \frac{1}{3}$.

Using $y_0(x) < x$ ($x > 0$) in (5) we get $y_0' < e^{-2x}$ and $y_0 < \frac{1}{2}(1 - e^{-2x})$.

Hence $L_0 < \frac{1}{2}$. A numerical computation gives (27). We also need

$$(28) \quad L < (1 + 2\sqrt{\alpha})^2 L_0 .$$

Let $0 < a \leq \frac{1}{2}$. The line $y = a - a^2 + (1 - 2a)(x - a)$ is tangent to the curve

$y = x - x^2$ in the point $(a, a - a^2)$. It follows by (25) that

$y_2(x) := y(x; a^2, 1 - 2a)$ equals $y_0(x)$ for some $x = b < a$ since $y_2(a) < y_0(a)$.

At $x = b$ we have $y_0(b) = y_2(b)$ and $y_0'(b) > y_2'(b)$. We show that $y_0(x) > y_2(x)$

for all $x > b$. Immediately to the right of $x = b$ we have $y_0(x) > y_2(x)$. Then

in some interval $(b, b + \delta)$

$$\begin{aligned} y_0'(x) &= y_0'(b) \exp \left[-2 \int_b^x s(y_0(s))^{-1} ds \right] > y_0'(b) \exp \left[-2 \int_0^x s(y_2(s))^{-1} ds \right] \\ &= y_0'(b) (y_2'(b))^{-1} y_2'(x) > y_2'(x) . \end{aligned}$$

By standard arguments it follows that $y_0(x) > y_2(x)$ ($x > b$).

Letting $x \rightarrow \infty$ we find $L(a^2, 1 - 2a) < L_0$. By (10) we have $L(a^2, 1 - 2a) = (1 - 2a)^2 L(a^2(1 - 2a)^{-2}, 1)$. Substituting $a = \alpha^{\frac{1}{2}}(1 + 2\alpha^{\frac{1}{2}})^{-1}$ we get (28).

Now we proceed with the proof of (22).

The function K defined by (29) satisfies (30).

$$(29) \quad K(x) := y'(x)/y_0'(x) .$$

$$(30) \quad K'(x)/K(x) = 2x((y_0(x))^{-1} - (y(x))^{-1}) , \quad K(0) = 1 .$$

Using (24) and the inequalities $y_0(x) < x$, $y_0'(x) < 1$ for $x > 0$, we see that

$$K'/K > 2\alpha(y_0 + \alpha)^{-1} y_0' ,$$

from which it follows by integration

$$K > (1 + y_0/\alpha)^{2\alpha} .$$

Using (29) and integrating we get

$$y > \alpha + \alpha(2\alpha + 1)^{-1} [(1 + y_0/\alpha)^{2\alpha+1} - 1] .$$

Letting $x \rightarrow \infty$ we find for all $\alpha > 0$

$$L > \alpha + \alpha(2\alpha + 1)^{-1} [(1 + L_0/\alpha)^{2\alpha+1} - 1] .$$

Using that $L_0 < \frac{1}{2}$ we derive

$$(31) \quad L > L_0 + 2L_0 \alpha \log(1 + L_0/\alpha) .$$

Now we turn to the proof of the right hand inequality of (22).

Let $u := y - y_0$. Then u satisfies

$$(32) \quad y u'' = -2x u' - y_0'' u \quad ; \quad u(0) = \alpha \quad , \quad u'(0) = 0 .$$

We want an upper bound for $u(\frac{1}{4})$.

From (24) we know that $u' > 0$, and also $y > \alpha + y_0$. Moreover, we have $-y'' \leq 2$, and, using (30), $y_0(x) > x(1 - x + \frac{1}{3}x^2) > \frac{3}{4}x$ on $[0, \frac{1}{4}]$. Hence

$$(33) \quad u'' \leq 2(\alpha + \frac{3}{4}x)^{-1}u \quad (0 \leq x \leq \frac{1}{4}) .$$

