Non σ-finite measures and product measures

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MATHEMATICS

NON σ-FINITE MEASURES AND PRODUCT MEASURES

BY

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1. *Introduction.* The present paper does not contain many new results; its principal aim is of a methodic (or, if one wants, didactic) nature. It shows how to build up the abstract theory of integration starting from one central notion, that of "measure", defined on as simple a collection of sets as possible, a "semiring" of sets, without the usual assumption of σ-finiteness. A product measure of two not necessarily σ-finite measures is introduced, and the integral of a non-negative function is then defined as the product measure of its ordinate set, generalizing thus a well-known procedure in Euclidean space. This method of handling integration stands in a rather sharp contrast to the fashion of considering an integral first and foremost as a linear functional, originally defined on a simple class of functions, and which is to be extended then to a more extensive class. However, some remarks made in the last section of this paper may serve perhaps as a hint to indicate that the contrast is not so great as sometimes believed or stated.

We shall give a concise sketch of the theory. The theorems whose proofs are standard, are marked by an asterisk. Their proofs are omitted, and may be found e.g. in [1] or [2].

2. *Measure on a semiring.* We consider a general set X, the elements of which will be called points. All other point sets mentioned in what follows are subsets of X. The empty set 0 is also considered to be a subset of X. We use standard notations: $A-B$ for the set of all points belonging to A and not to B, E' for the complement of E (hence $E'=X-E$), $\sum E_i$ and ΠE_i for union and intersection of an at most countable collection of sets E_i, $\bigcup_\alpha E_\alpha$ for the union of a possibly uncountable collection E_α, $A \subset B$ to indicate that A is a subset of B, and $x \in A$ to indicate that the point x belongs to the set A.

Definition of a semiring. *The collection Γ of point sets is called a semiring whenever*

(*a*) $0 \in \Gamma$.

(*b*) *If* $A \in \Gamma$, $B \in \Gamma$, *then* $AB \in \Gamma$.

(*c*) *If* $A \in \Gamma$, $B \in \Gamma$, $B \subset A$, *then* $A-B=\sum C_n$, *where* $\sum C_n$ *is a finite or countable union, all* C_n *are disjoint and all* $C_n \in \Gamma$.

This definition is slightly more general than those in [1] and [2] since, in the present version, in condition (*c*) we also admit countable unions.

Theorem 1. *If Γ is a semiring, and $A_1, \ldots, A_n \in \Gamma$, there exist disjoint sets $B_1, B_2, \ldots \in \Gamma$ such that each A_i is the union of a subsequence of the B_j.*

Proof. If S is a non-empty subset of the index set $1, 2, \ldots, n$, and S' the complementary subset, we define $D_S = \Pi_{k \in S, l \in S'}(A_k - A_l)$, hence in particular $D_S = \Pi_1^n A_k$ if S' is empty. Then D_S is a union of disjoint sets of Γ, and $D_{S_1} D_{S_2} = 0$ for $S_1 \neq S_2$. Since $A_i = \sum_S D_S$, where the summation is over all S such that $i \in S$, the proof is complete.

Definition of a σ-ring. *A non-empty collection Λ of point sets is called a σ-ring whenever*

(*a*) *If $A_n \in \Lambda$ $(n=1, 2, \ldots)$, then $\sum_1^\infty A_n \in \Lambda$.*

(*b*) *If $A \in \Lambda$, $B \in \Lambda$, then $A - B \in \Lambda$.*

* Theorem 2. *If Λ is a σ-ring, then*

(1) $0 \in \Lambda$.

(2) *If $A_n \in \Lambda$ $(n=1, 2, \ldots)$, then $\Pi_1^\infty A_n \in \Lambda$, lim sup $A_n \in \Lambda$ and lim inf $A_n \in \Lambda$.*

(3) *Λ is a semiring.*

We shall adopt the usual conventions with regard to operations with $\pm\infty$ in the extended real number system; in particular $0 \cdot (\pm\infty) = 0$.

Definition of measure on a semiring. *Let there be assigned to every set A of a semiring Γ a real number $\mu(A)$. This function $\mu(A)$ is called a measure on Γ whenever*

(*a*) *$\mu(0) = 0$ and $0 \leqslant \mu(A) \leqslant \infty$ for every $A \in \Gamma$.*

(*b*) *If $A \in \Gamma$, $A_n \in \Gamma$ $(n=1, 2, \ldots)$, $A \subset \sum_1^\infty A_n$, then $\mu(A) \leqslant \sum_1^\infty \mu(A_n)$.*

(*c*) *If $A \in \Gamma$, $A_n \in \Gamma$ $(n=1, \ldots, p)$ and disjoint, $A \supset \sum_1^p A_n$, then $\mu(A) \geqslant \sum_1^p \mu(A_n)$.*

* Theorem 3. *If $\mu(A)$ is a measure on the semiring Γ, then*

(1) *μ is monotone, that is, if $A \in \Gamma$, $B \in \Gamma$, $A \subset B$, then $\mu(A) \leqslant \mu(B)$.*

(2) *μ is countably additive, that is, if $A \in \Gamma$, $A_n \in \Gamma$ $(n=1, 2, \ldots)$ and disjoint, $A = \sum_1^\infty A_n$, then $\mu(A) = \sum_1^\infty \mu(A_n)$.*

These conditions, together with $\mu(0) = 0$, are also sufficient to ensure that $\mu(A)$ is a measure on Γ.

