

On the asymptotic behaviour of moments of infinitely dicisible distributions

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On the asymptotic behaviour of moments of
infinitely divisible distributions

by

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The Netherlands

Summary

Title: On the asymptotic behaviour of moments of infinitely divisible distributions

Abbreviated title: Asymptotic behaviour of moments

In this note the asymptotic behaviour of absolute moments of infinitely divisible (inf div) distributions is considered. It follows from the fact that tails of inf div distributions are bounded from below that their moments are bounded from below. It turns out that moments of inf div distributions also behave more regularly than moments in general, and a strong similarity is shown to exist between the moments of inf div distributions on the half-line and the probabilities of inf div distributions on the nonnegative integers. Regularity properties of the moment generating function φ are used to show that limits of the form $\log \alpha_n / (nk(n))$ exist, where $\alpha_n = \beta_n / n!$ with β_n the n -th absolute moment. Furthermore, it is shown that the boundedness of the Poisson spectrum has a simple characterization in terms of moment behaviour. Finally, this kind of behaviour is related to the order and type of the functions φ and $\log \varphi$.

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by

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1. Introduction

Tails of infinitely divisible (inf div) distribution functions (cdf's) have been studied in [3], [7] and [9]. The existence of moments of inf div distributions is discussed in [11]. In this paper new results are obtained concerning the asymptotic behaviour of moments of inf div cdf's.

In section 2 we collect various lemmas. In section 2 we are concerned with tails of inf div distributions, and give some new results on discrete distributions. In section 4 we obtain several results for moments of inf div cdf's. In section 5 relationships between the behaviour of entire characteristic functions and the moments of their cdf's are discussed.

We shall use the following notation. A cdf will be denoted by F (possibly with an index, its density (pdf) by f and its moment generating function (mgf) by φ . In section 5, we shall use φ to denote a characteristic function (cf). The tail of F is denoted by T , i.e. $T(x) = F(-x) + 1 - F(x)$ for $x > 0$. The moments of F we denote by μ_n , the absolute moments by β_n . It follows easily that

$$(1.1) \quad \beta_n = n \int_0^{\infty} x^{n-1} T(x) dx \quad (n=1,2,\dots).$$

Further we set $\alpha_n = \beta_n / n!$ and $a_n = \mu_n / n!$. It follows that, if the mgf φ is analytic in the neighbourhood of $z=0$, we have

$$(1.2) \quad \varphi(z) = \sum_0^{\infty} a_n z^n.$$

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As the quantities μ_n and a_n cannot be expected to behave very regularly (they may vanish for odd n), we shall mainly be concerned with the β_n and α_n . We shall need the simple relations

$$(1.3) \alpha_{2n} = a_{2n} ; \alpha_{2n} \geq |a_{2n-1}| \alpha_1.$$

We shall use the well-known fact (cf.[4]) that a cdf with $\beta_2 < \infty$ is inf div iff its mgf φ can be represented (uniquely) in the form

$$(1.4) \varphi(it) = \exp\{ict + \int_{-\infty}^{\infty} (e^{itu} - 1 - itu) u^{-2} dM(u)\},$$

where M is non decreasing and bounded with $M(-\infty) = 0$. M will be called the spectral function and the support of M will be referred to as the (Poisson) spectrum. We shall say that M has support on $[a,b]$ if $[a,b]$ is the smallest closed interval containing all points of increase of M . As the normal distributions (including the degenerate distributions) have smaller tails than any other inf div cdf, for our discussions it is no restriction to assume that the support of M is not concentrated at 0. So, from now on all distributions will be non-normal and non-degenerate.

2. Preliminaries

In this section we collect some results that we shall need later. The proofs not given can be found in the references shown.

Lemma 2.1 [7]. If F is an inf div cdf and if c is the smallest positive number (possibly ∞) such that $[-c,c]$ contains the support of M , then (taking $c^{-1}=0$, if $c=\infty$)

$$(2.1) \lim_{x \rightarrow \infty} \frac{-\log T(x)}{x \log x} = c^{-1}$$

Corollary 2.2 If F is inf div, then for every $\gamma > c^{-1}$ there is a constant A , and for every $\delta < c^{-1}$ there is a constant B such that

$$A \exp(-\gamma \log x) \leq T(x) \leq B \exp(-\delta x \log x) \quad (x > 0)$$

Lemma 2.3 [8]. If $(p_n)_{n=0}^{\infty}$ is a disistribution on $\{0,1,\dots\}$ with $p_0 > 0$ and $p_1 > 0$, then it is inf div iff the p_n satisfy

