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ANALYSIS OF OIL TRAPPING IN POROUS MEDIA FLOW*

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Abstract. We analyze a one-dimensional nonlinear convection-diffusion equation describing the flow of water and oil through a porous medium composed of two types of rock with different permeability. We prove existence, uniqueness, and regularity properties, as well as matching conditions between the two rock types.

Key words. degenerate parabolic equation, porous media flow, existence, uniqueness, qualitative properties, matching conditions

AMS subject classifications. Primary, 35K65; Secondary, 35B05, 35K10, 35K55, 76S05

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1. Introduction and problem formulation. It is well known that capillary forces, combined with spatial variations of rock properties, considerably reduce the recovery factor of an oil reservoir. For instance, it is difficult to remove oil from parts of the reservoir with small scale heterogeneities. Sometimes the oil may even remain trapped; see, for instance, [K, W]. This is clearly a difficult problem, mainly due to the complex nature of rock (soil) heterogeneities.

To understand oil trapping in heterogeneous media more quantitatively, [DMN] considered the case of a 2-phase water-oil flow which is perpendicular to an interface, separating two types of rock, across which the permeability changes abruptly. Under simplifying assumptions this leads to a one-dimensional flow problem which allowed them to investigate the role of convection and capillary diffusion in relation to the discontinuous permeability. They used formal asymptotics and numerical techniques. In this paper we will take their formulation as a starting point. The aim is to analyze the structure of the model equations resulting in existence, uniqueness, and regularity properties, as well as matching conditions between the two rock types.

Following [DMN] (further references are given there), the one-dimensional flow of water and oil through a porous medium is described by a nonlinear convection-diffusion equation for the reduced water saturation $S = S(x,t)$, with $0 \le S \le 1$. This equation has the form

$$\Phi\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left\{ q f_w(S) + k(x)H(s)\frac{\partial p}{\partial x}\right\} = 0, \tag{1.1}$$

where Φ (porosity) and q (discharge) are positive constants, and where the functions $f_w,\ H : [0,1] \to [0,\infty)$ satisfy $f_w(0) = 0,\ f_w(S) > 0$ for $0 < S \le 1$ (typically convex-concave behavior) and $H(0) = H(1) = 0,\ H(S) > 0$ for $0 < S < 1$. Further $k(x)$ denotes permeability and p capillary pressure. Situating the discontinuity in

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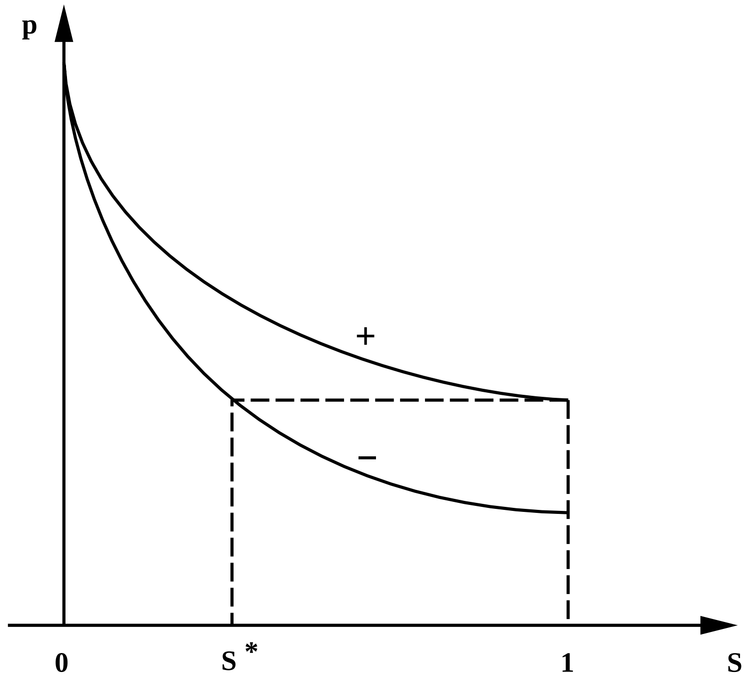

FIG. 1. *Capillary pressure curves for fine* (+) *and coarse* (−) *material. Here* $J(1) > 0$*, so an entry pressure exists.*

permeability at $x = 0$, we have

$$k(x) = \begin{cases} k^- & \text{for} \quad x < 0, \\ k^+ & \text{for} \quad x > 0. \end{cases} \tag{1.2}$$

Without loss of generality we take $0 < k^+ < k^- < \infty$. This means that coarse material occupies $\{x < 0\}$ and fine material $\{x > 0\}$. The flow is in positive x-direction.

For the capillary pressure the Leverett model [L] was used. With $\sigma > 0$ denoting interfacial tension, this means

$$p = p(x, S) = \sigma \frac{J(S)}{\sqrt{k(x)/\Phi}} \qquad \text{for} \quad 0 < S \le 1, \tag{1.3}$$

where the Leverett function J is strictly decreasing in $(0, 1]$ with $J(1) \ge 0$. The quantity $\sqrt{k/\Phi}$ may be associated with the mean pore diameter, and the J-Leverett function is typical for the lithology of the porous medium. When $J(1) > 0$, the medium has an entry pressure given by $J(1)/\sqrt{k/\phi}$. This is the minimum pressure needed for the oil to enter a medium that is saturated by water. In this paper we assume $J(1) > 0$ and show that the occurrence of an entry pressure causes trapping of oil at the interface when the medium changes from coarse to fine. Figure 1 shows two typical capillary pressure functions, the top curve for fine material ($x > 0$), the bottom curve for coarse material ($x < 0$).

Because k is discontinuous, the capillary pressure may be discontinuous as well. This makes the interpretation of (1.1) across $x = 0$ difficult. To circumvent this problem, [DMN] considered (1.1) for $x < 0$ and $x > 0$, with matching conditions at

$x = 0$. One condition is obvious. Conservation of mass across $x = 0$ requires that the fluxes to the left and right of $x = 0$ be equal:

$$(\widetilde{\mathrm{M}}_1) \qquad \left(qf_w + k^- H\frac{\partial p}{\partial x}\right)_{x=0^-} = \left(qf_w + k^+ H\frac{\partial p}{\partial x}\right)_{x=0^+}$$

for all $t > 0$. A condition related to the pressure was obtained by a formal regularization procedure. Replacing in (1.1) $k(x)$ by C^∞ approximations $k_n(x)$, according to

$$(1.4) \qquad k_n(x) = \begin{cases} k^- & \text{for} \quad x \le -\dfrac{1}{n}, \\ \varphi(nx) & \text{for} \quad -\dfrac{1}{n} < x < \dfrac{1}{n}, \\ k^+ & \text{for} \quad x \ge \dfrac{1}{n}, \end{cases}$$

with φ smooth ($\varphi(-1) = k^-$, $\varphi(1) = k^+$, and $\varphi' \le 0$), blowing up the transition region by $x \to nx$ and letting $n \to \infty$, we found the following. Let S^* be defined by the relation

$$(1.5) \qquad \frac{J(S^*)}{\sqrt{k^-}} = \frac{J(1)}{\sqrt{k^+}} > 0,$$

and let S^- and S^+ denote, respectively, the left and right limits of S at $x = 0$. Then for all $t > 0$ (see also Figure 1),

$$(\widetilde{\mathrm{M}}_2) \qquad \begin{cases} \dfrac{J(S^-)}{\sqrt{k^-}} = \dfrac{J(S^+)}{\sqrt{k^+}} & \text{if} \quad S^- \le S^* \quad \text{(pressure continuous)}, \\ S^+ = 1 & \text{if} \quad S^- > S^* \quad \text{(positive pressure jump)}. \end{cases}$$

Instead of analyzing (1.1) and conditions ($\widetilde{\mathrm{M}}_{1-2}$) in the form presented above, we shall consider a further simplified model problem, without losing essential characteristic features. We take in (1.1)

$$f(S) = S, \quad H(S) = 1 - S, \quad \text{and} \quad J(S) = 2 - S.$$

After a trivial scaling, the following equations result for the oil saturation $u = 1 - S$:

$$(1.6) \qquad u_t + f_x = 0 \qquad (u \ge 0),$$

$$(1.7) \qquad f = u - N_c\, k\, u\, p_x,$$

$$(1.8) \qquad p = \frac{1 + u}{\sqrt{k(x)}},$$

where f denotes the flux and N_c the dimensionless capillary number

$$N_c = \frac{\sigma\sqrt{K\phi}}{q\mu_w L}.$$

Here K is a characteristic k-value, L a characteristic length scale, and μ_w the water viscosity. By an additional scaling we may set $N_c = 1$. Further, k is given by (1.2) and the subscripts t and x denote partial differentiation.