We give some obvious examples, meant for future reference.

(1) X is real Euclidean space of m dimensions; Γ consists of 0 and all cells A (left open intervals $a_i < x_i \leqslant b_i$, $1 \leqslant i \leqslant m$); $\mu(A) = \Pi_1^m (b_i - a_i)$. The proof that μ is a measure on Γ is by the Heine–Borel theorem. Variants: Γ consists of 0 and all rational cells (rational endpoints); Γ consists of 0 and all integer cells (integer endpoints).

(2) X is arbitrary, but not empty; Γ consists of 0 and all one-point sets A; $\mu(A) = 1$.

(3) X is arbitrary, but containing an infinity of points; Γ consists

of 0, X, all finite sets and their complements; $\mu(0)=0$, $\mu(A)=n$ if A contains n points, $\mu(A)=\infty$ in all other cases.

(4) X is arbitrary, but not empty; Γ consists of all subsets of X; $\mu(0)=0$, $\mu(A)=\infty$ in all other cases.

(5) X is arbitrary, but containing an uncountable number of points; Γ consists of all subsets of X; $\mu(A)=0$ for countable A, $\mu(A)=\infty$ for uncountable A.

(6) X is two-dimensional real Euclidean space (the plane therefore); Γ consists of 0 and all "horizontal" left open linesegments $A(a<x\leqslant b;\ y=c)$; $\mu(A)=b-a$.

3. *Exterior measure.* We assume that $\mu(A)$ is a measure on the semiring Γ.

Definition of sequential covering. *The set $S\subset X$ is said to be sequentially covered by Γ whenever $S\subset\sum_1^\infty A_n$, where all $A_n\in\Gamma$.*

In 1, Ex (2), if X contains an uncountable number of points, every uncountable set fails to be sequentially covered. In 1, Ex (6), if the set S contains points (x, y) for an uncountable number of values of y, S fails to be sequentially covered.

Definition of exterior measure. *If $S\subset X$ is sequentially covered, the exterior measure $\mu^*(S)$ of S is defined by*

$$\mu^*(S)=\inf\sum\mu(A_n)\ \text{over all}\ \sum A_n\supset S,\ \text{all}\ A_n\in\Gamma.$$

If S fails to be sequentially covered, we define $\mu^*(S)=\infty$.

* Theorem 1. *We have*

(1) *μ^* is subadditive, that is, $\mu^*(\sum S_n)\leqslant\sum\mu^*(S_n)$.*

(2) *$\mu^*(0)=0$, and $0\leqslant\mu^*(S)\leqslant\infty$ for every S.*

(3) *μ^* is monotone, that is, if $S\subset T$, then $\mu^*(S)\leqslant\mu^*(T)$.*

(4) *If $A\in\Gamma$, then $\mu^*(A)=\mu(A)$.*

* Theorem 2. *(X, Γ, μ) and (X, Γ_1, μ_1) generate the same exterior measure in X if and only if $\mu^*=\mu_1$ on Γ_1 and $\mu_1^*=\mu$ on Γ.*

Comparison of the two variants of 1, Ex (1) shows that $\mu^*=\mu_1$ on Γ_1 alone is not sufficient.

4. *Measurable sets.* The subadditivity of μ^* shows that

$$\mu^*(S)\leqslant\mu^*(SE)+\mu^*(SE')$$

for any pair of sets S, E.

Definition of a measurable set. *The set $E\subset X$ is called μ-measurable (or shortly measurable) whenever*

$$\mu^*(S)=\mu^*(SE)+\mu^*(SE')$$

for every $S\subset X$. The collection of all measurable sets is denoted by Λ.

* Theorem 1. *If $E\in\Lambda$, then $E'\in\Lambda$. Hence $0\in\Lambda$ and $X\in\Lambda$.*

* Theorem 2. *If $E_n\in\Lambda$ $(n=1, 2, \ldots)$, then $\sum_1^\infty E_n\in\Lambda$ and $\Pi_1^\infty E_n\in\Lambda$.*

If all E_n are disjoint, then $\mu^*(S\sum_1^\infty E_n)=\sum_1^\infty \mu^*(SE_n)$ for any S. In particular $\mu^*(\sum_1^\infty E_n)=\sum_1^\infty \mu^*(E_n)$, which shows that μ^* is countably additive on Λ. Finally, if $E_1 \in \Lambda$, $E_2 \in \Lambda$, then $E_1-E_2 \in \Lambda$. Hence Λ is a σ-ring on which μ^* is countably additive.

* Theorem 3. *If $A \in \Gamma$, then $A \in \Lambda$; hence $\Gamma \subset \Lambda$. It follows that μ^* is a measure on Λ, an extension of μ on Γ.*

Without fear for confusion we may and shall write, if E is measurable, $\mu(E)$ instead of $\mu^*(E)$.