$$(2.2) \quad (n+1)p_{n+1} = \sum_0^n p_{n-k} r_k \quad (n=0,1,\dots),$$

with $r_0 > 0$ and $r_k \geq 0$ ($k=1,2,\dots$), or equivalently, iff $P(x)$, defined by $P(x) = \sum_0^{\infty} p_n x^n$, is of the form

$$(2.3) \quad P(x) = \exp\left\{\sum_1^{\infty} \lambda_n (x^n - 1)\right\},$$

where $\lambda_n = r_{n-1}/(n\lambda)$ with $\lambda = \sum_1^{\infty} \lambda_n < \infty$. (Here the measure generated by the λ_n has the same support as M in (1.4)).

Lemma 2.4 [8]. A cdf F on $[0,\infty)$ is inf div iff there exists a non-decreasing function K on $[0,\infty)$ with $\int (x+1)^{-1} dK < \infty$, such that

$$(2.4) \quad x dF = dF * K,$$

where $*$ denotes convolution.

Corollary 2.5. If F is an inf div cdf on $[0,\infty)$ having moments μ_n of all orders, then the quantities $a_n = \mu_n/n!$ satisfy

$$(2.5) \quad (n+1) a_{n+1} = \sum_0^n a_{n-k} b_k,$$

with $b_k > 0$ ($k=0,1,\dots$).

proof: from (2.4) we obtain

$$(2.6) \quad \mu_{n+1} = \sum_0^n \binom{n}{k} \mu_{n-k} \nu_k,$$

with $\nu_k = \int x^k dK > 0$, for all non-degenerate F . Putting $b_k = \nu_k/k!$, we obtain (2.5).

3. Tails and densities

From Lemma 2.1 we obtain the following slightly more precise result.

Theorem 3.1. If F is an inf div cdf, and if $[-a,b]$ is the smallest closed interval containing the support of M (a or b may be zero, both may be infinite), then

$$\lim_{x \rightarrow \infty} \frac{-\log F(-x)}{x \log x} = a^{-1}$$

(3.1)

$$\lim_{x \rightarrow \infty} \frac{-\log \{1-F(x)\}}{x \log x} = b^{-1}.$$

Proof: First, let $a > 0$ and $b < 0$. We put $M=M_-+M_0+M_+$, where M_-,M_0,M_+ have support on $[-a,-\epsilon]$, $[-\epsilon,\epsilon]$ and $[\epsilon,b]$ respectively. Now the corresponding cdf's can be taken such that F_- has support on $[-\infty,0]$, and F_+ on $[0,\infty)$ (cf[4], p.312). By Lemma 2.1, F_- and F_+ have (one-sided) tails satisfying (3.1). But again by Lemma 2.1, F_0 has tails that are essentially smaller than those of F_- and F_+ . As in $F=F_-*F_0*F_+$ the tails behave as the biggest tails of its components (see e.g.[11]), the theorem follows. The proof can be adapted in an obvious way if either a or b is zero.

For densities properties like those in (2.1) or (3.1) do not hold. As a counterexample consider $F = F_1 * F_2$ on $(0,\infty)$, where F_1 is absolutely continuous with $f_1(x)=(\pi x)^{-\frac{1}{2}} \exp(-x)$ and F_2 is the cdf of a geometric distribution. Now F has a density f given by $f(z) = \sum_{n \leq z} f_1(z-n)$, with poles at all nonnegative integers. It follows that $-\log f(x)/(x \log x)$ cannot have a limit as $x \rightarrow \infty$. For discrete densities on the half-line, however, we have

Theorem 3.2. If $(p_n)_0^\infty$ is an inf div distribution on $\{0,1,2,\dots\}$ with $p_0 > 0$ and $p_1 > 0$, then

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{-\log p_n}{n \log n} = N^{-1}$$

if N is the smallest positive number (necessarily integer, or possibly infinite) such that $[0,N]$ contains the support of M .

Proof: As $0 < p_n \leq 1-F(n-1)$, by Theorem 3.1 we have $\liminf_{n \rightarrow \infty} \frac{-\log p_n}{n \log n} \geq N^{-1}$.

