(A,B)-invariant subspaces and stabilizability spaces : some properties and applications

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1. *Introduction*

In the present paper a review is given of the important system theoretic concept of (A,B)-invariant subspace. The concept was introduced (with the name controlled invariant subspace) by Basile and Marro in 1969 [BM]. In 1970 this concept was rediscovered by Wonham and Morse [WM1] The concept turned out to be of fundamental importance for numerous applications and for many theoretic investigations. It was the basis of the geometric approach to linear multivariable systems propagated by Wonham and Morse [WM,Wn]. Since there is another important development in linear system theory, the polynomial matrix approach (see e.g. [Ro],[Wo],[WD]) it is useful to obtain polynomial representations, or frequency domain characterizations of (A,B)-invariant subspaces in order to bridge the two diverging branches. Results of this type were obtained in [EH], [FW] [Ha4,5] and some of them will be mentioned here. In addition some properties and applications of stabilizability subspaces, introduced in [Ha5], are discussed. Also the relation between strong observability and strong detectability introduced in [PS],[Mo] (for discrete time) and (A,B)-invariant subspaces is indicated.

2. *A-invariance*

Consider the time invariant linear differential equation

(2.1) $\quad \dot{x}(t) = Ax(t)$

in $X := \mathbb{R}^n$. A subspace $V \subseteq X$ is called <u>A-invariant</u> if for each initial value $x_0 \in V$ we have $x(t) \in V$ for $t \geq 0$.

If $v_1,\ldots,v_k$ is a basis of V, the matrix $V := [v_1,\ldots,v_k]$ is called a <u>basis matrix</u> of V. Obviously, $x \in V$ iff there exists $p \in \mathbb{R}^k$ such that $x = Vp$.

The following result can easily be proved.

(2.2) THEOREM. *Given equation* (2.1) *and a subspace* $V \subseteq X$, *the following statements are equivalent*

(i) V *is* A-*invariant*

(ii) $AV \subseteq V$, *i.e.*, $Ax \in V$ *for all* $x \in V$

(iii) *If* V *is a basis matrix of* $\mathcal{V}$, *then the matrix equation*

$$AV = VP$$

has a solution P. □

3. (A,B)-*invariance*

Here we consider the controlled system

$$(3.1) \qquad \mathring{x}(t) = Ax(t) + Bu(t) \qquad (t \geq 0)$$

with $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, where $\mathcal{X} := \mathbb{R}^n$, $\mathcal{U} := \mathbb{R}^m$. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is called <u>weakly invariant</u> if for each $x_0 \in \mathcal{V}$, there exists $u \in \Omega$ such that $\xi_u(t,x_0) \in \mathcal{V}$ for all $t \geq 0$. Here Ω denotes the set of piecewise continuous functions $u : \mathbb{R}_+ \to \mathcal{U}$ and $\xi_u(t,x_0)$ denotes the solution of (3.1) with initial value x_0 and control u.

For a given $x \in \mathcal{X}$, the formula

$$x = (sI - A)\xi(s) - B\omega(s)$$

will be called a <u>(ξ,ω)-representation</u> if $\xi(s)$ and $\omega(s)$ are strictly proper rational functions.

Then we have the following result.

(3.2) THEOREM. *Given system* (3.1) *and* $\mathcal{V} \subseteq \mathcal{X}$, *the following statements are equivalent*

(i) (<u>Open loop characterization</u>) $\mathcal{V}$ *is weakly invariant.*

(ii) (<u>Geometric characterization</u>) $\mathcal{V}$ *is* (A,B)-*invariant* i.e., $A\mathcal{V} \subseteq \mathcal{V} + B\mathcal{U}$.

(iii) (<u>Matrix characterization</u>) *If* V *is a basis matrix of* $\mathcal{V}$, *then there exist matrices* P *and* Q *such that* $AV = VP + BQ$.

(iv) (<u>Feedback characterization</u>) *There exists* $F : \mathcal{X} \to \mathcal{U}$ *such that* $\mathcal{V}$ *is* $(A + BF)$-*invariant, i.e.,* $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

(v) (<u>Frequency domain characterization</u>) *Every* $x \in \mathcal{V}$ *has a* (ξ,ω)-*representation with* $\xi(s) \in \mathcal{V}$.