If we consider Λ as a semiring, and start the process again, defining the exterior measure $\bar{\mu}^*$ by $\bar{\mu}^*(S)=\inf \sum\mu(E_n)$ over all $\sum E_n \supset S$, where all $E_n \in \Lambda$, it follows from 3, Th. 2 that $\bar{\mu}^*=\mu^*$. The same is true if we start from some semiring Γ_1 satisfying $\Gamma \subset \Gamma_1 \subset \Lambda$.

Theorem 4. *E is measurable if and only if EA is measurable for each $A \in \Gamma$ of finite measure.*

Proof. "Only if" is evident. Assume next that $EA \in \Lambda$ for each $A \in \Gamma$ satisfying $\mu(A)<\infty$. We have to prove that $\mu^*(S) \geqslant \mu^*(SE)+\mu^*(SE')$ for each S. This is true if $\mu^*(S)=\infty$; assume therefore that $\mu^*(S)<\infty$, so $S \subset A=\sum_1^\infty A_n$, where $A_n \in \Gamma$, $\mu(A_n)<\infty$. Since all $EA_n \in \Lambda$ by hypothesis, we have $EA \in \Lambda$, which implies $\mu^*(S)=\mu^*(SEA)+\mu^*\{S(EA)'\}$. But $SEA=SE$, $S(EA)'=S(E'+A')=SE'$, so that $\mu^*(S)=\mu^*(SE)+\mu^*(SE')$.

Corollary. *If $\mu(EA)=0$ for each $A \in \Gamma$ of finite measure, then either $\mu(E)=0$ or $\mu(E)=\infty$.*

Proof. We immediately observe that $\mu(EF)=0$ for each measurable F of finite measure; hence, if $\mu(E)<\infty$, the choice $F=E$ leads to $\mu(E)=0$.

We once more consider the examples of section 2.

(1) Λ is the collection of all Lebesgue measurable sets. The first variant gives the same collection Λ; for the second variant we have $E \in \Lambda$ if and only if $E=\sum_1^\infty A_n$, $A_n \in \Gamma$.

(2) and (3) Λ is the collection of all subsets of X, $\mu(E)$ is the number of points in E.

(4) and (5) $\Lambda=\Gamma$.

(6) If $E \subset X$, denote for any fixed y by E_y the set of all points (x, y) such that $(x, y) \in E$. Λ is the collection of all sets $\bigcup_y E_y$, where each E_y is linearly Lebesgue measurable (cf. Theorem 4 above); $\mu(E)=\infty$ if an uncountable number of the E_y are non-empty, $\mu(E)=\sum_y \mu(E_y)$ if there is only a countable number of non-empty E_y, where $\mu(E_y)$ is the linear Lebesgue measure of E_y.

This example also shows that the following statements are both false:

(a) If $\mu(EA)=0$ for each $A \in \Gamma$, then $\mu(E)=0$.

(b) If $\mu(EF)=0$ for each $F \in \Lambda$ of finite measure, then $\mu(E)=0$.

If E is a set having exactly one point on each "horizontal" line, then E satisfies the hypotheses but not the conclusions of (a) and (b).

5. *Properties of measurable sets*

* Theorem 1. *If $\mu^*(E)=0$, then E is measurable. Hence every subset of a set of measure zero is measurable and of measure zero.*

Corollary. *If every $A \in \Gamma$ satisfies either $\mu(A)=0$ or $\mu(A)=\infty$, then every set E is measurable and satisfies either $\mu(E)=0$ or $\mu(E)=\infty$.*

Proof. Combine the above theorem with the corollary of 4, Th. 4.

We adopt the usual "almost everywhere" conventions. Any set $O=\sum A_n$, where all $A_n \in \Gamma$, is called a σ-set. If $O_n (n=1, 2, \ldots)$ are σ-sets, then $\sum_1^\infty O^n$ and any finite intersection $\Pi_1^p O_n$ are σ-sets. Any set $\Pi_1^\infty O_n$, where all O_n are σ-sets, is called a σ_δ-set. The measurable set E is called of σ-finite measure whenever $E \subset \sum A_n$, where all $A_n \in \Gamma$ and all $\mu(A_n)<\infty$.

* Theorem 2. *If S is sequentially covered by Γ, then $\mu^*(S)=inf\ \mu(O)$ over all σ-sets O covering S. For any set S we have $\mu^*(S)=inf\ \mu(E)$ over all measurable E covering S (Take $E=X$ whenever $\mu^*(S)=\infty$). If E is of σ-finite measure, there exists a σ_δ-set $O_\delta=\Pi_1^\infty O_n$ covering E such that $\mu(O_\delta-E)=0$ and the sequence O_n is descending.*

* Theorem 3. *We have*

(*a*) *If the sequence $E_n \in \Lambda$ is ascending, and $E=lim\ E_n$, then $\mu(E)=lim\ \mu(E_n)$.*

(*b*) *If $E_n \in \Lambda$ is descending with limit E, and $\mu(E_n)<\infty$ for some n, then $\mu(E)=lim\ \mu(E_n)$.*

(*c*) *If $E_n \in \Lambda$, then μ $(lim\ inf\ E_n) \leqslant lim\ inf\ \mu(E_n)$. Furthermore, if $\mu(E_n+E_{n+1}+\ldots)<\infty$ for some n, then μ $(lim\ sup\ E_n) \geqslant lim\ sup\ \mu(E_n)$. Hence, if $E=lim\ E_n$ exists and $\mu(E_n+E_{n+1}+\ldots)<\infty$ for some n, then $\mu(E)=lim\ \mu(E_n)$.*

6. *Measurable functions.* Let Λ be a σ-ring of subsets of X. In this section the sets of Λ are called measurable sets, which is not meant to ndicate that Λ is necessarily the σ-ring of all measurable sets with respect to some measure μ. The function $f(x)$, defined on the measurable set E, and assuming values in the extended real number system, is called measurable whenever $E(f>a)$, i.e. the subset of E on which $f(x)>a$, is measurable for each finite a.