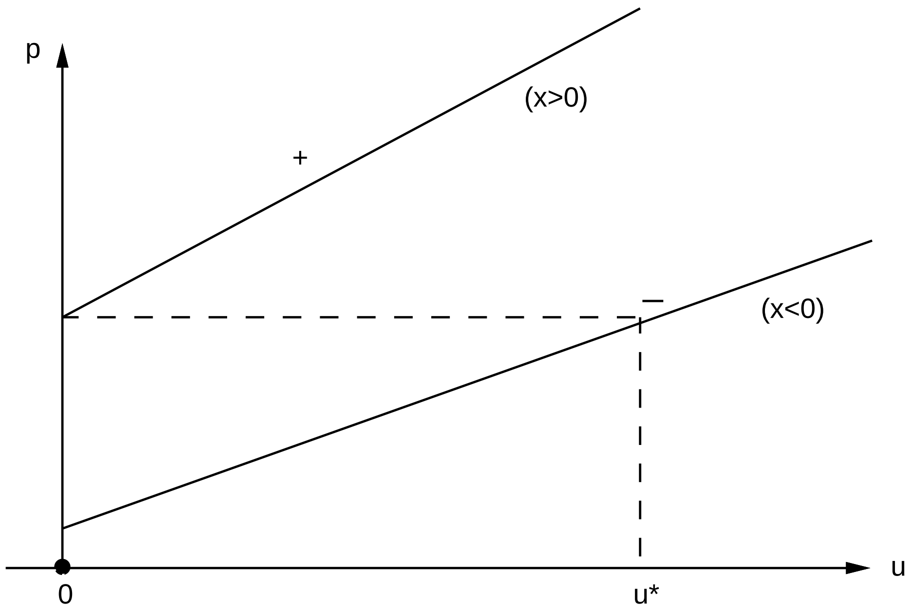

FIG. 2. *Transformed capillary pressures.*

We solve (1.6)–(1.8) in the subdomains

$$Q^{\pm} = \{(x,t) : x \in \mathbb{R}^{\pm},\ t \in (0,\infty)\},$$

with transformed matching conditions at $x = 0$. These are

$$[f] = 0 \qquad \text{in} \quad (0,\infty), \tag{M$_1$}$$

and (see Figure 2)

$$\begin{cases} \dfrac{1+u^-}{\sqrt{k^-}} = \dfrac{1+u^+}{\sqrt{k^+}} & \text{if} \quad u^- \geq u^* \\ u^+ = 0 & \text{if} \quad u^- < u^* \end{cases} \qquad \text{in} \quad (0,\infty),$$

or, equivalently,

$$u^+[p] = 0\ , \quad [p] \geq 0 \quad \text{in} \quad (0,\infty). \tag{M$_2$}$$

Here $u^* = \sqrt{\frac{k^-}{k^+}} - 1$. As before, $u^{\pm} = u^{\pm}(t) = u(0\pm,t),\ [u] = u^+ - u^-$, and f and p have similar notation.

At $t = 0$ we prescribe

$$u(\cdot,0) = u_0(\cdot) \quad \text{in} \quad \mathbb{R}, \tag{1.9}$$

with u_0 satisfying

$$\begin{cases} u_0 : \mathbb{R} \to [0,\infty), \quad \text{supp}(u_0) \subset \mathbb{R} \text{ is bounded;} \\ u_0 \text{ uniformly Lipschitz continuous in } \mathbb{R}\backslash\{0\}; \\ u_0^+[p_0] = 0, \quad f_0 := u_0 - \dfrac{\sqrt{k}}{2}(u_0^2)' \in BV(\mathbb{R}\backslash\{0\}). \end{cases} \tag{H}$$

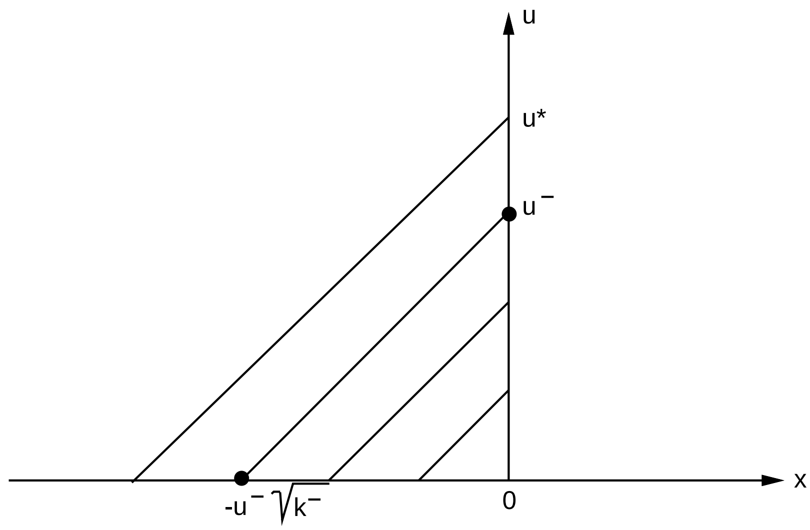

FIG. 3. *Admissible steady state solutions* ($u^- \leq u^*$).

The pressure condition at $t = 0$ is needed to construct an approximate sequence $\{u_{0n}\}$ for which the corresponding fluxes $f_{0n} := u_{0n} - k_n u_{0n}(p_{0n})'$ are uniformly bounded in $BV(\mathbb{R})$. This in turn will imply $f \in L^\infty((0,\infty);\ BV(\mathbb{R}))$, which is a crucial point in the existence proof. If the k_n are taken as in (1.4), then $[p_0] \geq 0$ is needed as well. We will return to this in section 2 and in the appendix.

For steady state solutions, the role of (M_2) can be seen explicitly. Assume $u = u(x)$ only, with $u(-\infty) = u(+\infty) = 0$. Then

$$f = u - k\, up' = 0 \qquad \text{in} \quad \mathbb{R}\backslash\{0\}. \tag{1.10}$$

Using $u \geq 0$, we obtain

$$u(x) = 0 \qquad \text{for} \quad x > 0.$$

Hence the first condition in (M_2) is always satisfied. Given any $u^- \geq 0$, we see that

$$u(x) = \left(u^- + \frac{1}{\sqrt{k^-}}\, x\right)_+ \tag{1.11}$$

satisfies (1.10) for $x < 0$. Here $(\ \cdot\ )_+ := \max\{\cdot, 0\}$. However, only for $u^- \in [0, u^*]$ we have $[p] \geq 0$. Thus we have a family of admissible steady state solutions, as shown in Figure 3.

Integrating the maximal steady state gives the maximal amount of oil that can be trapped to the left of the permeability discontinuity. It is given by

$$\bar{M} = \frac{1}{2}(u^*)^2\sqrt{k^-}. \tag{1.12}$$

Next we give the weak formulation of the trapping problem. Because the flux is expected to be continuous across $x = 0$, it will be defined globally in the formulation.

The saturation (and pressure) will be considered in the subdomains Q^- and Q^+ separately. Let

$$Q^0 := Q^- \cup Q^+ \quad \text{and} \quad Q := \mathbb{R} \times (0, \infty).$$

Combining the saturation equations and the matching conditions gives the following.

Problem P. Find $u : Q^0 \to [0, \infty)$, $f : Q \to \mathbb{R}$ such that

(i) u, $(u^2)_x \in L^\infty(Q^0)$; u is uniformly continuous in Q^0;

(ii) $f \in L^\infty((0, \infty); BV(\mathbb{R}))$;

(iii) $f = u - \frac{\sqrt{k}}{2}(u^2)_x$ a.e. in Q^0 and $\int_Q (u\zeta_t + f\zeta_x)dxdt + \int_{\mathbb{R}} u_0(x)\zeta(x, 0)dx = 0$ for all $\zeta \in H^1(Q) \cap C(\bar{Q})$, with compact support in $\bar{Q}$;

(iv) $u^+[p] = 0$ and $[p] \geq 0$ in $(0, \infty)$, where $p := \frac{1+u}{\sqrt{k}}$ in Q^0.

To prove existence we apply a k-regularization as in (1.4). This yields a sequence of approximating problems on Q for which we derive the necessary estimates. This is done in section 2. In section 3 we consider the limit $n \to \infty$ giving existence for Problem P, with u satisfying a porous media equation ($m = 2$) with linear convection in Q^0. Clearly (M_2) is satisfied. The weak equation in (iii) implies $[f] = 0$ a.e. in $(0, \infty)$. The comparison principle, with uniqueness as a consequence, is shown in section 4. In section 5 we give sufficient conditions for oil trapping; i.e., conditions that imply $u(x, t) = 0$ for $x > 0$ and for all $t > 0$. Finally, in section 6, we present some closing remarks about nonuniqueness, waiting times, and optimal regularity.

In a recent paper [DMP] considered oil transport in a multilayered porous medium. This work involves a discontinuous permeability which varies periodically in space. Using homogenization techniques they derived effective (upscaled) transport equations for the case where the periodicity length is small compared to the characteristic length L. In their analysis matching conditions ($\widetilde{\text{M}}_1$) and ($\widetilde{\text{M}}_2$) play a crucial role. They lead to a macroscopic irreducible oil saturation.