PROOF: (i) ⇒ (ii) If $x(t) \in \mathcal{V}$ for $t \geq 0$ then $\mathring{x}(0+) = \lim_{t \downarrow 0} t^{-1}(x(t) - x(0)) \in \mathcal{V}$. Hence $Ax_0 = \mathring{x}(0+) - Bu(0+) \in \mathcal{V} + B\mathcal{U}$.

(ii) $\Rightarrow$ (iii) similar as in theorem 2.1.

(iii) $\Rightarrow$ (iv) V is left invertible, say $V^+ V = I$. Take $F := -QV^+$. Then $(A + BF)V = VP$.

(iv) $\Rightarrow$ (v) Choose $\xi(s) := (sI - A - BF)^{-1}x_0$, $\omega(s) := F\xi(s)$.

(v) $\Rightarrow$ (i) Let $\tilde{\xi}(t)$, $\tilde{\omega}(t)$ be the time domain functions (inverse Laplace transforms) of $\xi(s)$, $\omega(s)$. Then $\overset{\circ}{\tilde{\xi}}(t) = A\tilde{\xi}(t) + B\tilde{\omega}(t)$, $\tilde{\xi}(0) = x_0$ and $\tilde{\xi}(t) \in V$ for all $t \geq 0$. □

Because of the equivalence of (i) and (iv), we may say that, if for every $x_0 \in V$ there exists a control u such that the trajectory stays in the space, then there exists a feedback law $u = Fx$, such that the state stays in V for all initial values.

4. *The largest weakly invariant subspace contained in* ker C

In this section we consider the controlled system with output equation:

(4.1) $\quad \overset{\circ}{x} = Ax + Bu, \; y = Cx.$

We are interested in a weakly invariant subspace contained in ker C which is as large as possible. We denote the system (4.1) briefly by (C,A,B) or by Σ.

(4.2) DEFINITION. *Given* $\Sigma = (C,A,B)$, *then* V_Σ *denotes the space of points* $x_0 \in X$ *such that there exists* $u \in \Omega$ *for which* $y_u(t,x_0) := C\xi_u(t,x_0) = 0$ *for all* $t \geq 0$.

The following result follows easily from the definition (for a proof see [Ha5]).

(4.3) THEOREM. V_Σ *is the largest* (A,B)*-invariant subspace contained in* ker C *that is,* V_Σ *is* (A,B)*-invariant,* $V_\Sigma \subseteq$ ker C *and for every* (A,B)*-invariant subspace* $V \subseteq$ ker C *we have* $V \subseteq V_\Sigma$. □

The following result, which is a direct consequence of theorem (3.1), gives some further properties of V_Σ:

(4.4) THEOREM. *Given* Σ, *and* $x \in X$ *the following statements are equivalent:*

(i) $x \in V_\Sigma$

(ii) x *has a* (ξ,ω)*-representation satisfying* $C\xi(s) = 0$

(iii) *These exists a strictly proper rational function* $\omega(s)$ *such that*

$$R(s)\omega(s) = -C(sI - A)^{-1}x_0,$$

where $R(s) := C(sI - A)^{-1}B$ *is the transfer matrix of* Σ.

(iv) *There exists a strictly proper rational solution* (ξ,ω) *of*

$$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \omega(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

□

5. *Applications of weakly invariant subspaces*

A. Strong observability. A system Σ is called strongly observable if for any two points $x_1, x_2 \in X$ we have: If there exist $u_1, u_2 \in \Omega$ such that $y_{u_1}(t,x_1) = y_{u_2}(t,x_2)$ for $t \geq 0$, then $x_1 = x_2$. In a strongly observable system the initial state is uniquely determined from the output alone. The concept was introduced for example in [PS], where it was termed perfect observabi

By linearity, the following is easily seen: Σ is strongly observable iff $y_u(t,x_0) = 0$ for all $t \geq 0$ implies $x_0 = 0$. Thus we have the equivalence of i) and ii) in

(5.1) THEOREM. *Let* rank $B = m$, *i.e., let* B *be injective. Then the following statements are equivalent:*

i) Σ *is strongly observable,*

ii) $V_\Sigma = 0$,

iii) $(C, A + BF)$ *is observable for every* F,

iv) $\operatorname{rank} \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = n + m$, *for every* $s \in \mathbb{C}$.