The standard theorems on measurability of sums, products, inf, sup, lim inf and lim sup of sequences of measurable functions are proved in the usual way.

7. *Product measure.* If X_1 and X_2 are point sets, and $A \subset X_1$, $B \subset X_2$, we denote by $A \times B$ the Cartesian product of A and B (in particular $A \times 0=0 \times B=0$). If $E \subset X_1 \times X_2$ and $x \in X_1$, the set of all $y \in X_2$ such that $(x, y) \in E$ is denoted by E_x. Similarly E_y for $y \in X_2$.

Theorem 1. *If Γ_1 and Γ_2 are semirings in X_1 and X_2 respectively, then the collection of all subsets $C=A \times B$ of $X=X_1 \times X_2$ for which $A \in \Gamma_1$, $B \in \Gamma_2$ is a semiring $\Gamma=\Gamma_1 \times \Gamma_2$.*

Proof. Trivial.

Theorem 2. *If* $\mu_1(A)$ *and* $\mu_2(B)$ *are measures on the semirings* Γ_1 *(in* X_1*) and* Γ_2 *(in* X_2*) respectively, and we define* $\mu(C)=\mu_1(A)\mu_2(B)$ *for each set* $C=A\times B$ *belonging to the semiring* $\Gamma=\Gamma_1\times\Gamma_2$ *(we remind the reader that* $0\cdot\infty=\infty\cdot 0=0$*), then* $\mu(C)$ *is a measure on* Γ.

The proof is divided into several lemmas which together show that μ has the desired properties.

Lemma α. *We have* $\mu(0)=0$ *and* $0\leqslant\mu(C)\leqslant\infty$ *for each* $C\in\Gamma$. *If* $C_1\in\Gamma$, $C_2\in\Gamma$ *and* $C_1\subset C_2$, *then* $\mu(C_1)\leqslant\mu(C_2)$.

Proof. Trivial.

Lemma β. *If* $C\in\Gamma$, $C_n\in\Gamma$ $(n=1,\dots,p)$, *all* C_n *are disjoint and* $C\supset\sum_1^p C_n$, *then* $\mu(C)\geqslant\sum_1^p\mu(C_n)$.

Proof. Let $C=A\times B$ and $C_n=A_n\times B_n$ $(n=1,\dots,p)$. Then there exist disjoint sets $D_1, D_2,\dots\in\Gamma_1$ such that each A_n is the union of a subsequence of the D_i, and disjoint sets $E_1, E_2,\dots\in\Gamma_2$ such that each B_n is the union of a subsequence of the E_j. If $A_n={}'\sum D_i$, $B_n={}'\sum E_j$, then $A_n\times B_n={}'\sum'\sum D_i\times E_j$, and

$$\mu(A_n\times B_n)=\mu_1(A_n)\mu_2(B_n)=\{'\textstyle\sum\mu_1(D_i)\}\{'\sum\mu_2(E_j)\}={}'\sum'\sum\mu(D_i\times E_j).$$

Hence, observing that each $D_i\times E_j$ is contained in at most one of the $A_n\times B_n$, we find

$$\textstyle\sum_1^p\mu(C_n)\leqslant\sum_{i=1}^\infty\sum_{j=1}^\infty\mu(D_i\times E_j)=\{\sum_1^\infty\mu_1(D_i)\}\{\sum_1^\infty\mu_2(E_j)\}\leqslant\mu_1(A)\mu_2(B)=\mu(C).$$

This proof also shows that if $C=\sum_1^p C_n$, then $\mu(C)=\sum_1^p\mu(C_n)$.

Lemma γ. *Let* $0<a<\infty$, $0<b<\infty$, $A\subset X_1$ *and* $\mu_1^*(A)>a$. *Furthermore, let* $V\subset X_1\times X_2$ *and* $\mu_2^*(V_x)>b$ *for almost every* $x\in A$. *Then, if* $C_k\in\Gamma$ $(k=1,2,\dots)$ *and* $V\subset\sum_1^\infty C_k$, *we have* $\sum_1^\infty\mu(C_k)>ab$.