We introduce a function v as follows: $v(0) = \alpha$, $v'(0) = 0$, v satisfies (32) with equality. Then $v(x) \geq u(x)$ on $[0, \frac{1}{4}]$. This follows by standard arguments using the differential (in-)equalities for $\varphi := u'/u$, $\psi := v'/v$. Since v is convex we have $v(s) \leq 4s v_1 + \alpha(1 - 4s)$ ($0 \leq s \leq \frac{1}{4}$) where $v_1 := v(\frac{1}{4})$. Hence $v''(s) \leq 2(\alpha + \frac{3}{4}s)^{-1}(4s v_1 + \alpha(1 - 4s))$. Integrating twice we find

$$v_1 - \alpha \leq \frac{1}{3}(v_1 - \alpha) - \frac{256}{9}\alpha(v_1 - \alpha - \frac{3}{16})[(\frac{1}{4} + \frac{4}{3}\alpha) \log(1 + 3(16\alpha)^{-1}) - \frac{1}{4}] ,$$

from which an upper bound for v_1 arises, thus also for $u(\frac{1}{4})$. We get

$$(34) \quad u(\frac{1}{4}) < \alpha + \alpha \log(1 + 3(16\alpha)^{-1}) .$$

Finally we shall show that $u(\infty)/u(\frac{1}{4}) < C$, where C is independent of α for $\alpha \in (0, \frac{1}{3})$. Defining $\varphi := u'/u$ we get from (32):

$$\varphi' = 2x(y_0 y)^{-1} y_0' - 2xy^{-1}\varphi - \varphi^2 =: F(x, \varphi) \quad , \quad \varphi(0) = 0 .$$

Let $\psi(x) := (y_0' + (2xy_0')^{\frac{1}{2}})(y_0)^{-1}$. Then $\psi'(x) - F(x, \psi(x)) = (2x)^{\frac{1}{2}}(y_0')^{3/2}(y_0)^{-2} + (2xy_0')^{\frac{1}{2}}(y_0)^{-1}(x(2(y)^{-1} - (y_0)^{-1}) + (2xy_0)^{-1})$. Clearly $\psi' - F(x, \psi) > 0$ for $0 < x < 2^{-\frac{1}{2}}$. Further $\psi' - F(x, \psi) > 0$ for $x \geq 2^{-\frac{1}{2}}$ provided that $2(y)^{-1} - (y_0)^{-1} > 0$ on $[2^{-\frac{1}{2}}, \infty)$. This condition is fulfilled, since by (25) and (28), we have

$$2y_0(2^{-\frac{1}{2}}) > (7 - 3\sqrt{2})(3\sqrt{2})^{-1} > (1 + 2\sqrt{\alpha})^2 \frac{1}{2} > (1 + 2\sqrt{\alpha})^2 L_0 > L > y(x) ,$$

and the second inequality holds if $0 < \alpha \leq \frac{1}{3}$.

Further, $F(x,0) > 0$ ($x > 0$), and $\varphi > 0$ ($x > 0$). It follows (see [3] p. 177) that $0 < \varphi < \psi$. Therefore,

$$\begin{aligned} \log(u(\infty)/u(\frac{1}{4})) &< \int_{\frac{1}{4}}^{\infty} \psi(x) dx = \log(L_0/y_0(\frac{1}{4})) + \\ &+ \int_{\frac{1}{4}}^{\infty} (2xy_0')^{\frac{1}{2}} (y_0')^{-1} dx = \log(L_0/y_0(\frac{1}{4})) + 2(2y_0'(\frac{1}{4}))^{\frac{1}{2}} - \\ &- 2^{-\frac{1}{2}} \int_{\frac{1}{4}}^{\infty} x^{-3/2} (y_0')^{\frac{1}{2}} dx < \log(L_0/y_0(\frac{1}{4})) + 2(2y_0'(\frac{1}{4}))^{\frac{1}{2}} . \end{aligned}$$

Using $y_0(\frac{1}{4}) > 37/192$ (from (25)) and $y_0'(\frac{1}{4}) < e^{-\frac{1}{2}}$ (see proof of (26)) we find

$$(35) \quad u(\infty)/u(\frac{1}{4}) < 47 L_0 .$$

Combining (21), (31), (34) and (35) we can summarize the results of this section by

$$(36) \quad 2 L_0 \log(1 + L_0 \gamma^2) < L(\gamma) - L_0 \gamma^2 < 47 L_0 \log(1 + 3(16)^{-1} \gamma^2) + 47 L_0$$

($\gamma > 1.8$) .