On the other hand, by (2.3) we have

$$p_n = \sum_{m_1+2m_2+\dots+nm_n=n} e^{-\lambda} \frac{\lambda_1^{m_1}}{m_1!} \dots \frac{\lambda_n^{m_n}}{m_n!} \geq e^{-\lambda} \frac{\lambda_1^r}{r!} \frac{\lambda_N^{m_N}}{m_N!},$$

where $m_N = (n-r)/N$ and r is chosen such that m_N is an integer.

By Stirling's formula it now easily follows that $\limsup_{n \rightarrow \infty} \frac{-\log p_n}{n \log n} \leq N^{-1}$.

Corollary 3.3. If $(a_n)_0^\infty$ is a sequence of positive numbers having a positive radius of convergence, and satisfying

$$(n+1)a_{n+1} = \sum_0^n a_{n-k} b_k,$$

with $b_k > 0$ for all k , then

$$\lim_{n \rightarrow \infty} \frac{-\log a_n}{n \log n} = 0$$

Proof: For suitable constants A and B the sequence $p_n = A a_n B^n$ satisfies the conditions of Lemma 3.2 with $N = \infty$ (cf. Lemma 2.3)

4. Moments

From Corollaries 2.5 and 3.3 we obtain

Theorem 4.1. If F is an inf div cdf on $[0, \infty)$ (not concentrated at zero) with moments $\mu_n < \infty$, and if the sequence $a_n = \mu_n/n!$ has a positive radius of convergence, then

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{-\log a_n}{n \log n} = 0,$$

or equivalently

$$(4.2) \quad \lim_{n \rightarrow \infty} \frac{\log \mu_n}{n \log n} = 1.$$

Both the probabilities of compound geometric distributions on the non-negative integers and the quantities $\mu_n/n!$ for compound geometric distributions on the half line satisfy equations of the form

$a_{n+1} = \sum_0^n a_{n-k} b_k$. Using a theorem in [2] we then obtain

$$(4.3) \quad \lim_{n \rightarrow \infty} \frac{-\log a_n}{n} = \log \gamma.$$

where $\gamma = \sup \{x \mid \sum_{k=0}^{\infty} b_k x^k \leq 1\} \geq 1$. We omit the details.

More general and more precise results than (4.1) and (4.2) will be proved subsequently. To put these results in a clearer perspective, we conclude this section by showing that limits like (3.4) do not exist in general, not even in the case of infinite divisibility. It is well known that for any random variable (r.v) X the function $v(t) = E|X|^t$ is log-convex, and hence that $\lim (\log v(t)/t)$ exists as $t \rightarrow \infty$. We shall see that much more cannot be said in general.

Theorem 4.2. Let ℓ be a function defined on $(0, \infty)$, that is positive, increasing and such that $\ell(t) = O(\log t)$ and $\ell(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then there exists a random variable X with $v(t) = E|X|^t < \infty$ for all $t > 0$ and such that $\log v(t)/(t\ell(t))$ does not have a limit as $t \rightarrow \infty$.

proof: Under the conditions of Theorem 4.1, $\log v(t)/(t\ell(t)) = [\log v(t)/(t \log t)](\log t/\ell(t))$ will not tend to a limit if $\log t/\ell(t)$ does not have a limit (see(4.2.)). So, let $\log t/\ell(t) \rightarrow a > 0$ (possibly infinite). Now, let Y have an exponential distribution with $EY = 1$ and put $X = Yh(Y)$, where

$$(4.4) \quad h(x) = \begin{cases} 1 & t_{2n} < x < t_{2n+1} \\ 0 & \text{otherwise} \end{cases},$$

with $t_{n+1} = \ell^{-1}(t_n)$ for $n=1,2,\dots$ and t_1 such that $\ell(t) < t$ for $t \geq t_1$. Here ℓ^{-1} denotes the inverse function of ℓ . Now an easy estimation yields

$$v(t_{2n}) = \int_0^{\infty} x^{t_{2n}} h(x) e^{-x} dx \geq (t_{2n})^{t_{2n}} e^{-t_{2n}^{-1}},$$

and therefore $\limsup_{n \rightarrow \infty} \frac{\log v(t_{2n})}{t_{2n} \ell(t_{2n})} \geq \limsup_{n \rightarrow \infty} \frac{\log t_{2n}}{\ell(t_{2n})} = a > 0$.