Proof. We may assume that the C_k are disjoint (replace, if necessary, $\sum C_k$ by $C_1+(C_2-C_1)+\{(C_3-C_1)-C_2\}+\dots$). Now write $S_n=\sum_1^n C_k$. For any $x\in X_1$ the measure $\mu_2(S_{nx})$ is an ascending function of n; hence, since $\mu_2^*(V_x)>b$ almost everywhere on A and since $V\subset\sum_1^\infty C_k$, we have $\mu_2(S_{nx})>b$ for $n>n_x$ almost everywhere on A. Writing $D_n=\{x|\mu_2(S_{nx})>b\}$, the set D_n is μ_1-measurable (since $\mu_2(S_{nx})$ assumes only a finite number of differen values, each on a measurable set), the sequence D_n is ascending, and almost every $x\in A$ belongs to some D_n. Hence, if $D=\lim D_n$, we find on account of $\mu_1^*(A)>a$ that $\mu_1(D)>a$. This shows that $\mu_1(D_N)>a$ for some suitable N. A subdivision (similarly as in the preceding lemma) leads now easily to $\sum_1^N\mu(C_k)>ab$.

Lemma δ. *If* $C\in\Gamma$, $C_k\in\Gamma$ $(k=1,2,\dots)$ *and* $C\subset\sum_1^\infty C_k$, *then* $\mu(C)\leqslant\sum_1^\infty\mu(C_k)$.

Proof. For $\mu(C)=0$ there is nothing to prove; assume therefore that $\mu(C)>0$. Then, if $C=A\times B$, there exist numbers a, b such that $0<a<\mu_1(A)$, $0<b<\mu_2(B)$. We have to prove that $\sum_1^\infty\mu(C_k)>ab$ for all such pairs a, b. This is achieved by taking $V=C$ in the preceding lemma.

The measure $\mu(C)$ obtained in this way on $\Gamma=\Gamma_1\times\Gamma_2$ may now be extended to the σ-ring Λ of all μ-measurable sets in $X_1\times X_2$. There is one question which immediately arises: Assuming that Γ_1 and Γ_2 are not identical with the σ-rings Λ_1 and Λ_2 of all μ_1-measurable and all μ_2-measurable sets respectively, we can first extend μ_1 and μ_2 from Γ_1 and Γ_2 to Λ_1 and Λ_2 and then form, in the way sketched above, a measure $\bar{\mu}$ on $\Lambda_1 \times \Lambda_2$. What is in this case the connection between μ (on $\Gamma_1 \times \Gamma_2$) and $\bar{\mu}$ (on $\Lambda_1 \times \Lambda_2$)? Do they generate the same exterior measure (and therefore the same measurable sets) in $X_1 \times X_2$? It is well-known that the answer to the last question is affirmative in case X_1 is of σ-finite μ_1-measure and X_2 of σ-finite μ_2-measure, but the same need not be true in the general case, as the following example shows:

X_1 is the straight line, Γ_1 consists of 0 and all cells, μ_1 is Lebesgue measure; X_2 is also the straight line, Γ_2 consists of all subsets, $\mu_2(0)=0$ and $\mu_2(B)=\infty$ for any non-empty B. We consider in $X_1 \times X_2$ a set $E=E_1 \times E_2$ consisting of one single point (x, y). Obviously

$$\bar{\mu}(E)=\mu_1(E_1)\mu_2(E_2)=0\cdot\infty=0.$$

But the best we can do to cover E by sets $A \times B \in \Gamma_1 \times \Gamma_2$ is to take $A \times B=A \times E_2$ where A is a cell, and $\mu(A \times B)=\mu_1(A)\mu_2(B)=\infty$ implies $\mu^*(E)=\infty$. Hence $\mu^*(E)\neq\bar{\mu}(E)$. This failure is obviously due to the fact that E_2, although consisting of only one point, is not of σ-finite measure.

In order to avoid these difficulties we assume that $\Gamma_1=\Lambda_1$, $\Gamma_2=\Lambda_2$, and that the product measure $\mu=\mu_1 \times \mu_2$ on $\Lambda_1 \times \Lambda_2$ is defined by

$$\mu(E_1 \times E_2)=\mu_1(E_1)\mu_2(E_2).$$

This measure μ is then extended to the σ-ring Λ of all μ-measurable sets in $X_1 \times X_2$.

If we consider Cartesian products of more than two point sets with measures defined on them, it is not possible to circumvent the stated difficulties, since the question of associativity arises, essentially the same question as the one under which conditions Fubini's theorem on repeated integrals holds. It seems that, in order to get a satisfactory result, it is necessary to assume that all point sets under consideration are of σ-finite measure.

Theorem 3. *If $E \subset X_1 \times X_2$ is measurable and of σ-finite measure (it is not assumed that $X_1 \times X_2$ itself is of σ-finite measure), then E_x is μ_2-measurable for almost every $x \in X_1$, that is, if P is the set of all x for which E_x is not μ_2-measurable, then $\mu_1(P)=0$. If $\mu(E)=0$, then $\mu_2(E_x)=0$ for almost every $x \in X_1$.*

Proof. We first assume that $\mu(E)=0$, and we denote, if b is such that $0<b<\infty$, the set of all x such that $\mu_2^*(E_x)>b$ by A. Let us assume that $\mu_1^*(A)>0$, so that $0<a<\mu_1^*(A)$ for some a. Observe next that under the hypotheses and with the notations of Lemma γ we have $\mu^*(V)\geqslant ab$, so that, taking $V=E$ in the present case, this lemma yields $\mu^*(E)\geqslant ab>0$, in contradiction with $\mu(E)=0$. Hence $\mu_1(A)=0$.