Remark. Numerical experiments suggest that

$$L - L_0 \gamma^2 \sim C \log \gamma \quad (\gamma \rightarrow \infty) .$$

6. Bounds for the limit if $\gamma < 0$.

Integrating (1) in the form $y'' = -2xy^{-1}y'$ over $[1, x]$ we get

$$(37) \quad y(x) = 1 + \gamma x + 2 \int_0^x (s^2 - xs)(y(s))^{-1} y'(s) ds .$$

Writing $s^2 - xs = (s - \frac{1}{2}x)^2 - \frac{1}{4}x^2$ and taking exponentials we arrive at

$$(38) \quad y(x) = \exp[2x^{-2} + 2\gamma x^{-1} - 2x^{-2} y(x) + I(x)] ,$$

where

$$(39) \quad I(x) := 4x^{-2} \int_0^x (s - \frac{1}{2}x)^2 (y(s))^{-1} y'(s) ds .$$

Since $I(x) < 0$ we have

$$(40) \quad y(x) < \exp[2x^{-2} + 2\gamma x^{-1}] \quad (x > 0) .$$

Obviously, we have the inequality $L < y(2|\gamma|^{-1}) < \exp[-\frac{1}{2}\gamma^2]$, but we can do much better. First we will show that

$$(41) \quad L \geq y(2|\gamma|^{-1}) (1 - \frac{1}{4}\gamma^2 \exp[-8\sqrt{2}|\gamma|^{-3} \exp[\frac{1}{2}\gamma^2]]) .$$

This inequality states that $y(2|\gamma|^{-1})$ is a very good approximation of L for large $|\gamma|$.

Secondly, we will obtain sharp inequalities for $y(2|\gamma|^{-1})$ which, by (41), will result in sharp inequalities for L .

Utilizing (5) with $x = \alpha := 2|\gamma|^{-1} - 2\sqrt{2}|\gamma|^{-2}$ and $x = \beta := 2|\gamma|^{-1}$ respectively, and (40) we have

$$\begin{aligned} y'(\beta) &> \gamma \exp[-(\beta^2 - \alpha^2) (y(\alpha))^{-1}] > \\ &> \gamma \exp[-(\beta^2 - \alpha^2) \exp -2\alpha^{-2} - 2\gamma\alpha^{-1}] > \\ &> \gamma \exp[-8\sqrt{2}|\gamma|^{-3} \exp[\frac{1}{2}\gamma^2]] . \end{aligned}$$

Using this in (8) with $a = 2|\gamma|^{-1}$ we get (41).

In order to obtain good estimates for $y(2|\gamma|^{-1})$ we have, by (38), to find good estimates for $I(2|\gamma|^{-1})$. Therefore we put

$$(42) \quad u(t) := \gamma y(|\gamma|^{-1} t) (y'(|\gamma|^{-1} t))^{-1} .$$

Then, with $\varepsilon := 2|\gamma|^{-2}$ we get

$$(43) \quad \dot{u}(t) = -1 + \varepsilon t u(t) \exp \left[\int_0^t (u(s))^{-1} ds \right] , \quad u(0) = 1 ,$$

and

$$(44) \quad I(2|\gamma|^{-1}) = \int_0^2 (t-1)^2 (u(t))^{-1} dt .$$

The solution u of (43) has the following global behaviour. It starts with the value 1, decreases till $t = t_0$, where it attains a minimum $u_0 := u(t_0) > 0$ and thereafter it increases very rapidly. Since $f(t) := u(t) \exp \left[\int_0^t (u(s))^{-1} ds \right]$ is increasing and $f(t_0) = (\varepsilon t_0)^{-1}$ we have $\dot{u}(t) \leq -1 + t/t_0$ ($0 \leq t \leq t_0$), from which it follows that $u(t) \leq 1 - t + \frac{1}{2}t^2/t_0$, whence $0 < u_0 < 1 - \frac{1}{2}t_0$. Hence $t_0 < 2$.