2. The approximate problem. In this section we study the approximate equation in which k is replaced by the smooth function k_n, defined by (1.4). Together with k we also need to approximate the initial value u_0. We construct approximations u_{0n}, so that the corresponding fluxes

$$f_{0n} := u_{0n} - k_n u_{0n} p'_{0n}, \qquad p_{0n} := \frac{1 + u_{0n}}{\sqrt{k_n}} \tag{2.1}$$

have a uniformly bounded total variation. In addition we require that each u_{0n} is strictly positive to eliminate the degeneracy of the equation at points where u vanishes. The existence of such u_{0n} is given in the following lemma. Since the proof is quite technical, it is given in the appendix.

LEMMA 2.1. *Let $n \in \mathbb{N}$ and let k_n be defined by* (1.4). *Suppose u_0 satisfies hypothesis* (H) *and in addition*

$$[p_0] = \frac{1 + u_0^+}{\sqrt{k^+}} - \frac{1 + u_0^-}{\sqrt{k^-}} \geq 0. \tag{2.2}$$

Then there exist $u_{0n} \in W^{1,\infty}(\mathbb{R})$ and $\varepsilon_n \in \mathbb{R}^+$ such that

(i) *$u_{0n} \geq \varepsilon_n > 0$ in $\mathbb{R}$, and $u_{0n}(x) = \varepsilon_n$ for $|x|$ sufficiently large;*

(ii) *u_{0n} is uniformly bounded in $\mathbb{R}$, and f_{0n}, defined by* (2.1)*, is uniformly bounded in $BV(\mathbb{R})$;*

(iii) *As* $n \to \infty$,

$$u_{0n} \to u_0 \quad \textit{uniformly in } \mathbb{R}\backslash\{0\}$$

and

$$u_{0n} - \varepsilon_n \to u_0 \quad \textit{in } L^1(\mathbb{R}).$$

For each $n \in \mathbb{N}$ we consider the approximate problem

$$(\mathrm{P}_n) \qquad \begin{cases} u_t + u_x = (k_n u\, p_x)_x, \quad p = \dfrac{1+u}{\sqrt{k_n}} \quad \text{in } \ Q, \\ u(x,0) = u_{0n}(x) \quad \text{for } x \in \mathbb{R}. \end{cases}$$

In the remainder of this section we prove the following results.

THEOREM 2.2. *Let* u_{0n} *be given by Lemma* 2.1. *Then problem* (P_n) *has a solution* $u_n \in C^\infty(Q) \cap C(\bar{Q})$ *such that*

(i) $0 < u_n \le \mathcal{C}$ *in* Q, *where* $\mathcal{C}$ *does not depend on* n;

(ii) $f_n := u_n - k_n u_n(\frac{1+u_n}{\sqrt{k_n}})_x$ *is uniformly bounded in* $L^\infty([0,\infty); BV(\mathbb{R}))$;

(iii) u_n *is uniformly continuous in* $\{\mathbb{R}\backslash(-\varepsilon,\varepsilon)\} \times [0,\infty)$ *for all* $\varepsilon > 0$.

Proof. Since $u_{0n} \ge \varepsilon_n > 0$ in $\mathbb{R}$, problem (P_n) is nondegenerate at $t = 0$. Hence it has a unique local (with respect to t) classical solution u_n; see, for instance, [LSU] and [F]. This solution can be continued as long as it remains bounded and bounded away from zero. Let $Q_{T_n} := \mathbb{R} \times (0, T_n)$ denote the maximal existence domain for u_n.

A positive lower bound follows from the maximum principle. Indeed, if we set $L_n := \max_{\mathbb{R}} |(\sqrt{k_n})''|$ we observe that the solution of the initial value problem

$$(\mathrm{LB}) \qquad \begin{cases} s' = -L_n s(1+s) \quad \text{for} \quad t > 0, \\ s(0) = \varepsilon_n \end{cases}$$

is a subsolution for problem (P_n). Hence if s_n denotes the solution of (LB), we have

$$u_n(x,t) \ge s_n(t) > 0 \quad \text{for} \quad (x,t) \in Q_{T_n}. \tag{2.3}$$

Before proving a uniform upper bound for u_n, we observe that the flux f_n is uniformly bounded in Q_{T_n}. A straightforward calculation yields for f_n the linear equation

$$f_t = a_n f_{xx} + b_n f_x, \tag{2.4}$$

where

$$a_n(x,t) := u_n\sqrt{k_n}, \qquad b_n(x,t) := -\frac{f_n}{u_n} - \frac{u_n k_n'}{2\sqrt{k_n}}. \tag{2.5}$$

Hence, by the maximum principle

$$||f_n||_{L^\infty(Q_{T_n})} \le ||f_{0n}||_{L^\infty(\mathbb{R})} \le \mathcal{C} \tag{2.6}$$

for all $n \in \mathbb{N}$.

We use this estimate to demonstrate a uniform upper bound for u_n in Q_{T_n}. As a first observation we note that (2.6) implies the differential inequality

$$|u_n - \sqrt{k^-}\, u_n u_{nx}| \le \mathcal{C} \quad \text{in} \quad \left(-\infty, -\frac{1}{n}\right] \times [0, T_n). \tag{2.7}$$

Then the upper bound for u_n in this set is immediate if we can control the decay of u_n as $x \to -\infty$. This decay results from the following argument.

Let $\bar{u}_n$ be a steady state solution satisfying

$$\begin{cases} u - k_n u\, p' = \varepsilon_n, \qquad p = \dfrac{1+u}{\sqrt{k_n}} \quad \text{in } \mathbb{R}, \\ u(\pm\infty) = \varepsilon_n. \end{cases}$$

Clearly, $\bar{u}_n(x) = \varepsilon_n$ for all $x \geq \frac{1}{n}$. The corresponding pressure $\bar{p}_n$ satisfies

$$\begin{cases} k_n(p\sqrt{k_n} - 1)p' = p\sqrt{k_n} - 1 - \varepsilon_n \quad \text{for} \quad x < \dfrac{1}{n}, \\ p\left(\dfrac{1}{n}\right) = \dfrac{1+\varepsilon_n}{\sqrt{k^+}}. \end{cases}$$

At points where $\bar{p}_n' = 0$, we must have $\bar{p}_n > 0$ and $\bar{p}_n'' < 0$. We use this to obtain $\bar{p}_n' > 0$ and $\bar{p}_n > \frac{1+\varepsilon_n}{\sqrt{k^-}}$ on $(-\infty, \frac{1}{n})$, and $\bar{p}_n(x) \to \frac{1+\varepsilon_n}{\sqrt{k^-}}$ as $x \to -\infty$. In particular, $\bar{u}_n(x) \to \varepsilon_n$ exponentially as $x \to -\infty$ and $\bar{u}_n - \varepsilon_n \in L^1(\mathbb{R})$, uniformly in $n \in \mathbb{N}$.

Now using Lemma 2.1(iii) and an argument as in the proof of Theorem 4.1, one finds for $t > 0$ the L^1-contraction

$$\int_{\mathbb{R}} |u_n(x,t) - \bar{u}_n(x)| dx \leq \int_{\mathbb{R}} |u_{0n}(x) - \bar{u}_n(x)| dx.$$

This inequality controls the behavior of u_n as $|x| \to \infty$. Combined with (2.7) it gives the upper bound in $(-\infty, -\frac{1}{n}] \times (0, T_n)$. Arguing similarly for $x > \frac{1}{n}$, we conclude that for all $n \in \mathbb{Z}^+$

$$u_n(x,t) \leq \mathcal{C} \quad \text{for} \quad |x| \geq \frac{1}{n}, \quad 0 \leq t < T_n. \tag{2.8}$$

To obtain the upper bound in the remaining strip $[-\frac{1}{n}, \frac{1}{n}] \times [0, T_n)$ we express (2.6) in terms of the pressure p_n:

$$|p_n\sqrt{k_n} - 1 - k_n(p_n\sqrt{k_n} - 1)p_{nx}| \leq \mathcal{C}. \tag{2.9}$$

By (2.8), $p_n(\pm\frac{1}{n}, t)$ is uniformly bounded. Then (2.9) implies that p_n, and thus u_n, is uniformly bounded as well.

The uniform upper bound, together with lower bound (2.3), guarantees existence for all $t > 0$. Hence, $T_n = \infty$ for each $n \in \mathbb{N}$. This completes the proof of (i).

The proof of (ii) is a direct consequence of Lemma 2.1(ii) and the total variation estimate for the flux in Lemma 2.4 below.