On the other hand, using the fact that $x^\alpha e^{-x}$ is maximal at $x = \alpha$,

$$v(t_{2n+1}) = \sum_1^{\infty} \int_{t_{2k}}^{t_{2k+1}} x^{t_{2n+1}} e^{-x} dx \leq n(t_{2n+1})^{t_{2n+1}} e^{-t_{2n}} + \int_{t_{2n+2}}^{\infty} x^{t_{2n+1}} e^{-x} dx.$$

As the last integral tends to zero, it follows that (we have $\ell(t_{2n+1})=t_{2n}$)

$$\liminf_{n \rightarrow \infty} \frac{\log v(t_{2n+1})}{t_{2n+1}^{\ell(t_{2n+1})}} \leq \liminf_{n \rightarrow \infty} \frac{\log(t_{2n+1})}{t_{2n}} = 0$$

We now use the relations (2.6) to construct a similar counter example for the moments μ_n of an inf div cdf F . We do this for the special case that $\ell(t) = \log t$. First we define this cdf F by (2.4), where for K we take the cdf of a random variable X with moments v_n . As in the previous example take Y exponential with $EY = 1$ and define $X = Y^3 h(Y)$, where h is defined by (4.4) with $t_1=2$ and $t_{n+1}=t_n^3$. As in the previous example we easily see that

$$(4.5) \quad \limsup_{n \rightarrow \infty} \log v_n / (n \log n) \geq 3, \quad \liminf_{n \rightarrow \infty} \log v_n / (n \log n) \leq 1.$$

From (2.6) it follows by induction, using the fact that the v_n are increasing (we have $X \geq 8^3$) and that $v_k v_{n-k} \leq v_n$, that

$$(4.6) \quad v_n \leq \mu_{n+1} (n+1)! v_n.$$

From (4.5) and (4.6) we obtain $\limsup \log \mu_n / (n \log n) \geq 3$ and $\liminf \mu_n / (n \log n) \leq 2$.

From now on we shall be concerned with the absolute moments β_n of general cdf's, and we write $\alpha_n = \beta_n / n!$. We start with some simple observations. For notation we refer to section 1.

We shall be concerned with the asymptotic behaviour of the absolute moments of a random variable X , i.e. with $\beta_n = E|X|^n$ and with $\alpha_n = \beta_n / n!$. We start with some simple observations; for notation we refer to section 1.

Lemma 4.3. Let F be an inf div cdf with a mgf that is analytic in a neighbourhood of zero, then for any function k with $k(n) \rightarrow \infty$ as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \frac{\log \alpha_n}{nk(n)} = \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{nk(n)} \leq 0.$$

Proof: This follows at once from the definition of radius of convergence, and the relations (1.3).

Lemma 4.4. If F is inf div (and non-normal), then

$$\liminf_{n \rightarrow \infty} \frac{\log \alpha_n}{n \log n} \geq 0.$$

Proof: From Corollary 2.2 we obtain

$$\beta_n \geq A_n \int_n^{n+1} x^{n-1} e^{-\gamma x} \log x dx \geq A_n^n e^{-\gamma(n+1)\log(n+1)},$$

from which the result easily follows.

From the preceding two lemma's we obtain (compare Corollary 3.4)

Theorem 4.5. If β_n ($n=1,2,\dots$) are the absolute moments of an inf div cdf with a mgf that is analytic in a neighbourhood of zero, then

$$(4.7) \quad \lim_{n \rightarrow \infty} \frac{\log \alpha_n}{n \log n} = 0,$$

or equivalently

$$(4.8) \quad \lim_{n \rightarrow \infty} \frac{\log \beta_n}{n \log n} = 1$$

Remark: Theorem 4.5 may not hold in one of two ways: the limit in (4.8) may not exist or it may be different from one. Instances of the first way were given in the counter example of Theorem 4.2 and the one following that theorem; in the first case the mgf is analytic, but the cdf is not inf div, in the second case vice versa. Convergence of (4.8) to $\alpha > 0$ occurs for $\beta_n = E Y^{\alpha n}$, where Y is exponential; for $0 < \alpha < 1$ the cdf is not inf div, but the mgf is analytic, for $\alpha > 1$, it is the other way around. An instance where the limit in (4.8) is infinite is provided by the log-normal distribution, which was recently shown to be inf div [10].

The following theorem gives a more detailed result.