We need the relations

$$(45) \quad t(\dot{u} + 1)^{-1} = \varepsilon^{-1} - \int_0^t s(u(s))^{-1} ds ,$$

$$(46) \quad \int_0^t s^2(u(s))^{-1} ds = \varepsilon^{-1} + (u + t) \left(\int_0^t s(u(s))^{-1} ds - \varepsilon^{-1} \right) ,$$

the correctness of both is easily checked by differentiation. Substituting $t_0 = t_0$ in (42), (45) and (46) we obtain

$$(47) \quad \int_0^{t_0} (u(s))^{-1} ds = -\log(\varepsilon t_0 u_0) ,$$

$$(48) \quad \int_0^{t_0} s(u(s))^{-1} ds = \varepsilon^{-1} - t_0 ,$$

$$(49) \quad \int_0^{t_0} s^2(u(s))^{-1} ds = \varepsilon^{-1} - t_0^2 - u_0 t_0 .$$

We will now prove that $t_0 > 1$ for $\varepsilon > 0$ sufficiently small. For suppose that $t_0 \leq 1$. Then on the one hand we have

$$\begin{aligned} t_0^2 + u_0 t_0 - t_0 &= \int_0^{t_0} (s - s^2)(u(s))^{-1} ds \geq \int_0^{t_0} (s - s^2) ds = \\ &= \frac{1}{2} t_0^2 - \frac{1}{3} t_0^3 , \end{aligned}$$

which implies $u_0 > 1 - \frac{1}{2} t_0 - \frac{1}{3} t_0^2 \geq \frac{1}{6}$. On the other hand we have

$$\varepsilon^{-1} - 1 \leq \int_0^{t_0} s(u(s))^{-1} ds \leq \int_0^{t_0} (u(s))^{-1} ds = -\log(\varepsilon t_0 u_0) < -\log(\varepsilon u_0) ,$$

implying that $u_0 < \varepsilon^{-1} \exp[1 - \varepsilon^{-1}]$ which contradicts $u_0 \geq \frac{1}{6}$ for $0 < \varepsilon < 0.23$.

In the sequel we need the inequalities

$$(50) \quad (\varepsilon t_0)^{-1} + \log(\varepsilon t_0) - 1 < \log(u_0^{-1}) < \varepsilon^{-1} + \log(\varepsilon t_0) + 1 - t_0 .$$

Using (47), (48) and the inequality $u(s) \geq 1 - s$ ($0 \leq s \leq 1$) we can write

$$\begin{aligned} \varepsilon^{-1} - t_0 &= \int_0^{t_0} s(u(s))^{-1} ds < t_0 \int_0^{t_0} (u(s))^{-1} ds = -t_0 \log(\varepsilon t_0 u_0) ; \\ \varepsilon^{-1} - t_0 &> \int_0^1 s(u(s))^{-1} ds + \int_1^{t_0} (u(s))^{-1} ds = \\ &= \int_0^1 (s - 1)(u(s))^{-1} ds + \int_0^{t_0} (u(s))^{-1} ds > -1 + \log(\varepsilon t_0 u_0)^{-1} , \end{aligned}$$

from which (50) follows.

The next stage in our treatment is the proof of

$$(51) \quad t_0 - 1 < \frac{1}{2}\varepsilon + 4\varepsilon^2 |\log \varepsilon| \quad (0 < \varepsilon \leq 0.05) .$$

Since $\ddot{u} = t^{-1}(\dot{u} + 1) + u^{-1}(\dot{u} + 1)^2 > 0$ on $[0, t_0]$ we have

$$(52) \quad u(t) \leq t_0 + u_0 - t \quad (0 \leq t \leq t_0) .$$

Therefore

$$\begin{aligned} (53) \quad \int_1^{t_0} (s^2 - s)(u(s))^{-1} ds &\geq \int_1^{t_0} (s^2 - s)(t_0 + u_0 - s)^{-1} ds = \\ &= (t_0 - 1) [(u_0 + t_0) \log(u_0^{-1}(u_0 + t_0 - 1)) + \frac{1}{2} - \frac{3}{2} t_0 - u_0] > \\ &> (t_0 - 1) [t_0 \log(u_0^{-1}(t_0 - 1)) + \frac{1}{2} - \frac{3}{2} t_0 - u_0] . \end{aligned}$$

Using (48), (49), (50) and the inequality $u(t) \geq 1 - t$ on $[0,1]$ we find

$$\begin{aligned} t_0(1 - t_0 - u_0) &= \int_0^1 (s^2 - s)(u(s))^{-1} ds + \int_1^{t_0} (s^2 - s)(u(s))^{-1} ds > \\ &> -\frac{1}{2} + \int_1^{t_0} (s^2 - s)(u(s))^{-1} ds , \end{aligned}$$

which with (53) leads to

$$(t_0 - 1)[\varepsilon^{-1} + t_0 \log(\varepsilon t_0(t_0 - 1)) - \frac{3}{2}t_0 + \frac{1}{2} + u_0(t_0 - 1)^{-1}] < \frac{1}{2}$$

from which (51) follows.