PROOF. i) $\Leftrightarrow$ ii) have been observed before.

ii) $\Rightarrow$ iv). Suppose that for some $s_0 \in \mathbb{C}$ we have

$$\operatorname{rank} \begin{bmatrix} s_0 I - A & -B \\ C & 0 \end{bmatrix} < n + m.$$

Then there exist vectors $x_0 \in \mathcal{X}$, $u_0 \in \mathcal{U}$ not both zero, such that

$$\begin{bmatrix} s_0 I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

If we define $\xi(s) := (s - s_0)^{-1} x_0$, $\omega(s) := (s - s_0)^{-1} u_0$, then a straightforward calculation shows that

$$x_0 = (sI - A)\xi(s) - B\omega(s), \; C\xi(s) = 0,$$

hence that $x_0 \in \mathcal{V}_\Sigma$. Since $\mathcal{V}_\Sigma = 0$ this implies $x_0 = 0$. Consequently $Bu = (s_0 I - A)x_0 = 0$. Because of rank $B = m$ we conclude that $u_0 = 0$ which is a contradiction to the assumption that x_0 and u_0 were not both zero.

iv) $\Rightarrow$ iii) A system is observable iff

$$\operatorname{rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n \qquad (s \in \mathbb{C})$$

(see [Ha1]). If iv) is satisfied then we have for every F,

$$\operatorname{rank} \begin{bmatrix} sI - A - BF & -B \\ C & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} = n + m$$

Hence $\operatorname{rank} \begin{bmatrix} sI - A - BF \\ C \end{bmatrix} = n \qquad (s \in \mathbb{C})$

so that $(C, A + BF)$ is observable.

iii) $\Rightarrow$ i) Suppose that $\mathcal{V}_\Sigma \neq 0$ and choose F such that $(A + BF)\mathcal{V}_\Sigma \subseteq \mathcal{V}_\Sigma$. Then $\mathcal{V}_\Sigma$ is an $(A + BF)$-invariant subspace contained in ker C, which contradicts the observability of $(C, A + BF)$ $\square$

REMARK. If rank B $\neq$ m then condition iv) can never be satisfied. However, it is easily seen that the theorem remains valid if we replace condition iv) by

iv)': $$\operatorname{rank}\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = n + \operatorname{rank} B$$

for all $s \in \mathbb{C}$. □

The property of strong observability is important in the following situation: Suppose we have a system with two types of input, a control u and a disturbance q which is completely unknown. Then the following question arises: Is it possible to determine the state uniquely from u and y? It is easily seen that this is the case iff $\Sigma_1 := (C,A,E)$ is strongly observable, where we assume that the system equations are

$$\dot{x} = Ax + Bu + Eq$$

$$y = Cx.$$

The foregoing theorem gives necessary and sufficient conditions for this to be the case.

B <u>Output null controllability</u>. We denote $\langle A|B\rangle$ the set of null controllable points (or equivalently, the set of reachable points) in X. A point $x_0 \in X$ is called <u>output null controllable</u> if there exists $u \in \Omega$ such that for some $T > 0$ we have

$$y_u(t,x_0) = 0 \qquad \text{for } t \geq T$$

Let S denote the space of output null controllable points. The following results are shown in [Ha4]:

(5.2) THEOREM. (i) $S = \langle A|B\rangle + V_\Sigma$,
(ii) $x \in S$ *iff there exist rational functions* ξ,ω *such that* $x = (sI - A)\xi - B\omega$ *and* $C\xi$ *is a polynomial.* □

Σ is called output null controllable if $S = X$, i.e., if $\langle A|B\rangle + V_\Sigma = X$.

(5.3) COROLLARY. Σ *is output null controllable iff there exist rational matrices* P *and* Q *satisfying*

$$(sI - A)P(s) - BQ(s) = I,$$

$CP(s)$ *is a polynomial.* □

C <u>Left invertibility</u>. Σ is called left invertible if

$$y_{u_1}(.,0) = y_{u_2}(.,0) \Rightarrow u_1 = u_2.$$

By linearity, an equivalent condition is :

$$y_u(.,0) = 0 \Rightarrow u = 0.$$

(5.4) THEOREM. (Compare [Wn.Ex.4.1] and [SP]). *The following statements are equivalent*

(i) Σ *is left invertible*

(ii) rank $B = m$ *and* $V_\Sigma \cap B\mathcal{U} = 0$

(iii) rank $\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = n + m$

for almost all $s \in \mathbb{C}$.