We conclude by proving (iii). The boundedness of u_n and the flux estimate (2.7) imply that u_n is uniformly Hölder continuous (exponent $\frac{1}{2}$) with respect to x in $\{(x,t) : x < -\frac{1}{n},\ t > 0\}$. The same result holds in $\{(x,t) : x > \frac{1}{n},\ t > 0\}$. The smoothness and boundedness of the coefficients in the u_n-equation allow us to apply [G1], yielding that u_n is uniformly Hölder continuous (exponent $\frac{1}{4}$) with respect to t in $\{(x,t) : |x| > \frac{1}{n},\ t > 0\}$. Since, for fixed $\varepsilon > 0$, $\frac{1}{n} < \varepsilon$ for n large enough, this proves (iii) and completes the proof of Theorem 2.2. ∎

Remark 2.3. It is not difficult to show that the steady states $\bar{u}_n$, corresponding to $k = k_n$ and $\bar{u}_n(\pm\infty) = \varepsilon_n$, approximate the maximal steady state in Figure 3. In essence this follows from $\bar{u}_n(x) = \varepsilon_n$ for all $x \geq \frac{1}{n}$ and, using the pressure equation,

$$0 < \bar{p}_n\left(\frac{1}{n}\right) - \bar{p}_n\left(-\frac{1}{n}\right) = \int_{-\frac{1}{n}}^{+\frac{1}{n}} \frac{1}{k_n} \frac{\bar{p}_n(x)\sqrt{k_n} - 1 - \varepsilon_n}{\bar{p}_n(x)\sqrt{k_n} - 1}\, dx \to 0$$

as $n \to \infty$.

It remains to prove the following lemma used in the proof of Theorem 2.2.

LEMMA 2.4. *Let u_{0n} be given by Lemma* 2.1 *and let u_n be the corresponding solution of problem* (P_n). *Then*

$$TV_{\mathbb{R}}(f_n(t)) \le TV_{\mathbb{R}}(f_{0n}) \qquad \textit{for all} \quad t > 0.$$

Proof. Each flux f_n satisfies the linear problem

$$\begin{cases} f_t = a_n f_{xx} + b_n f_x & \text{in } Q, \\ f(x,0) = f_{0n}(x) & \text{for} \quad x \in \mathbb{R}, \end{cases}$$

where a_n and b_n, defined in (2.5), are bounded functions and where f_{0n} has uniformly bounded variation. First we proceed formally. Let us fix $\varepsilon > 0$ and calculate (dropping the subscript n)

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \left\{ \sqrt{f_x^2 + \varepsilon} - \sqrt{\varepsilon} \right\} &= \int_{\mathbb{R}} \frac{f_x}{\sqrt{f_x^2 + \varepsilon}} (a f_{xx} + b f_x)_x \\ &= -\varepsilon \int_{\mathbb{R}} \frac{f_{xx}(a f_{xx} + b f_x)}{(f_x^2 + \varepsilon)^{3/2}}. \end{aligned}$$

Integrating in time gives, for any $t > 0$,

$$\begin{aligned} \int_{\mathbb{R}} \left\{ \sqrt{f_x^2(t) + \varepsilon} - \sqrt{\varepsilon} \right\} - \int_{\mathbb{R}} \left\{ \sqrt{{f_{0n}'}^2 + \varepsilon} - \sqrt{\varepsilon} \right\} &= -\varepsilon \int_{\mathbb{R} \times (0,t)} \frac{a f_{xx}^2 + b f_x f_{xx}}{(f_x^2 + \varepsilon)^{3/2}} \\ &\le - \int_{\mathbb{R} \times (0,t)} \frac{\varepsilon f_x}{(f_x^2 + \varepsilon)^{3/2}} \, b f_{xx}. \end{aligned}$$

Since

$$\left| \frac{\varepsilon f_x}{(f_x^2 + \varepsilon)^{3/2}} \right| \le 1$$

and

$$\frac{\varepsilon f_x}{(f_x^2 + \varepsilon)^{3/2}} \to 0, \quad \text{pointwise in } Q \text{ as } \varepsilon \to 0,$$

the boundedness of b and Lebesgue's dominated convergence theorem imply

$$\int_{\mathbb{R}} |f_x(t)| \le \int_{\mathbb{R}} |f_{0n}'|,$$

provided $f_{xx} \in L^1(\mathbb{R} \times (0,t))$. To complete the proof of the lemma we need to make this argument rigorous.

It is enough to apply a mollifier to the initial function f_{0n} of the linear flux problem. This ensures the smoothness up to $t = 0$ necessary to carry out the above calculations. $\square$

3. Existence for Problem P. Let u_n be the solution of problem (P_n) as stated in Theorem 2.2. By a standard argument there exist a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, and $u \in L^\infty(Q) \cap C((\mathbb{R}^- \cup \mathbb{R}^+) \times [0,\infty))$ such that

$$u_n \to u \quad \text{in } C_{\text{loc}}((\mathbb{R}^- \cup \mathbb{R}^+) \times [0,\infty))$$

as $n \to \infty$. We show the following theorem.

THEOREM 3.1. *u is a solution of Problem* P.

Proof. Clearly u is a (weak) solution of the equation

$$u_t + u_x = \frac{1}{2}\sqrt{k^\pm}(u^2)_{xx} \quad \text{in } Q^\pm$$

and

$$f = u - \frac{1}{2}\sqrt{k^\pm}\,(u^2)_x \in L^\infty([0,\infty);\ BV(\mathbb{R}^\pm)).$$

The boundedness of u and f implies that u^2 is uniformly Lipschitz continuous with respect to x in Q^0. Hence the following quantities are well defined for each $t > 0$:

$$u^\pm(t), \qquad f^\pm(t), \qquad \text{and} \qquad p^\pm(t) = \frac{1 + u^\pm(t)}{\sqrt{k^\pm}}.$$

Using the equation

$$u_t + f_x = 0 \qquad \text{a.e. in } Q^\pm$$

and again the boundedness of f, we obtain as in [DP] that the functions

$$t \to u^\pm(t)$$

are continuous in $[0,\infty)$.

Next we claim

$$f^+(t) = f^-(t) \quad \text{for almost all } t > 0. \tag{3.1}$$

Indeed, using the asymptotic behavior of $u_n(x,t)$ as $|x| \to \infty$, we find, for $n \to \infty$,

$$\int_{\mathbb{R}} (u_n(x,t) - \varepsilon_n)dx = \int_{\mathbb{R}} (u_{0n}(x) - \varepsilon_n)dx \to \int_{\mathbb{R}} u_0(x)dx$$

and hence

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx \quad \text{for all} \quad t > 0,$$

which expresses conservation of mass. This identity implies

$$\begin{aligned} 0 &= \lim_{\delta \to 0^+} \left(\int_{-\infty}^{-\delta} u(x,t)dx + \int_{\delta}^{\infty} u(x,t)dx - \int_{-\infty}^{-\delta} u_0(x)dx - \int_{\delta}^{\infty} u_0(x)dx \right) \\ &= \int_0^t (f^+(s) - f^-(s))ds \quad \text{for all} \quad t > 0. \end{aligned}$$

Together with the equations in $Q^\pm$, equality (3.1) implies the weak form (iii) of Problem P.

It remains to prove

$$u^+[p] = 0 \quad \text{and} \quad [p] \geq 0 \quad \text{for all} \quad t > 0. \tag{3.2}$$

For this purpose we study u_n and p_n in the interval $(-\frac{1}{n}, \frac{1}{n})$. Since k_n changes rapidly there, we make the blow-up

$$y = nx \qquad \text{for} \quad -\frac{1}{n} < x < \frac{1}{n}.$$

Knowing that the fluxes f_n are uniformly bounded, we obtain

$$|u_n - nk_n u_n (p_n)_y| \leq \mathcal{C}.$$

Thus for appropriate $\mathcal{C} > 0$ we have

$$|u_n (p_n)_y| \leq \frac{\mathcal{C}}{n},$$

or

$$\left| (u_n^2)_y - \frac{\varphi'}{\varphi} u_n (1 + u_n) \right| \leq \frac{\mathcal{C}}{n} \tag{3.3}$$

for all $-1 < y < 1$ and $t > 0$.

Hence u_n^2 are Lipschitz continuous in $[-1, 1]$ uniformly with respect to n and t. Up to a subsequence, $u_n \to u$ uniformly in $[-1, 1]$, as $n \to \infty$, for all $t > 0$; in particular $u(-1, t) = u^-(t)$ and $u(1, t) = u^+(t)$. In addition, it follows easily from (3.3) that u satisfies

$$(u^2)_y = \frac{\varphi'}{\varphi} u(1 + u) \tag{3.4}$$

in $\{(y, t) : -1 < y < 1, \ t > 0\}$. Since φ' is nonpositive, u is decreasing. Thus, if $u^+(t) > 0$, we have $u(y, t) > 0$ in $[-1, 1]$ and (3.4) reduces to

$$u_y = \frac{\varphi'}{2\varphi}(1 + u). \tag{3.5}$$

A straightforward calculation gives $[p] = 0$.