Denoting the set of all x such that $\mu_2^*(E_x)>n^{-1}$ by A_n, and the set of all x

such that $\mu_2^*(E_x) > 0$ by P, we have now $P = \sum A_n$, hence $\mu_1(P) = \sum \mu_1(A_n) = 0$. This completes the proof in case $\mu(E) = 0$.

Let now $E = O$ be an arbitrary σ-set $\sum A_n \times B_n$. Then, for every x, the set O_x is a countable union of sets B_n, and therefore μ_2-measurable.

Finally, if E is an arbitrary μ-measurable set of σ-finite measure, there exists a descending sequence O_n $(n = 1, 2, \ldots)$ of σ-sets covering E and a set S of measure zero such that $E + S = \Pi O_n$ and $ES = 0$. Hence, for every x,

$$E_x + S_x = (\Pi O_n)_x = \Pi O_{nx}, \quad E_x S_x = 0.$$

Since all O_{nx} are μ_2-measurable for every x and S_x is μ_2-measurable for almost every x, the set E_x is μ_2-measurable for almost every x.

If E is not of σ-finite measure, the theorem is not necessarily true, not even in the case that one of the spaces X_1 or X_2 is of σ-finite measure. We give two examples.

Example 1. X_1 consists of one point, $\mu_1(X_1) = \infty$; X_2 is the straight line with ordinary Lebesgue measure. Every $E \subset X_1 \times X_2$ is measurable, since every $C \in \Lambda_1 \times \Lambda_2$ satisfies either $\mu(C) = 0$ or $\mu(C) = \infty$ (cf. 5, Th. 1, Cor.). Take $M \subset X_2$ such that M is not μ_2-measurable. Then $E = X_1 \times M$ is μ-measurable, but "every" E_x, being equal to M, fails to be μ_2-measurable. The set of x for which E_x is not μ_2-measurable, is therefore of infinite measure.

Example 2. (X_1, Λ_1, μ_1) is Example (6) in section 2 and section 4; X_2 is the straight line with ordinary Lebesgue measure. Since the points of X_1 are already denoted by (x, y), we denote the points of X_2 by z. We remind the reader that any set $F \subset X_1$ may be written as $F = \bigcup_y F_y$, and that F is μ_1-measurable if and only if all F_y are μ_1-measurable. In the same way every $E \subset X_1 \times X_2$ may be written as $E = \bigcup_y E_y$, and E is μ-measurable if and only if all E_y are μ-measurable. Let now S be a subset of X_1 such that every S_y fails to be μ_1-measurable, and let $\{\alpha\}$ be the subset of X_2 consisting of the single point $z = \alpha$. Consider now the subset $\bigcup_y (S_y \times \{y\})$ of $X_1 \times X_2$. Every $E_y = S_y \times \{y\}$ is μ-measurable (in fact, $\mu(E_y) = 0$), so that E is μ-measurable. Every $E_z = S_z$ however fails to be μ_1-measurable.

In Theorem 3 we derived that $\mu(E) = 0$ implies $\mu_2(E_x) = 0$ for almost every x. We shall now prove a result which goes in the converse direction.

Theorem 4. *Let $E \subset X_1 \times X_2$ be a μ-measurable set such that $\mu_2(E_x) = 0$ for almost every $x \in X_1$, and let $A \times B \in \Lambda_1 \times \Lambda_2$. Then, if $D = (A \times B) - E$, we have $\mu(D) = \mu(A \times B)$. Hence, if E is of σ-finite measure, $\mu(E) = 0$.*

Proof. If $\mu(A \times B) = 0$, there is nothing to prove; assume therefore that $\mu(A \times B) = \mu_1(A)\mu_2(B) > 0$, and let $0 < a < \mu_1(A)$, $0 < b < \mu_2(B)$. Taking $V = D$ in Lemma γ, we obtain $\mu(D) \geqslant ab$ for all such a and b; hence $\mu(D) = \mu(A \times B)$.

If E is not of σ-finite measure, $\mu_2(E_x) = 0$ for all x, even simultaneously with $\mu_1(E_y)=0$ for all y, does not necessarily imply $\mu(E)=0$, as the following example shows:

X_1 is the straight line with $\mu_1(A)=0$ for any countable A and $\mu_1(A)=\infty$ for any uncountable A; μ_2 is ordinary Lebesgue measure on the straight line X_2. If $E \subset X_1 \times X_2$ is the "diagonal" consisting of all points (x, x), then $\mu_2(E_x)=0$ for all $x \in X_1$, $\mu_1(E_y)=0$ for all $y \in X_2$, and nevertheless $\mu(E)=\infty$.

8. *The ordinate set of a non-negative function.* Let μ be a measure in X and m Lebesgue measure on the straight line R_1. The product measure $\mu \times m$ in $X \times R_1$ is denoted by $\bar{\mu}$.