PROOF. (i) $\Rightarrow$ (iii) If the condition of (iii) is not satisfied then there exist rational functions ξ and ω not both zero, which may be supposed to be strictly proper, such that

$$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \omega(s) \end{bmatrix} = 0$$

Let $\tilde{\xi}(t)$ and $\tilde{\omega}(t)$ be the inverse laplace transform of these rational functions. Then

$$(*) \qquad \mathring{\tilde{\xi}}(t) = A\tilde{\xi}(t) + B\tilde{\omega}(t), \ \tilde{\xi}(0) = 0$$

$$C\tilde{\xi}(t) = 0.$$

By invertibility we must have $\omega(t) = 0$. But then $\xi(t)$ must be zero because of (*) contradicting our assumption.

iii) $\Rightarrow$ ii) If iii) holds, then obviously rank $B = m$. Let $x_0 \in V_\Sigma \cap B\mathcal{U}$. Then there exist $u_0 \in \mathcal{U}$ and ξ, ω, strictly proper such that

$$x_0 = Bu_0 = (sI - A)\xi - B\omega$$

Hence $$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} \xi \\ u_0 + \omega \end{bmatrix} = 0$$

Hence, by iii), $\xi = 0$, $u_0 + \omega = 0$. Since ω is strictly proper this implies $u_0 = 0$ and therefore $x_0 = 0$.

ii) $\Rightarrow$ i) Choose F such that $(A + BF)\mathcal{V}_\Sigma \subseteq \mathcal{V}_\Sigma$. Suppose that $y_u(t,0) = 0$ for some $u \in \Omega$. The function $v(t) := u(t) - F\xi_u(t,0)$, satisfies

$$Bv(t) = \mathring{\xi}_u(t,0) - (A + BF)\xi_u(t,0) \in \mathcal{V}_\Sigma \qquad (t \geq 0)$$

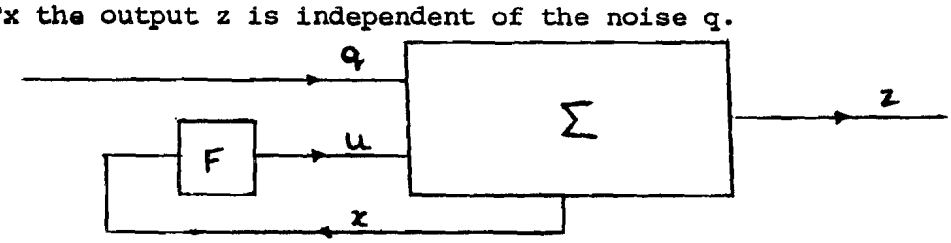
since $\mathring{\xi}_u(t,0) \in \mathcal{V}_\Sigma$ and $\mathcal{V}_\Sigma$ is $(A + BF)$-invariant. But obviously $Bv(t) \in B\mathcal{U}$ and hence $Bv(t) = 0$, so that $\mathring{\xi}_u(t,0) = (A + BF)\xi_u(t,0)$. Because of $\xi_u(0,0) = 0$, this implies $_u(t,0) = 0$. In addition, because of rank $B = m$, and $Bu(t) = \mathring{\xi}_u(t,0) - A\xi_u(t,0) = 0$ we have $u(t) = 0$. □

It follows from theorem 5.4 that the (ξ,ω)-representation satisfying $C\xi = 0$ of each element $x_0 \in \mathcal{V}_\Sigma$ is unique iff Σ is left invertible.

D. <u>Disturbance decoupling</u>. We consider the system

$$\mathring{x} = Ax + Bu + Eq, \qquad z = Dx$$

and we ask whether it is possible to find $F : \mathcal{X} \to \mathcal{U}$ such that with the feedback $u = Fx$ the output z is independent of the noise q.



This is the so-called <u>disturbance decoupling problem</u> (DDP), see [WM]. The following is proved in [Ha5]:

(5.5) THEOREM. *The following statements are equivalent*

(i) DDP *has a solution* F.