Next we show that

$$(54) \quad \int_{t_0}^2 (t - 1)^2 (u(t))^{-1} dt \leq 3\varepsilon^2 \quad (0 < \varepsilon \leq 0.05) .$$

Since $\ddot{u} = 3(\dot{u} + 1)^2 (tu)^{-1} + (\dot{u} + 1)^2 (\dot{u} + 2)u^{-2} > 0$ we have $\ddot{u}(t) > \ddot{u}(t_0) = t_0^{-1} + u_0^{-1} > u_0^{-1}$ for $t > t_0$ whence $u(t) > u_0 + \frac{1}{2}u_0^{-1}(t^2 - t_0^2)$ ($t > t_0$).

Now, using this latter inequality and (50), (51) we have

$$\begin{aligned} (55) \quad \int_{t_0}^2 (s - 1)^2 (u(s))^{-1} ds &= \int_{t_0}^2 [(t_0 - 1)^2 + 2(t_0 - 1)(s - t_0) + \\ &\quad + (s - t_0)^2] (u(s))^{-1} ds \leq \\ &\leq (t_0 - 1)^2 2^{-\frac{1}{2}} \pi + (2(t_0 - 1)u_0 + u_0^2) \log(1 + \frac{1}{2}u_0^{-2}) < 3\varepsilon^2 \quad (0 < \varepsilon \leq 0.05) . \end{aligned}$$

Finally, we are able to estimate $\int_0^{t_0} (1 - s)^2 (u(s))^{-1} ds$ as follows, using

(48), (51) and (55),

$$\begin{aligned} \int_0^{t_0} (1-s)^2 (u(s))^{-1} ds &< \int_0^1 (1-s)^2 (1-s)^{-1} ds + \\ &+ (t_0 - 1)^2 \int_0^{t_0} s (u(s))^{-1} ds \leq \\ &\leq \frac{1}{2} + \left(\frac{1}{4}\varepsilon^2 + 4\varepsilon^2 |\log \varepsilon| + 16\varepsilon^4 \log^2 \varepsilon\right) \left(\frac{1}{\varepsilon} - t_0\right) \leq \frac{1}{2} + \frac{1}{3}\varepsilon \quad (0 < \varepsilon \leq 0.05), \end{aligned}$$

and, using (52) and (51) and (50)

$$\begin{aligned} \int_0^{t_0} (1-s)^2 (u(s))^{-1} ds &> \int_0^1 (1-s)^2 (u_0 + t_0 - s)^{-1} ds = \\ &= \frac{1}{2} - (t_0 - 1) + [(u_0 + 2t_0 - 2)u_0 + (t_0 - 1)^2] \log(u_0 + t_0) (u_0 + t_0 - 1)^{-1} - u_0 > \\ &> \frac{1}{2} - (t_0 - 1) - u_0 \quad \frac{1}{2} - \frac{1}{2}\varepsilon - 4\varepsilon^2 |\log \varepsilon| - \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}} \quad (0 < \varepsilon \leq 0.05). \end{aligned}$$

Combining the last two inequalities with (38), (41), (44), (54) and (55) we can obtain

$$(56) \quad 1 - \gamma^{-2} \quad L(\gamma) \exp\left[\frac{1}{2}\gamma^2 + \frac{1}{2}\right] < 1 + \gamma^{-2} \quad (\gamma < -7).$$

Remark. Numerical experiments suggest that

$$L(\gamma) \exp\left[\frac{1}{2}\gamma^2 + \frac{1}{2}\right] - 1 \sim C\gamma^{-2} \quad (\gamma \rightarrow -\infty).$$

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