Definition of ordinate set. *If $f(x) \geqslant 0$ is defined on the set $S \subset X$, the ordinate set F of $f(x)$ is the set of all $(x, y) \in X \times R_1$ such that $x \in S$ and $0<y<f(x)$. Observe that if $f(x)=0$ for some $x \in S$, the point $(x, 0)$ is not counted as belonging to F.*

Theorem 1. *If $E \subset X$ is μ-measurable, and $f(x) \geqslant 0$ is defined on E, then $f(x)$ is a μ-measurable function on E if and only if its ordinate set F is $\bar{\mu}$-measurable.*

Proof. First assume that F is $\bar{\mu}$-measurable. We have to prove that $E_a=E(f>a)$ is μ-measurable for each finite a, and by 4, Th. 4 it is sufficient for this purpose to prove that E_aD is μ-measurable for each μ-measurable $D \subset X$ of finite measure. We may therefore restrict ourselves to the case that E is of finite μ-measure and F of σ-finite $\bar{\mu}$-measure. For any $y \in R_1$ the set F_y is the set of all $x \in E$ such that $f(x)>y>0$, hence $F_y=E(f(x)>y)$ for $y>0$, and F_y is empty for $y \leqslant 0$. Since on account of 7, Th. 3 the set F_y is μ-measurable for almost every y, this shows that $E(f(x)>y)$ is μ-measurable for all $y=r_n$ in a suitably chosen sequence dense in R_1. This however is sufficient to ensure the measurability of $f(x)$.

Conversely, if $f(x)$ is μ-measurable on E, and r_n is a dense sequence in R_1, we define the set $E_{(n)}$ $(n=1, 2, \ldots)$ by $E_{(n)}=\{x|x \in E,\ f(x)>r_n\}$ and the set B_n by $B_n=\{y|0<y \leqslant r_n\}$. Then $S=\sum E_{(n)} \times B_n$ is $\bar{\mu}$-measurable, and it will be sufficient to prove that $F=S$. If $(x, y) \in S$, then $(x, y) \in E_{(k)} \times B_k$ for some k, hence $0<y \leqslant r_k<f(x)$, so $(x, y) \in F$. Conversely, if $(x, y) \in F$, then $0<y<f(x)$, so that $0<y<r_k<f(x)$ for some r_k. But then $(x, y) \in E_{(k)} \times B_k$, hence $(x, y) \in S$.

Sometimes the ordinate set of $f(x) \geqslant 0$ is defined as the set F^* of all (x, y) such that $0<y \leqslant f(x)$ if $f(x)<\infty$, and $0<y<f(x)$ if $f(x)=\infty$. The set $G=F^*-F$, i.e. the set of all (x, y) such that $0<y=f(x)<\infty$, is called the graph of $f(x)$. The following theorem shows that F is measurable if and only if F^* is measurable, and moreover that measurability of F implies measurability of G.

Theorem 2. *We have*

(*a*) *If $F \subset H \subset F^*$, $F \subset H_1 \subset F^*$, and H is measurable, then H_1 is measurable.*

(*b*) *If F is measurable, then $\bar\mu(F)=\bar\mu(F^*)$, but $\bar\mu(G)=0$ is not necessarily true.*

Proof. (a) Write $H(\alpha)$ for the set of all points $(x, \alpha y)$, where $(x,y)\in H$. Evidently

$$F^*=\lim_{n\to\infty} H(1+1/n),\quad F=\lim_{n\to\infty} H(1-1/n),$$

and so both F^* and F are measurable.

In order to prove the measurability of H_1-F, we consider any set $E\subset X\times R_1$ of finite $\bar\mu$-measure. Writing $K=(H_1-F)E$, we derive from 7, Th. 4 that $\bar\mu(K)=0$, since each K_x consists of one point at most. Now 4, Th. 4 shows that H_1-F is measurable.

(b) If $\bar\mu(G)>0$, we have $F\supset G(^1/_2)+G(^1/_3)+\ldots$, and so $\bar\mu(F)=\infty$. Therefore $\bar\mu(F^*)=\bar\mu(F)$. If $\bar\mu(G)=0$, then $\bar\mu(F^*)=\bar\mu(F)$ is trivial.

If (cf. the example at the end of section 7) X is the straight line with $\mu(E)=0$ for any countable E, and $\mu(E)=\infty$ for any uncountable E, and $f(x)=x$ for $x\geqslant 0$, then $\bar\mu(G)=\infty$. Nevertheless $f(x)$ is measurable. In fact, any subset of X is measurable, and so any function defined on X is measurable.

Theorem 3. *If the measurable functions $f_n(x)\geqslant 0$ $(n=1, 2, \ldots)$ have the ordinate sets F_n, and $h(x)=\sup f_n(x)$, $k(x)=\inf f_n(x)$, $p(x)=\limsup f_n(x)$, $q(x)=\liminf f_n(x)$ have the ordinate sets H, K, P and Q respectively, then $h(x)$, $k(x)$, $p(x)$ and $q(x)$ are measurable by section 6. Furthermore $H=\sum_1^\infty F_n$, and K, P, Q are contained in and of the same measure as $\Pi_1^\infty F_n$, $\limsup F_n$ and $\liminf F_n$ respectively.*

Proof. We give the proof for P. It is easily seen that $P\subset \limsup F_n \subset P^*$. Hence, in view of the preceeding theorem, $\bar\mu(P)=\bar\mu$ (lim sup F_n).