(ii) $E\mathcal{Q} \subseteq \mathcal{V}_\Sigma$; *here* $\mathcal{Q}$ *is the space in which the disturbance* q *takes its values and* $\Sigma := (D,A,B)$.

(iii) *There exists a strictly proper rational matrix* $Q(s)$ *such that*

$$R_2(s) = R_1(s)Q(s),$$

where $R_1(s) := D(sI - A)^{-1}B$ *is the control to output transfer matrix and* $R_2(s) := D(sI - A)^{-1}$ *is the noise to output transfer matrix.*

(iv) *There exists strictly proper* $X(s), U(s)$ *such that*

$$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix} \qquad \square$$

The equivalence (i) $\Leftrightarrow$ (ii) is proved in [WM]. See also [Wn, section 4.1].

6. *Stabilizability subspaces*

We consider stability from a general point of view, i.e., we assume that we are given a set $\mathbb{C}^- \subseteq \mathbb{C}$ such that $\mathbb{C}^- \cap \mathbb{R} \neq \emptyset$ and we denote by $\mathbb{C}^+$ the complement of $\mathbb{C}^-$ in $\mathbb{C}$. A rational function will be called <u>stable</u> if it has no poles in $\mathbb{C}^+$. A (ξ,ω)-representation will be called <u>stable</u> if ξ and ω are stable rational functions. We consider again the system given by (3.1). A subspace V of X will be called a <u>stabilizability subspace</u> if there exists $F : X \to U$ such that $(A + BF)V \subseteq V$ and $\sigma(A + BF)|_V \subseteq \mathbb{C}^-$. Obviously a stabilizability subspace is weakly invariant. We have the following frequency domain characterization:

(6.1) THEOREM. *V is a stabilizability subspace iff each point in V has a stable (ξ,ω) representation such that $\xi(s) \in V$.*
For a proof see [Ha5].

Now we assume that we are given a system $\Sigma = (C,A,B)$ and we introduce the stabilizability analogue of V_Σ.

(6.2) DEFINITION. *V_Σ^- denotes the set of points for which there exists a stable (ξ,ω)-representation satisfying* $C\xi(s) = 0$.

(6.3) THEOREM. *V_Σ^- is the largest stabilizability subspace contained in* ker C.

Obviously, $V_\Sigma^- \subseteq V_\Sigma$. We have the following properties: (see [Ha4,5]):

(6.4) THEOREM (i) $x \in V_\Sigma^-$ *iff there exists strictly proper stable ξ,ω such that*

$$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \omega \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

(ii) *If the system is detectable, (i.e.,* $\operatorname{rank}\begin{bmatrix} sI - A \\ C \end{bmatrix} = n$ *for* $s \in \mathbb{C}^+$*), then* $x \in V_\Sigma$ *iff there exists a strictly proper stable* ω *such that*

$$R(s)\,\omega(s) = C(sI - A)^{-1}x_0$$

where

$$R(s) := C(sI - A)^{-1}B.$$

The latter result is no longer true if the detectability condition is omitted (for an example, see [Ha5]).

7. *Applications of* V_Σ^-

A. <u>Strong detectability</u>. A system Σ is called strongly detectable if for any $u_1, u_2 \in \Omega$ and for any $x_1, x_2 \in X$ we have: $y_{u_1}(t,x_1) = y_{u_2}(t,x_2)$ $(t \geq 0)$ implies $\xi_{u_1}(t,x_1) - \xi_{u_2}(t,x_2) \to 0$ $(t \to \infty)$.

In the case of a strongly detectable system, it is possible to get based on the output alone an estimate of the state the error of which tends to zero as $t \to \infty$. By linearity we may say that Σ is strongly detectable iff $y(t) = 0$ $(t \to 0)$ implies $x(t) \to 0$ $(t \to \infty)$, for each input u and initial value x_0.

(7.1) THEOREM. *If* rank $B = m$, *the following statements are equivalent*

(i) Σ *is strongly detectable*

(ii) $\operatorname{rank}\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = n + m \qquad (\operatorname{Re} s \geq 0)$

(iii) $(C, A + BF)$ *is detectable for all* F *with respect to the stability set* $\mathbb{C}^- := \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$

(iv) Σ *is left invertible and* $V_\Sigma^- = V_\Sigma$

PROOF. The equivalence (i) $\Leftrightarrow$ (ii) is proved in [BH]. The proof of (ii) $\Rightarrow$ (iii) is completely analogous to the proof of iv) $\Rightarrow$ iii) in Theorem 5.1.