Theorem 4.6. Let F be a (non-normal), inf div cdf with spectral function M . Then M has a bounded spectrum if and only if F has absolute moments β_n of all orders and if, putting $\alpha_n = \beta_n/n!$,

$$(4.9) \quad \lim_{n \rightarrow \infty} \alpha_n^{1/n} \log n = c < \infty,$$

in which case $c = \max(a, b)$, where $[-a, b]$ is the support of M .

proof: First, let M have support on $[-a, b]$, with $\max(a, b) = c$. Then by Corollary 2.2, for every $\gamma > c^{-1}$ we have (cf. (1.1))

$$\begin{aligned} \beta_n &\geq A \int_0^\infty x^{n-1} e^{-\gamma x} \log x \, dx = A \gamma^{-n} \int_0^\infty x^{n-1} e^{-x} \log(x/\gamma) \, dx \geq \\ &A \gamma^{-n} \int_{n/\log n}^{1+n/\log n} x^{n-1} e^{-x} \log x/\gamma \, dx \geq A \gamma^{-n} (n/\log n)^{n-1} e^{-n}, \end{aligned}$$

from which, using Stirling's formula, it follows that $\liminf \alpha_n^{1/n} \log n \geq \gamma^{-1}$.

On the other hand by Corollary 2.2 we have

$$\begin{aligned} \beta_n &\leq B \delta^{-n} \int_0^\infty x^{n-1} e^{-x} \log(x/\delta) \, dx \leq n^2 B \delta^{-n} x_n^{n-1} e^{-x_n} \log(x_n/\delta) + \\ &+ B \delta^{-n} \int_{n^2}^\infty x^{n-1} e^{-x} \log(x/\delta) \, dx, \end{aligned}$$

where x_n is the point where the integrand is maximal. The last integral tends to zero, and it is easily verified that $x_n \sim n/\log n$ as $n \rightarrow \infty$.

It then follows that $\limsup \alpha_n^{1/n} \log n \leq \delta^{-1}$, and the first part of the theorem is proved. The converse is easily obtained by contradiction.

Corollary 4.7. If F is a non-normal cdf with a bounded Poisson Spectrum, then

$$(4.10) \quad \lim_{n \rightarrow \infty} \frac{\log \alpha_n}{n \log \log n} = -1.$$

Relation (4.10) implies (4.7).

5. Entire characteristic functions

Some of the results concerning moments of inf div distributions were suggested by properties of entire functions. In this section we show how theorems about the coefficients of entire functions can be used to generalize some of our previous results.

The following two theorems are well known (cf.[1], p.p. 9-12),

Theorem 5.1. The entire function $\psi(z) = \sum_{n=0}^{\infty} a_n z^n$ is of finite order iff

$$\rho = \limsup_{n \rightarrow \infty} \frac{n \log n}{-\log |a_n|}$$

is finite; the order of ψ is then equal to ρ .

Theorem 5.2. Let

$$\gamma = \limsup_{n \rightarrow \infty} n |a_n|^{\rho/n}$$

If $0 < \gamma < \infty$, then the function $\psi(z)$ is of finite order ρ and type τ iff $\gamma = e\tau\rho$. If $\gamma = 0$ or $\gamma = \infty$, then ψ is respectively of order and type $(\rho, 0)$ or of order and type not less than (ρ, ∞) , and conversely.

Similar theorems concerning the order and type of logarithms of entire functions can be obtained. Let $\psi(z)$ be an entire function without zeros, let $\chi(z) = \log \psi(z)$, and let $M(r, \psi)$ denote the maximum value of $|\psi(z)|$ for $|z| = r$. Let

$$\lambda_{\psi} = \limsup_{r \rightarrow \infty} \frac{\log \log \log M(r, \psi)}{\log r},$$

and let

$$\eta_{\psi} = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, \psi)}{r_{\psi}^{\lambda_{\psi}}}.$$

Let ρ_{ψ} and τ_{ψ} denote the order and type of ψ . Ruegg [6] has proved that $\rho_{\chi} = \rho$, where $\rho > 0$, if and only if $\rho_{\psi} = \infty$ and $\lambda_{\psi} = \rho$. In a similar manner it can be proved that if $\rho_{\chi} > 0$, then $\tau_{\psi} = \tau$ if and only if $\eta_{\psi} = \tau$. Once this is known, theorems 5.1 and 5.2 can be generalized to obtain the following theorems.