Next suppose $u^+(t) = 0$. We have to show $[p] \geq 0$. If $u^-(t) = 0$, we get

$$[p] = \frac{1}{\sqrt{k^+}} - \frac{1}{\sqrt{k^-}} > 0.$$

If $u^-(t) > 0$, define $\tilde{y} := \sup\{y \in [-1, 1] : u(y, t) > 0\}$ and solve (3.5) in $[-1, \tilde{y}]$. This gives

$$\frac{1 + u^-}{\sqrt{k^-}} = \frac{1}{\sqrt{\varphi(\tilde{y})}} \leq \frac{1}{\sqrt{k^+}},$$

which implies $[p] \geq 0$ and $u^-(t) \leq u^*$. $\square$

4. The comparison principle. We start with some preliminary observations for solutions (u, f) of Problem P. Choosing test functions with support in $Q^\pm$, we obtain

$$\int_{Q^\pm} u\zeta_t + \int_{Q^\pm} \left(u - \frac{\sqrt{k^\pm}}{2}(u^2)_x\right)\zeta_x = 0.$$

Thus away from $x = 0$ we have two weak equations of "porous media" type ($m = 2$) with linear convection, implying

$$u_t + \left(u - \frac{\sqrt{k^\pm}}{2}(u^2)_x\right)_x = 0 \quad \text{a.e. in } Q^\pm \tag{4.1}$$

and

$$\operatorname{supp}(u(t)) \text{ is bounded in } \mathbb{R} \tag{4.2}$$

for all $t \in [0, \infty)$. Further, using hypothesis (H), we can apply the Bernstein argument of [A] in the truncated domain

$$Q^\delta := \mathbb{R}\backslash(-\delta, \delta) \times (0, \infty) \qquad (\text{for } \delta > 0, \text{ fixed})$$

to obtain

$$||u_x||_{L^\infty(Q^\delta)} \leq C(\delta). \tag{4.3}$$

We use this to derive an estimate on u_t in Q^δ. Let u be a smooth solution of (4.1) in the sense of the usual "porous media" approximations, and let $\xi : \mathbb{R} \to [0, 1]$ be an even C^1 cut-off function satisfying

$$\xi(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x \leq \delta/2, \\ 1 & \text{for} \quad \delta \leq x \leq L, \\ 0 & \text{for} \quad x \geq L + 1 \end{cases}$$

for any $L > \delta$. Multiplying (4.1) by $\xi^2 u_t$ gives

$$\int_{Q_\tau} \xi^2 u_t^2 = -\int_{Q_\tau} \xi^2 u_t u_x - \int_{Q_\tau} \xi\xi' \sqrt{k} u_t (u^2)_x - \int_{Q_\tau} \xi^2 \frac{\sqrt{k}}{2}\, u_{xt}\, (u^2)_x,$$

where $Q_\tau = \mathbb{R} \times (0, \tau)$ with $\tau > 0$ arbitrarily chosen. Using $u_{xt}(u^2)_x = u(u_x^2)_t$, the last integral becomes

$$\int_{\mathbb{R}} \xi^2 \frac{\sqrt{k}}{2}\, u u_x^2 \Big|_0^\tau - \int_{Q_\tau} \xi^2 \frac{\sqrt{k}}{2}\, u_t u_x^2.$$

Then (i) of Problem P and (4.3) in $Q^{\delta/2}$ give

$$\int_{Q_\tau} \xi^2 u_t^2 \leq C(\delta, \tau),$$

implying

$$u_t \in L^2_{\text{loc}}(\bar{Q^\delta}). \tag{4.4}$$

We are now in a position to prove the following theorem.

THEOREM 4.1. *Let* (u_1, f_1) *and* (u_2, f_2) *be weak solutions of Problem* P *corresponding to initial values* u_{01} *and* u_{02}*, respectively. Then* $u_{01} \le u_{02}$ *in* $\mathbb{R}$ *implies* $u_1 \le u_2$ *in* Q^0.

Proof. Let $\tau > 0$ be arbitrary. In the weak equation for the difference

$$\int_Q \{(u_1 - u_2)\zeta_t + (f_1 - f_2)\zeta_x\} + \int_{\mathbb{R}} \zeta(u_{01} - u_{02}) = 0,$$

we take the test function

$$\zeta = \xi\psi S_\varepsilon(u_1^2 - u_2^2),$$

where the following hold:

(i) ξ is an even C^1 cut-off function near $x = 0$,

$$\xi(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le \delta/2, \\ 1 & \text{for} \quad x \ge \delta, \end{cases} \qquad \xi'(x) \ge 0 \text{ for } \delta/2 < x < \delta.$$

(ii) ψ is a C^1 cut-off function near $t = \tau$,

$$\psi(t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le \tau - \mu, \\ 0 & \text{for} \quad \tau \le t, \end{cases} \qquad \psi'(t) \le 0 \text{ for } \tau - \mu < t < \tau.$$

(iii) $S_\varepsilon : \mathbb{R} \to [0, 1]$ is given by

$$S_\varepsilon(r) = \begin{cases} 0, & r \le 0, \\ \dfrac{r}{\sqrt{r^2 + \varepsilon^2}}, & r > 0. \end{cases}$$

Here δ, μ, and ε are small positive parameters. Note that for $\varepsilon \searrow 0$

$$\left. \begin{aligned} &S_\varepsilon(r) \to \chi_{\{r>0\}} := \begin{cases} 1, & r > 0, \\ 0, & r \le 0, \end{cases} \\ &rS'_\varepsilon(r) \to 0 \end{aligned} \right\} \quad \text{pointwise in } \mathbb{R}. \tag{4.5}$$

Integrating the first term by parts gives

$$\begin{aligned} &\int_{Q_\tau} (u_1 - u_2)_t \, \xi\psi S_\varepsilon(u_1^2 - u_2^2) \\ &\quad = \int_{Q_\tau} (f_1 - f_2)\psi \Big\{ \xi' S_\varepsilon(u_1^2 - u_2^2) + \xi S'_\varepsilon(u_1^2 - u_2^2)(u_1^2 - u_2^2)_x \Big\} \\ &\quad \le \int_{Q_\tau} (f_1 - f_2) \, \psi\xi' S_\varepsilon(u_1^2 - u_2^2) + \int_{Q_\tau} (u_1 - u_2) \, \psi\xi S'_\varepsilon(u_1^2 - u_2^2)(u_1^2 - u_2^2)_x. \end{aligned}$$

For fixed $\mu, \delta > 0$, we first let $\varepsilon \searrow 0$. Using (4.5), we have

$$(u_1 - u_2)\psi\xi S'_\varepsilon(u_1^2 - u_2^2) \to 0 \quad \text{pointwise in } Q_\tau.$$

Hence by (4.4) we obtain

$$\int_{Q_\tau} \xi\psi((u_1 - u_2)_+)_t \le \int_{Q_\tau} (f_1 - f_2)\psi\xi' \chi_{\{u_1 > u_2\}}.$$

Next we let $\mu \searrow 0$. This gives

$$
\begin{aligned}
\int_{\mathbb{R}} & \xi(u_1 - u_2)_+(\tau) \\
& \le \int_0^\tau \left\{ \int_{-\delta}^{-\delta/2} (f_1 - f_2)\xi' \chi_{\{u_1 > u_2\}} + \int_{\delta/2}^{\delta} (f_1 - f_2)\xi' \chi_{\{u_1 > u_2\}} \right\} \qquad (4.6) \\
& =: \int_0^\tau \{I_\delta^- + I_\delta^+\}.
\end{aligned}
$$

Let $t \in (0, \tau)$ be chosen such that f^-, f^+ exist. Consider the possibilities:

(i) $u_1^+ \ne u_2^+$, say $u_1^+ > u_2^+$. Then $u_1 > u_2$ in a right neighborhood of $x = 0$ and $\chi_{\{u_1>u_2\}} = 1$ in $(\delta/2, \delta)$ for δ sufficiently small. The pressure conditions (M_2) give $u_1^- > u_2^-$: if $u_2^+ > 0$, then $[p_1] = [p_2] = 0$ implies $u_1^- > u_2^-$; if $u_2^+ = 0$, then $u_2^- \le u^*$, while $u_1^- > u^*$. Therefore also $\chi_{\{u_1>u_2\}} = 1$ in $(-\delta, -\delta/2)$. As a consequence

$$\lim_{\delta \searrow 0} (I_\delta^- + I_\delta^+) = [f_1] - [f_2] = 0.$$

(ii) $u_1^+ = u_2^+$. Now we need to compare the corresponding fluxes. Suppose that $f_1^+ = f_2^+$. Then

$$\sup_{(\delta/2, \delta)} (f_1 - f_2)\chi_{\{u_1>u_2\}} \to 0 \quad \text{as} \quad \delta \searrow 0,$$

and the same applies in $(-\delta, -\delta/2)$. Thus again

$$\lim_{\delta \searrow 0} (I_\delta^- + I_\delta^+) = 0.$$

If $f_1^+ > f_2^+$, then $(u_1^2)_x < (u_2^2)_x$ and therefore $u_1 < u_2$ in $(\delta/2, \delta)$. Thus

$$I_\delta^- + I_\delta^+ = I_\delta^- \le 0 \qquad \text{for } \delta > 0 \text{ sufficiently small.}$$

Finally, if $f_1^+ < f_2^+$, then $(u_1^2)_x > (u_2^2)_x$ and $u_1 > u_2$ in $(\delta/2, \delta)$. Thus $\lim_{\delta \searrow 0} I_\delta^+ = f_1^+ - f_2^+$. Furthermore, since

$$(f_1 - f_2)\xi' \chi_{\{u_1>u_2\}} \le (f_1 - f_2)\xi' \quad \text{in } (-\delta, -\delta/2),$$

$$\limsup_{\delta \searrow 0} (I_\delta^- + I_\delta^+) \le 0.$$

Combining these results, we obtain from (4.6)

$$u_1(\cdot, \tau) - u_2(\cdot, \tau) \le 0 \qquad \text{in} \quad \mathbb{R}\backslash\{0\},$$

which proves the theorem. $\square$

As an immediate consequence we have the following.