(iii) $\Rightarrow$ (ii) Let $\operatorname{rank}\begin{bmatrix} s_0I - A & -B \\ C & 0 \end{bmatrix} < n + m$ for some $s_0 \in \mathbb{C}$. Then there

exist $x_0 \in X$, $u_0 \in U$ not both zero such that $(s_0 I - A)x_0 = Bu_0$, $Cx_0 = 0$. Because of rank $B = m$ this implies $x_0 \neq 0$. Let F be an arbitrary map $X \to U$ satisfying $Fx_0 = u_0$. Then we have $(sI_0 - A - BF)x_0 = 0$, $Cx_0 = 0$. Because of the detectability of $(C, A+BF)$ we must have $s_0 \in \mathbb{C}^-$.

(ii) $\Rightarrow$ (iv): (ii) obviously implies that Σ is left invertible. Let $x \in V_\Sigma$. Then there exist strictly proper (ξ,ω) such that

$$(*) \qquad \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \omega \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

We show that (ξ,ω) is stable. Let s_0 be a pole of (ξ,ω). Then there exists $k \in \mathbb{N}$ such that $(s - s_0)^k(\xi,\omega) = (x_0,u_0) + (s - s_0)(\tilde{\xi},\tilde{\omega})$ where $(x_0,u_0) \neq 0$ and $(\tilde{\xi},\tilde{\omega})$ are rational functions with no pole at $s = s_0$. Substituting this into $(*)$ yields:

$$(s - s_0)^k \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \left(\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} + (s - s_0) \begin{bmatrix} \tilde{\xi} \\ \tilde{\omega} \end{bmatrix} \right)$$

For $s = s_0$ we obtain $\begin{bmatrix} s_0 I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$

Because of (ii) we must have $s_0 \in \mathbb{C}^-$.

Hence $x_0 \in V_\Sigma^-$.

(iv) $\Rightarrow$ (ii) Suppose that for some $s \in \mathbb{C}$ there exist $(x_0,u_0) \neq 0$ such that $(s_0 I - A)x_0 = Bu_0$, $Cx_0 = 0$. Define $\xi = (s - s_0)^{-1}x_0$, $\omega := (s - s_0)^{-1}u_0$. Then $(sI - A)\xi - B\omega$ is a (ξ,ω)-representation of x_0 satisfying $C\xi(s) = 0$. Since Σ is invertible, such a representation is unique. Also $x_0 \in V_\Sigma = V_\Sigma^-$. Hence (ξ,ω) must be stable and consequently $s_0 \in \mathbb{C}^-$. □

Ordinary detectability is well kwown to be a necessary and sufficient condition for the existence of an observer (see, e.g. [Ha2]). Accordingly, one expects that strong detectability is necessary and sufficient for the existence of an observer whose input is only the output and not the input of the original system. This is not the case, however, as follows from the following example.

(7.2) EXAMPLE. Let $n = 2$, $m = r = 1$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1,0]$. Then

$$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

has full rank for every $s \in \mathbb{C}$. Hence Σ is strongly observable and in particular strongly detectable. The system equations are

$$\mathring{x}_1 = x_2, \quad y = x_1$$
$$\mathring{x}_2 = u$$

An observer with y as input and $(\hat{x}_1, \hat{x}_2)$ as output would give an asymptotically improving estimate of $x_2 = \mathring{y}$ based on knowledge of y. That is, the observer would contain a differentiating element: this is intuitively impossible, and an exact proof of this can easily be provided (see [BH]). □

The condition needed for the existence of a "strong observer" is considerably stronger than strong observability, viz: $y \to 0$ $(t \to \infty)$ implies $x(t) \to 0$ $(t \to \infty)$. See [BH] for further details. What can be constructed for strongly detectable system is an integrating strong observer.