9. *The integral.* The introduction of the concept of an integral offers now no difficulties. As before, we assume that μ is a measure in X, m Lebesgue measure on the straight line, and $\bar\mu=\mu\times m$. If $E\subset X$ is μ-measurable, and $f(x)\geqslant 0$ is defined and μ-measurable on E, the integral $\int_E f(x)d\mu$ is defined to be the $\bar\mu$-measure of the ordinate set F of $f(x)$. If $\int_E f d\mu<\infty$ we say that $f(x)$ is integrable over E. Obviously, in this case, the subset of E on which $f(x)>0$ is of σ-finite measure. It is an immediate consequence of the countable additivity of $\bar\mu$ that $\int_D f d\mu$ is for $D\subset E$ a countably additive set function on E, and that the monotone convergence theorem holds. The relation $\int_E(f+g)d\mu=\int_E f d\mu+\int_E g d\mu$ is first proved for $g(x)$ a step function, and then by approximation for any measurable $g(x)\geqslant 0$. Fatou's theorem follows by combining 5, Th. 3(c) and 8, Th. 3.

Next, for a non-positive μ-measurable $f(x)$, we define $\int_E f d\mu=-\int_E(-f)d\mu$. Finally, for an arbitrary real μ-measurable $f(x)$, we define

$$f^+(x)=f(x) \text{ and } f^-(x)=0 \text{ on } E(f\geqslant 0),$$
$$f^-(x)=f(x) \text{ and } f^+(x)=0 \text{ on } E(f<0),$$

and $f(x)$ is said to have an integral over E if and only if one at least of the functions $f^+(x)$ and $f^-(x)$ is integrable over E. In this case we define

$\int_E f d\mu = \int_E f^+ d\mu + \int_E f^- d\mu$. Whenever both $f^+(x)$ and $f^-(x)$ are integrable over E, the function $f(x)$ is called integrable over E. The standard theorems on integrable functions follow easily (the dominated convergence theorem for example by combining once more 5, Th. 3(c) and 8, Th. 3).

Of course, one can also introduce the integral without using the product measure of the ordinate set. If $E(f(x) \neq 0)$ is of σ-finite measure one defines the integral first for step functions, and then by one of the standard methods for all measurable functions (cf. [1]). If $f(x) \geqslant 0$, and $E(f > 0)$ is not of σ-finite measure, one defines $\int_E f d\mu = \infty$.

The theory of the Daniell integral fits naturally in the pattern which we have developed so far. There we have a vector space L of bounded real functions on a set X such that L is closed under the lattice operations $f \cup g = \max(f, g)$ and $f \cap g = \min(f, g)$. On L we assume to be defined a finite real functional $I(f)$ satisfying $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for real α, β; $I(f) \geqslant 0$ for $f(x) \geqslant 0$; $\lim I(f_n) = 0$ if $f_n(x)$ is pointwise monotone decreasing to zero on X.

Defining, for any $f(x) \geqslant 0$ belonging to L, the ordinate set F by $F = \{(x, y) | x \in X,\ 0 \leqslant y < f(x)\}$, the collection of all sets $F - G$, where $f(x) \geqslant g(x)$, is a semiring Γ. Writing $\mu(F-G) = I(f-g)$, we shall prove that μ is a measure on Γ. Evidently $\mu(0) = 0$ and $0 \leqslant \mu(A) \leqslant \infty$ for any $A \in \Gamma$. Let now $A \in \Gamma$, $A_n \in \Gamma$ $(n = 1, 2, \ldots)$ and $A \subset \sum_1^\infty A_n$. Since we wish to prove that $\mu(A) \leqslant \sum_1^\infty \mu(A_n)$, it is no restriction of the generality to assume immediately that all A_n are disjoint and that $A = \sum_1^\infty A_n$. If A corresponds with $f(x) - g(x)$ and A_n with $f_n(x) - g_n(x)$, the sequence

$$r_k(x) = \{f(x) - g(x)\} - \sum_{n=1}^k \{f_n(x) - g_n(x)\}$$

is pointwise monotone decreasing to zero, hence $\lim I(r_k) = 0$, which implies $\mu(A) = \sum_1^\infty \mu(A_n)$. Finally, if $A \in \Gamma$, $A_n \in \Gamma$ $(n = 1, \ldots, p)$, all A_n are disjoint and $A \supset \sum_1^p A_n$, then $f(x) - g(x) \geqslant \sum_1^p \{f_n(x) - g_n(x)\}$, hence

$$\mu(A) \geqslant \sum_1^p \mu(A_n).$$

Extending this measure μ to the collection Λ of all μ-measurable sets, and considering in particular those sets $F \in \Lambda$ which are ordinate sets of functions $f(x) \geqslant 0$, we extend the linear functional I to these $f(x)$ by defining $I(f) = \mu(F)$. The extension to differences of such $f(x)$ follows immediately, and the thus defined $I(f)$ is called the Daniell integral of $f(x)$ over X. If the original vector space L has the additional property that $f \in L$ implies $f \cap 1 \in L$, it is possible in a well-known way to introduce a measure ν in X such that $I(f) = \int_X f d\nu$ for all f for which $I(f)$ is defined.

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REFERENCES

1. Halmos, P. R., Measure Theory (New York, 1950).
2. Zaanen, A. C., Linear Analysis (Amsterdam–Groningen, 1953).