Corollary 4.2. *Problem* P *has at most one solution* (u, f).

5. Oil trapping. The steady state solutions shown in Figure 3 suggest that oil may be trapped at the interface between coarse and fine material. Indeed, if $u_0(x) = 0$ for $x > 0$ and if for some $u^- \in (0, u^*]$

$$u_0(x) \leq \left(u^- + \frac{1}{\sqrt{k^-}}\,x\right)_+ \qquad \text{for} \quad x < 0,$$

then the comparison principle guarantees

$$u(x,t) \leq \left(u^- + \frac{1}{\sqrt{k^-}}\,x\right)_+ \qquad \text{for all} \quad (x,t) \in Q^-$$

and

$$u = 0 \quad \text{in} \quad Q^+.$$

The following theorem explains trapping in terms of the oil mass. For convenience, let

$$\bar{u}(x) := \begin{cases} \left(u^* + \frac{1}{\sqrt{k^-}}\,x\right)_+ & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x > 0 \end{cases}$$

denote the maximal admissible steady state having $\bar{M}$, given by (1.12), as corresponding mass.

THEOREM 5.1. *Let u_0 satisfy hypothesis* (H) *and let*

$$\int_{-\infty}^{x} u_0(s)ds \geq \int_{-\infty}^{x} \bar{u}(s)ds \qquad \text{for} \quad x < 0.$$

Then the solution of Problem P *satisfies*

$$\int_{-\infty}^{0} u(s,t)ds \geq \bar{M} \qquad \text{for all} \quad t > 0.$$

Proof. Fix any $\delta > 0$ and set

$$V_\delta(x,t) = \int_{-\infty}^{x} u(s,t)ds + \delta \qquad \text{for} \quad (x,t) \in \bar{Q}.$$

Then $V_\delta \in C(\bar{Q}), V(\cdot,t) \in C^1((-\infty,0]) \cup C^1([0,\infty))$ for all $t > 0$, and

$$V_\delta = \delta \ \ \text{to the left of the support of } u \text{ in } Q^-,$$
$$V_\delta = \int_{\mathbb{R}} u_0(s)ds + \delta \ \ \text{to the right of the support of } u \text{ in } Q^+.$$

As a consequence $V_\delta \geq \bar{M}$ in Q^+, and it satisfies

$$V_t + V_x - \sqrt{k^-}V_x V_{xx} = 0 \qquad \text{a.e. in} \ \ Q^-. \tag{5.1}$$

Setting

$$\bar{v}(x) := \int_{-\infty}^{x} \bar{u}(s)ds \qquad \text{for} \quad x \in (-\infty, 0],$$

we have

$$V_\delta > \bar{v} \qquad \text{in} \quad Q_t^- := (-\infty, 0] \times (0, t)$$

for t sufficiently small. Let

$$t_0 = \sup\{t > 0 : V_\delta > \bar{v} \quad \text{in} \ \ Q_t^-\}.$$

Below we show $t_0 = \infty$. Suppose $t_0 < \infty$. Then there exists $(x_0, t_0) \in \bar{Q^-}$ such that

$$V_\delta > \bar{v} \qquad \text{in} \ \ Q_{t_0}^- \tag{5.2}$$

and

$$V_\delta(x, t_0) \geq \bar{v}(x) \qquad \text{for all} \ \ x \in (-\infty, 0] \ \ \text{with} \ \ V_\delta(x_0, t_0) = \bar{v}(x_0). \tag{5.3}$$

We first rule out $x_0 = 0$.

If $x_0 = 0$, we distinguish the three following cases:

(i) $u(0^-, t_0) > u^*$. Then we have

$$\frac{\partial V_\delta}{\partial x}(0^-, t_0) = u(0^-, t_0) > u^* \ = \frac{d\bar{v}}{\partial x}(0^-).$$

This contradicts (5.3).

(ii) $u(0^-, t_0) < u^*$. By continuity there exists $\varepsilon > 0$ such that $u(0^-, t) < u^*$ and $u(0^+, t) = 0$ for $t_0 - \varepsilon < t < t_0$. Since $f^-(t) = f^+(t) \leq 0$ for almost all $t \in (t_0 - \varepsilon, t_0)$ (see also section 6), we find from integrating the u-equation in $(-\infty, 0) \times (t_0 - \varepsilon, t_0)$

$$\int_{-\infty}^0 u(s, t_0) ds - \int_{-\infty}^0 u(s, t_0 - \varepsilon) ds = -\int_{t_0 - \varepsilon}^{t_0} f^-(t) dt \geq 0.$$

Hence

$$V_\delta(0, t_0 - \varepsilon) \leq V_\delta(0, t_0) = \bar{v}(0),$$

which contradicts (5.2).

(iii) $u(0^-, t_0) = u^*$. Then $V_\delta(0^-, t_0) = \bar{v}(0)$ as well as

$$\frac{\partial V_\delta}{\partial x}(0^-, t_0) = \frac{d\bar{v}}{dx}(0^-) = u^*.$$

Using (5.1) locally in Q^- and the strong maximum principle, we again obtain a contradiction.

Hence $x_0 \neq 0$ and $V_\delta(0, \cdot) > \bar{v}(0)$ in $[0, t_0]$. We then apply the comparison principle to (5.1) in $Q_{t_0}^-$ to find $V_\delta > \bar{v}$ in $(-\infty, 0] \times [0, t_0]$. This shows that $t_0 = \infty$. As a consequence $V_\delta > \bar{v}$ in $\bar{Q^-}$ for any $\delta > 0$, which implies the assertion of the theorem. $\square$

Similarly we show the following.

THEOREM 5.2. *Let u_0 satisfy hypothesis* (H) *and let*

$$\int_x^\infty u_0(s) ds \leq \int_x^\infty \bar{u}(s) ds \qquad \textit{for } x \in \mathbb{R}.$$

Then

$$u = 0 \qquad \textit{in } \bar{Q}^+.$$

6. Closing remarks. In this section we briefly discuss some qualitative properties of solutions of Problem P.

6.1. Nonuniqueness. In the proof of the comparison principle, implying uniqueness, we have used the pressure condition

$$[p] \geq 0. \tag{6.1}$$

By means of a counterexample we show here that uniqueness fails if we drop condition (6.1). Let u_0 satisfy the structural properties

$$(\tilde{\mathrm{H}}) \qquad \begin{cases} u_0(x) = 0 \quad \text{if} \quad x > 0, \quad u_0 \not\equiv \bar{u} \quad \text{in} \ \mathbb{R}, \\ \bar{u}(x) \leq u_0(x) \leq (u^* + \delta x)_+ \quad \text{if} \ \ x < 0 \ \ \text{for some} \ \ 0 < \delta < \dfrac{1}{\sqrt{k^-}}. \end{cases}$$

Based on the results of section 5, we expect that the corresponding solution u of Problem P will have a nontrivial component in Q^+; i.e., $u \not\equiv 0$ in Q^+. We will construct a second solution $\tilde{u}$ which solves Problem P, except condition (6.1), and which satisfies $\tilde{u} \equiv 0$ in Q^+. This construction is based on a modification of k. Instead of (1.2) we consider

$$\tilde{k}_n(x) = \begin{cases} k^- & \text{for} \quad x < 0, \\ \kappa & \text{for} \quad 0 < x < \dfrac{1}{n}, \\ k^+ & \text{for} \quad x > \dfrac{1}{n}, \end{cases} \tag{6.2}$$

where $0 < \kappa < k^+ < k^-$, and we let $n \to \infty$.