(7.3) DEFINITION. *A system Σ_1 is called an* <u>*integrating strong observer*</u> *of Σ if Σ_1, fed with the output of Σ, yields an output $\hat{x}$ such that for some polynomial p, with zeroes in $\mathbb{C}^-$ only, we have*

$$p(D)\hat{x}(t) - x(t) \to 0 \qquad (t \to \infty)$$

Here D denotes the differentiation operator, $D = d/dt$.

This concept is related to the concept of integrating inverse, as introduced by Sain and Massey ([SM]. Also see [Ha3]). It is shown in [SM] that a system has an integrating left inverse if and only if the system is left invertible. Here an <u>integrating left inverse</u> is a system Σ_1 with transfer matrix $R_1(s)$ such that $R_1(s)R(s) = s^{-k}I$ for some k, where $R(s)$ denotes the transfer matrix of Σ_1. It turns out that strong detectability is the condition needed for the existence of a *stable* integrating left inverse; i.e. of a stable transfer matrix R_1 satisfying $R_1(s)R(s) = p(s)^{-1}I$ for some polynomial p with zeroes in $\mathbb{C}^-$ only.

(7.4) THEOREM. *Let* rank B = m. *The following statements are equivalent:*

(i) Σ *is strongly detectable*

(ii) *There exists a strong integrating observer of* Σ

(iii) *There exists a stable integrating left inverse of* Σ

A proof is given in [BH].

B. Disturbance decoupling with internal stability. The problem considered is the same as DDP (section 5.D.) with the additional requirement that $\sigma(A + BF) \subseteq \mathbb{C}^-$ The following result is proved in [Ha5].

(7.5) THEOREM. *For the* DDP *with internal stability to have a solution it is necessary that* Σ *be stabilizable. If* Σ *is stabilizable, the following statements are equivalent:*

(i) *The* DDP *with internal stability has a solution*

(ii) $E\mathcal{Q} \subseteq V_\Sigma^-$

(iii) *There exist stable rational strictly proper matrix functions* X(s), U(s) *such that*

$$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

(iv) *(If* (D,A) *is detectable): There exists a stable strictly proper matrix* Q(s) *such that*

$$R_2(s) = R_1(s)Q(s)$$

where R_1, R_2 *are defined as in theorem* 5.5.

8. *Output stabilization*

(8.1) DEFINITION. R_Σ *denotes the set of points* $x \in X$ *for which there exists a* (ξ,ω)*-representation such that* $C\xi(s)$ *is stable.*

R_Σ is a strongly invariant subspace, i.e. for all $u \in \Omega$, $x_0 \in R_\Sigma$ we have $\xi_u(t,x_0) \in R_\Sigma$.

In particular

$$R_\Sigma = \langle A|B\rangle + V_\Sigma + X^-(A) = S_\Sigma + X^-(A)$$

where $X^-(A)$ denotes the space corresponding to the unstable eigenvalues of A (see [Ha5]). This subspace can be used for the solution of some problems connected with output stabilization. We consider the problem of constructing $F : X \to U$ such that with $u = Fx$, the output will be stable with arbitrary initial state and zero input. Hence F has to be determined such that $C(sI - A - BF)^{-1}$ is stable. This problem is called the <u>output stabilization problem</u> (see [Wn. section 4.4]). The following result is proved in [Ha5]:

(8.2) THEOREM. *The following statements are equivalent:*

(i) *The output stabilization problem has a solution*

(ii) $R_\Sigma = X$

(iii) $V_\Sigma + \langle A|B\rangle \supseteq X^+(A)$

(iv) *There exist rational functions* $X(s)$, $U(s)$ *such that* $(sI - A)X + BU = I$ *and* $CX(s)$ *is stable.*

The equivalence of (i) and (iii) has been shown in [Wn]. A somewhat more general problem is the <u>disturbance stabilization problem.</u> We start from the same system as in DDP, but now we want to determine $F : X \to U$ such that with $u = Fx$ the i/o map $q \mapsto y$ is stable, i.e. such that $C(sI - A - BF)^{-1}E$ is stable.

We have the following result

(8.3) THEOREM. *The following statements are equivalent:*

(i) *The disturbance stabilization problem has a solution*

(ii) $EQ \subseteq R_\Sigma$

(iii) *There exist rational matrices* X *and* U *such that* $(sI - A)X + BU = E$ *and* CX *is stable*

For a proof, see [Ha5].

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