Theorem 5.3. Let $\psi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function without zeros and let $\chi(z) = \log \psi(z)$. Then $\chi(z)$ is an entire functions of finite order iff

$$\rho' = \limsup_{n \rightarrow \infty} \frac{n \log \log n}{-\log |a_n|}$$

is finite; the order of χ is then ρ' .

Theorem 5.4. Let

$$\gamma' = \limsup_{n \rightarrow \infty} (\log n) |a_n|^{\rho'}$$

If $0 < \gamma' < \infty$, then the function χ is of order ρ' and type τ' iff $\tau' = \gamma'$. If $\gamma' = 0$ or $\gamma' = \infty$, then χ is respectively of order and type $(\rho', 0)$ or of order and type not less than (ρ', ∞) , and conversely.

The proofs of Theorems 5.1 and 5.2 depend on the fact that $(K/x)^x$ is maximal for $x = K/e$. The proofs of Theorem 5.3 and 5.4 depend on the fact that $(K/\log x)^x$ is, for large K , maximal for a value of x close to e^K . The proofs of Theorem 5.3 and 5.4 are then similar to those of Theorems 5.1 and 5.2. Since they are quite long, they are omitted.

It is known (cf [4], p.202) that a cdf has bounded support if and only if its cf is entire with order one and intermediate type. We now prove a related theorem for inf div cdf's. This is proved here because it is of some independent interest, even though it is not directly related to other theorems in this paper.

Theorem 5.5. A (non-normal) inf div distribution with characteristic function φ has a bounded Poisson spectrum iff $\log \varphi$ is an entire function of order 1 and intermediate type.

proof: First (cf.(1.4)) let M have support on $[-a, b]$ with $c = \max(a, b) > 0$. We can write $\log \varphi = \log \varphi_1 + \log \varphi_2$, where

$$\log \varphi_1(z) = icz + \int_{-\epsilon}^{\epsilon} (e^{iuz} - 1 - iuz) u^{-2} dM(u)$$

and $\log \varphi_2(z) = \lambda \int_{-a}^b (e^{iuz} - 1) dG(u)$ where G is a cdf. Now, $\log \varphi_2$ is easily seen to be of order 1 and type c (cf. [5], p.1250), where as for φ_1 we have

$$|\log \varphi_1(z)| \leq |cz| + \int_{-\epsilon}^{\epsilon} |z|^2 e^{|uz|} dM(u) \leq |cz| + |z|^2 e^{\epsilon|z|} \int_{\epsilon}^{\epsilon} dM(u).$$

It follows that $\log \varphi_1$ has order and type at most 1 and ϵ , and hence that $\log \varphi$ has order 1 and type c . Conversely, if $\log \varphi$ is entire of order 1 and type c , then by the same argument the support of M must be confined to $[-c, c]$.

Ostrovskii has proved (cf. [4], p.224) that an entire characteristic function without zeros must have a logarithm of at least order one and intermediate type. This theorem can be combined with Theorem 5.3 to yield the following theorem which generalizes Theorem 4.5 and Corollary 4.7. for notation we refer to section 1.

Theorem 5.6: Let φ be a characteristic function without zeros. Then its coefficients a_n satisfy

$$(5.1) \quad \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{n \log n} \geq 0,$$

$$(5.2) \quad \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{n \log \log n} \geq -1;$$

if φ is analytic in a neighbourhood of zero, then

$$(5.3) \quad \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{n \log n} = 0$$

Proof: If φ is not entire, then (5.1) and (5.2) follow from the fact that the radius of convergence of a_n is finite, i.e. that $\limsup |a_n|^{1/n} > 0$.

If φ is entire, the only case where the condition that $\varphi \neq 0$ is needed, then (5.2) and hence (5.1) follow from Theorem 5.3 combined with Ostrovskii's result. Finally, (5.3) follows from (5.1) combined with the observation that now $\limsup |a_n|^{1/n} < \infty$.

It should be noted that these results are weaker than those in the previous section. It would be of interest to know if characteristic functions without zeros exist having the property that $\liminf (\log \alpha_n / (n \log n)) < 0$ or $\liminf (\log \alpha_n) / (n \log \log n) < -1$. Such cf's would not be inf div; it follows easily from Theorem 4.6 that for an inf div cf $\lim \alpha_n^{1/n} \log n$ exists. Similarly, it would be of interest to know whether cf's without zeros exist such that $\liminf \alpha_n^{1/n} \log n < \limsup \alpha_n^{1/n} \log n$.

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