Theorem 6.1. *Let u_0 satisfy hypotheses* (H) *and* ($\tilde{\mathrm{H}}$) *and let u denote the unique solution of Problem* P*. Then*

(i) *$u \not\equiv 0$ in Q^+;*

(ii) *there exists a second solution $\tilde{u}$ of Problem* P*, except* (6.1)*, which satisfies $\tilde{u} \equiv 0$ in Q^+.*

Proof. We first show that $u \not\equiv 0$ in Q^+. Arguing by contradiction, we assume

$$u(0^+, t) = 0 \qquad \text{for all} \quad t > 0.$$

Using $[p] \geq 0$ and $u \geq \bar{u}$ in Q, we conclude

$$u(0^-, t) = u^* \qquad \text{for all} \quad t > 0.$$

Hence u solves in Q^- the problem

$$(\mathrm{P}^-) \qquad \begin{cases} u_t + (u - \sqrt{k^-} u u_x)_x = 0 & \text{in} \quad Q^-, \\ u(0,t) = u^* & \text{for} \quad t > 0, \\ u(x,0) = u_0(x) & \text{for} \quad x < 0. \end{cases}$$

Now observe that $\bar{z} := (u^* + \delta x)_+$ is a supersolution for problem (P$^-$). Hence the solution $z(x,t)$ of problem (P$^-$) with initial data $z(\cdot, 0) = \bar{z}(x)$ is decreasing with respect to time and converges to a steady state solution $s(x)$. By comparison $s \geq \bar{u}$ in $\mathbb{R}^-$, but since $\bar{u}$ is maximal we have

$$s = \bar{u} \qquad \text{in} \ \ \mathbb{R}^-.$$

Using

$$\bar{u}(x) \leq u(x,t) \leq z(x,t) \qquad \text{for all} \quad (x,t) \in Q^-,$$

we obtain

$$\lim_{t\to\infty} u(x,t) = \bar{u}(x) \qquad \text{uniformly in} \ \ x < 0.$$

Combining this result with $u \equiv 0$ in Q^+, we find

$$\lim_{t\to\infty} \int_{-\infty}^{+\infty} u(x,t)ds \to \int_{-\infty}^{+\infty} \bar{u}(x)dx < \int_{-\infty}^{+\infty} u_0(x)dx,$$

which contradicts mass conservation for u.

Next we use (6.2) to explain the construction of $\tilde{u}$. As a first observation we note that the class of steady state solutions of the equation

$$\left(u - \tilde{k}_n\left(\frac{1+u}{\sqrt{\tilde{k}_n}}\right)'\right)' = 0 \qquad \text{in} \quad \mathbb{R},$$

having compact support and satisfying (M_1) and (M_2), has the same structure as the one shown in Figure 3, but with $u^* = \sqrt{\frac{k^-}{k^+}} - 1$ replaced by $\tilde{u}^* = \sqrt{\frac{k^-}{\kappa}} - 1$. In particular this class does not depend on n. For κ sufficiently small we find, for $\bar{\tilde{u}}$, the maximal steady state

$$u_0 \leq \bar{\tilde{u}} \qquad \text{in} \quad \mathbb{R}.$$

As a consequence, the solution $\tilde{u}_n$ of the problem

$$\begin{cases} u_t + \left(u - \tilde{k}_n\left(\dfrac{1+u}{\sqrt{\tilde{k}_n}}\right)_x\right)_x & \text{in} \quad Q, \\ u(x,0) = u_0(x) \qquad \text{for} \quad x \in R \end{cases}$$

satisfies

$$\tilde{u}_n(x,t) \leq \bar{\tilde{u}}(x) \qquad \text{for all} \quad (x,t) \in Q.$$

In particular

$$\tilde{u}_n \equiv 0 \qquad \text{in} \quad Q^+$$

for all $n \in \mathbb{Z}^+$. Finally, letting $n \to \infty$, $\tilde{u}_n$ converges along subsequences to a function $\tilde{u} = \tilde{u}(x,t)$ which satisfies all properties required for Problem P except (6.1). □

6.2. Waiting times and optimal regularity. Numerical simulations reported in [DMN] show that the right free boundary of u has a "waiting time" when it reaches the permeability discontinuity. The free boundary becomes stagnant there, while the oil saturation increases. It continues whenever the pressure exceeds the entry pressure of the low permeable region.

The following makes this precise.

THEOREM 6.2. *Let* u_0 *satisfy hypothesis* (H) *and let* $\text{supp}(u_0) \subset \mathbb{R}^-$. *Further, let the solution* u *of Problem* P *satisfy* $u \not\equiv 0$ *in* Q^+. *Set*

$$t_1 := \lim_{\varepsilon\to 0} \sup\{\tau > 0 : u \equiv 0 \ \ \textit{in} \ \ (-\varepsilon,\infty) \times (0,\tau)\}$$

and

$$t_2 := \sup\{\tau > 0 : u \equiv 0 \quad in \quad \mathbb{R}^+ \times (0, \tau)\}.$$

Then

$$0 < t_1 < t_2 < \infty \qquad (t_2 - t_1 \text{ is the waiting time})$$

and

$$u(0^-, t_1) = 0, \qquad u(0^-, t_2) = u^*.$$

Proof. Clearly t_1 and t_2 are well defined. Continuity of $u^\pm(t)$ and (M_2) imply directly $t_2 > t_1$ and $u(0^-, t_1) = 0$.

Suppose $u(0^-, t_2) < u^*$. By continuity, there exists $\delta > 0$ such that $u(0^-, t) < u^*$, and thus $u(0^+, t) = 0$, for $t_2 \le t < t_2 + \delta$. Thus $u \equiv 0$ in $\mathbb{R}^+ \times (0, t_2 + \delta)$, contradicting the definition of t_2. ∎

Next we consider the case where the oil initially is positioned in the fine material ($x > 0$). If the initial position is sufficiently close to the interface at $x = 0$, diffusion may drive the oil towards $x = 0$, i.e., against the flow, where it will penetrate the coarse material. This follows from the transformation $y = x - t,\ t = t$ and by considering an appropriate subsolution for the resulting porous media equation; see [G2].

Supposing the oil reaches $x = 0$, we have the following result.

THEOREM 6.3. *Let u_0 satisfy hypothesis* (H) *and let* $\mathrm{supp}(u_0) \subset \mathbb{R}^+$*. Further, let the solution u of Problem* P *satisfy $u \not\equiv 0$ in Q^-. Set*

$$t_1 := \sup\{\tau > 0 : u \equiv 0 \quad in \quad \mathbb{R}^- \times (0, \tau)\}.$$
$$t_2 := \sup\{\tau > 0 : u(0^+, t) = 0 \quad for \quad 0 < t < \tau\}.$$

Then

$$0 < t_1 < t_2 \le \infty.$$

In addition, there exists $t \in (t_1, t_2)$ such that for some $A > 0$

$$u(x, t) = A\sqrt{x}(1 + o(1)) \qquad as \quad x \to 0^+.$$

Proof. By the finite speed of propagation we have $t_1 > 0$. Continuity of $u(0^-, \cdot)$ implies $u(0^-, t_1) = 0$ and $u(0^-, t) \le u^*$ and hence $u(0^+, t) = 0$ for all t in an upper neighborhood of t_1. Hence $t_2 > t_1$. If $u(0^-, t) \le u^*$ for all $t > 0$, we have $t_2 = \infty$. Since $u \not\equiv 0$ in $\mathbb{R}^- \times (t_1, t_2)$ and $u(0^+, \cdot) = 0$ in (t_1, t_2), there exists $t \in (t_1, t_2)$ such that

$$f(t) = f^-(t) = f^+(t) < 0.$$

Hence, for this t fixed, setting $f(t) = -\mathcal{C}(\mathcal{C} > 0)$,

$$u - \sqrt{k^+}uu_x = -\mathcal{C}(1 + o(1)) \qquad \text{as} \quad x \to 0^+,$$

giving

$$\frac{1}{2}u^2(x, t) = \frac{\mathcal{C}}{\sqrt{k^+}}x(1 + o(1)) \qquad \text{as} \quad x \to 0^+. \qquad ∎$$

Appendix. Proof of Lemma 2.1. Let $\varepsilon_n > 0$ be such that

$$\varepsilon_n = o\left(\frac{1}{n}\right) \qquad \text{as} \quad n \to \infty,$$

and set

$$u_{0n}(x) = \begin{cases} \sqrt{u_0^2(x - \frac{1}{n}) + \varepsilon_n^2} & \text{if} \quad x > \dfrac{1}{n}, \\ \sqrt{(u_0^+)^2 + \varepsilon_n^2} & \text{if} \quad x = \dfrac{1}{n}, \end{cases}$$

where $u_0^+ = \lim_{x \searrow 0} u_0(x)$. Since $|u'_{0n}(x)| \le |u'_0(x - \frac{1}{n})|$ for $x > \frac{1}{n}$, the uniform Lipschitz continuity of u_0 in $\mathbb{R}^+$ implies

$$u_{0n} \text{ is uniform Lipschitz continuous in } [\tfrac{1}{n}, \infty].$$

Since

$$f_0 = u_0 - \frac{1}{2}\sqrt{k^+}(u_0^2)' \qquad \text{in} \quad \mathbb{R}^+,$$

$$f_{0n} = u_{0n} - \frac{1}{2}\sqrt{k^+}(u_{0n}^2)' \qquad \text{in} \quad \left[\frac{1}{n}, \infty\right),$$

the total variation of $(u_0^2)'$ in $\mathbb{R}^+$, $TV_{\mathbb{R}^+}((u_0^2)')$, is bounded, and since $(u_{0n}^2)'(x) = (u_0^2)'(x - \frac{1}{n})$,

$$TV_{(\frac{1}{n},\infty)}(f_{0n}) \to TV_{\mathbb{R}^+}(f_0) \qquad \text{as} \quad n \to \infty. \tag{A.1}$$

In order to extend u_{0n} to the interval $[-\frac{1}{n}, \infty]$ we distinguish two different cases: $u_0^+ > 0$ and $u_0^+ = 0$. At this point we remind the reader that the constant u^* is defined by

$$\frac{1 + u^*}{\sqrt{k^-}} = \frac{1}{\sqrt{k^+}}\ , \qquad \text{i.e.,} \quad u^* = \sqrt{\frac{k^-}{k^+}} - 1.$$

(i) *Case* $u_0^+ > 0$. We define u_{0n} in $[-\frac{1}{n}, \frac{1}{n})$ by the relation $p_{0n} = p_{0n}(\frac{1}{n})$ in $[-\frac{1}{n}, \frac{1}{n})$, i.e.,

$$u_{0n}(x) = -1 + \sqrt{\frac{k_n(x)}{k^+}}\left(1 + \sqrt{(u_0^+)^2 + \varepsilon_n^2}\right).$$

In particular, as $n \to \infty$,

$$\begin{aligned} u_{0n}(-\tfrac{1}{n}) &= -1 + \sqrt{\frac{k^-}{k^+}}\left(1 + \sqrt{(u_0^+)^2 + \varepsilon_n^2}\right) \\ &\to -1 + \sqrt{\frac{k^-}{k^+}}(1 + u_0^+) = u_0^-, \end{aligned} \tag{A.2}$$

where we have used, by hypothesis (H), $[p_0] = 0$ if $u_0^+ > 0$. Since $u_{0n}(-\frac{1}{n}) > u_0^-$, there exist $\delta_n > 0$ such that

$$u_{0n}\left(-\frac{1}{n}\right) = \sqrt{(u_0^-)^2 + \delta_n^2}. \tag{A.3}$$

It follows directly from the construction of u_{0n} that

$$TV_{(-\frac{1}{n},\frac{1}{n})}(f_{0n}) = -u_{0n}\Big(\frac{1}{n}\Big) + u_{0n}\Big(-\frac{1}{n}\Big) \to -[u_0] \quad \text{as} \quad n\to\infty \tag{A.4}$$

and

$$f_{0n}\Big(\frac{1}{n}+\Big) - f_{0n}\Big(\frac{1}{n}-\Big) = -\frac{1}{2}\sqrt{k^+}(u_{0n}^2)'\Big(\frac{1}{n}+\Big) = -\frac{1}{2}\sqrt{k^+}(u_0^2)'(0^+). \tag{A.5}$$

(ii) *Case* $u_0^+ = 0$. Since $[p_0]\geq 0$, $u_0^+ = 0$ implies that

$$0 \leq u_0^- \leq -1+\sqrt{\frac{k^-}{k^+}} = u^*.$$

Hence

$$(1+\varepsilon_n)\sqrt{k^-} > \sqrt{k^-} \geq \sqrt{k^+}(1+u_0^-),$$

and there exist $\delta_n > 0$ such that

$$\begin{cases} \delta_n \to 0 \quad \text{as} \quad n\to\infty, \\ (1+\varepsilon_n)\sqrt{k^-} > \sqrt{k^+}\Big(1+\sqrt{(u_0^-)^2+\delta_n^2}\Big), \\ \sqrt{(u_0^-)^2+\delta_n^2} > \varepsilon_n. \end{cases}$$

These two inequalities imply that for some $\kappa_n \in (k^+, k^-)$

$$(1+\varepsilon_n)\sqrt{k^-} = \sqrt{\kappa_n}\left(1+\sqrt{(u_0^-)^2+\delta_n^2}\right).$$

Then there exists $x_n \in (-\frac{1}{n},\frac{1}{n})$ such that

$$k_n(x_n) = \kappa_n,$$

and we define u_{0n} in $[-\frac{1}{n},\frac{1}{n})$ by the relations

$$u_{0n}(x) \equiv u_{0n}\Big(\frac{1}{n}\Big) \quad (=\varepsilon_n) \qquad \text{if} \quad x_n \leq x < \frac{1}{n}$$

and

$$p_{0n}(x) \equiv p_{0n}(x_n) \quad \left(=\frac{1+\varepsilon_n}{\sqrt{\kappa_n}}\right) \qquad \text{if} \quad -\frac{1}{n} < x < x_n.$$

By the definition of κ_n and p_{0n}, the latter relation can be written as

$$u_{0n}(x) = -1+\sqrt{\frac{k_n(x)}{k^-}}\Big(1+\sqrt{(u_0^-)^2+\delta_n^2}\Big) \qquad \text{if} \quad -\frac{1}{n} \leq x < x_n.$$

In particular we have

$$u_{0n}' \leq 0 \text{ in } \Big(-\frac{1}{n}, x_n\Big) \text{ and } u_{0n}\Big(-\frac{1}{n}\Big) = \sqrt{(u_0^-)^2+\delta_n^2} \to u_0^- \qquad \text{as } n\to\infty, \tag{A.6}$$

and

$$TV_{(-\frac{1}{n},x_n)}(f_{0n}) = TV_{(-\frac{1}{n},x_n)}(u_{0n}) \to -[u_0] \qquad \text{as } n \to \infty. \tag{A.7}$$

Since $|k_n^-| \le \frac{\mathcal{C}}{n}$ and $\varepsilon_n = o(\frac{1}{n})$ as $n \to \infty$, and since

$$f_{0n}(x) = \varepsilon_n + \frac{1}{2}\varepsilon_n(1+\varepsilon_n)\frac{k_n'(x)}{\sqrt{k_n(x)}} \qquad \text{if} \quad x_n < x < \frac{1}{n},$$

it follows that

$$TV_{(x_n,\frac{1}{n})}(f_{0n}) \to 0 \qquad \text{as} \quad n \to \infty. \tag{A.8}$$

In addition, as $n \to \infty$,

$$f_{0n}\left(\frac{1}{n}+\right) - f_{0n}\left(\frac{1}{n}-\right) \to -\frac{1}{2}\sqrt{k^+}(u_0^2)'(0^+) \tag{A.9}$$

and

$$f_{0n}(x_n^+) - f_{0n}(x_n^-) = \frac{1}{2}\varepsilon_n(1+\varepsilon_n)\frac{k_n'(x_n)}{\sqrt{\kappa_n}} \to 0. \tag{A.10}$$

Combining (A.6)–(A.10) gives

$$TV_{(-\frac{1}{n},\frac{1}{n})}(f_{0n}) \to -[u_0] \qquad \text{as} \quad n \to \infty. \tag{A.11}$$

Finally we have to define $u_{0n}(x)$ for $x < -\frac{1}{n}$. In view of (A.3) and (A.6) it seems natural to set

$$u_{0n}(x) = \sqrt{u_0^2\left(x+\frac{1}{n}\right)+\delta_n^2} \qquad \text{if} \quad x < -\frac{1}{n}. \tag{A.12}$$

Arguing as in the interval $(\frac{1}{n},\infty)$, we obtain as $n \to \infty$

$$TV_{(-\infty,-\frac{1}{n})}(f_{0n}) \to TV_{\mathbb{R}^-}(f_0) \tag{A.13}$$

and

$$f_{0n}\left(\left(-\frac{1}{n}\right)+\right) - f_{0n}\left(\left(-\frac{1}{n}\right)-\right) \to \frac{1}{2}\sqrt{k^-}(u_0^2)'(0^-). \tag{A.14}$$

Combining (A.1), (A.13), and (A.14) with, respectively, (A.4), (A.5) if $u_0^+ > 0$ and (A.9), (A.11) if $u_0^+ = 0$, we find

$$TV_{\mathbb{R}}(f_{0n}) \to TV_{\mathbb{R}}(f_0) \quad \text{as} \quad n \to \infty.$$

Now, if $\delta_n = \varepsilon_n$, u_{0n} satisfies all properties of Lemma 2.1. In general, however, $\delta_n \ne \varepsilon_n$ and we have to correct the construction of u_{0n} in $(-\infty, -\frac{1}{n})$. Since $u_{0n}(-\frac{1}{n}) > u_{0n}(\frac{1}{n}) \ge \varepsilon_n$, we can still use definition (A.12) in a neighborhood of $x = -\frac{1}{n}$. Since k_n is constant in $(-\infty, -\frac{1}{n})$, the expression for the flux is simply

$$f_{0n} = u_{0n} - \frac{1}{2}\sqrt{k^-}(u_{0n}^2)' \qquad \text{in} \quad \left(-\infty, -\frac{1}{n}\right).$$

Therefore it is not difficult to change slightly the definition of u_{0n} such that $u_{0n} \ge \varepsilon_n$ in $\mathbb{R}$ and $u_{0n}(x) = \varepsilon_n$ for $-x$ sufficiently large. We leave the details to the reader.

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