ANALYSIS OF OIL TRAPPING IN POROUS MEDIA FLOW

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Abstract. We analyze a one-dimensional nonlinear convection-diffusion equation describing the flow of water and oil through a porous medium composed of two types of rock with different permeability. We prove existence, uniqueness, and regularity properties, as well as matching conditions between the two rock types.

Key words. degenerate parabolic equation, porous media flow, existence, uniqueness, qualitative properties, matching conditions

AMS subject classifications. Primary, 35K65; Secondary, 35B05, 35K10, 35K55, 76S05

DOI. 10.1137/S0036141002407375

1. Introduction and problem formulation. It is well known that capillary forces, combined with spatial variations of rock properties, considerably reduce the recovery factor of an oil reservoir. For instance, it is difficult to remove oil from parts of the reservoir with small scale heterogeneities. Sometimes the oil may even remain trapped; see, for instance, [K, W]. This is clearly a difficult problem, mainly due to the complex nature of rock (soil) heterogeneities.

To understand oil trapping in heterogeneous media more quantitatively, [DMN] considered the case of a 2-phase water-oil flow which is perpendicular to an interface, separating two types of rock, across which the permeability changes abruptly. Under simplifying assumptions this leads to a one-dimensional flow problem which allowed them to investigate the role of convection and capillary diffusion in relation to the discontinuous permeability. They used formal asymptotics and numerical techniques. In this paper we will take their formulation as a starting point. The aim is to analyze the structure of the model equations resulting in existence, uniqueness, and regularity properties, as well as matching conditions between the two rock types.

Following [DMN] (further references are given there), the one-dimensional flow of water and oil through a porous medium is described by a nonlinear convection-diffusion equation for the reduced water saturation

\[ S = S(x,t), \quad 0 \leq S \leq 1. \]

This equation has the form

\[ \Phi \frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \left\{ q f_w(S) + k(x) H(S) \frac{\partial p}{\partial x} \right\} = 0, \]

where \( \Phi \) (porosity) and \( q \) (discharge) are positive constants, and where the functions

\[ f_w, \quad H : [0,1] \to [0,\infty) \]

satisfy \( f_w(0) = 0, \quad f_w(S) > 0 \) for \( 0 < S < 1 \) (typically convex-concave behavior) and \( H(0) = H(1) = 0, \quad H(S) > 0 \) for \( 0 < S < 1 \). Further \( k(x) \) denotes permeability and \( p \) capillary pressure. Situating the discontinuity in

*Received by the editors May 9, 2002; accepted for publication December 13, 2002; published electronically July 8, 2003.
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Fig. 1. Capillary pressure curves for fine (+) and coarse (−) material. Here $J(1) > 0$, so an entry pressure exists.

Permeability at $x = 0$, we have

$$k(x) = \begin{cases} k^- & \text{for } x < 0, \\ k^+ & \text{for } x > 0. \end{cases}$$  \hspace{1cm} (1.2)

Without loss of generality we take $0 < k^+ < k^- < \infty$. This means that coarse material occupies $\{x < 0\}$ and fine material $\{x > 0\}$. The flow is in positive $x$-direction.

For the capillary pressure the Leverett model [L] was used. With $\sigma > 0$ denoting interfacial tension, this means

$$p = p(x, S) = \sigma \frac{J(S)}{\sqrt{k(x)/\Phi}} \quad \text{for} \quad 0 < S \leq 1,$$  \hspace{1cm} (1.3)

where the Leverett function $J$ is strictly decreasing in $(0, 1]$ with $J(1) \geq 0$. The quantity $\sqrt{k/\Phi}$ may be associated with the mean pore diameter, and the $J$-Leverett function is typical for the lithology of the porous medium. When $J(1) > 0$, the medium has an entry pressure given by $J(1)/\sqrt{k/\Phi}$. This is the minimum pressure needed for the oil to enter a medium that is saturated by water. In this paper we assume $J(1) > 0$ and show that the occurrence of an entry pressure causes trapping of oil at the interface when the medium changes from coarse to fine. Figure 1 shows two typical capillary pressure functions, the top curve for fine material ($x > 0$), the bottom curve for coarse material ($x < 0$).

Because $k$ is discontinuous, the capillary pressure may be discontinuous as well. This makes the interpretation of (1.1) across $x = 0$ difficult. To circumvent this problem, [DMN] considered (1.1) for $x < 0$ and $x > 0$, with matching conditions at
$x = 0$. One condition is obvious. Conservation of mass across $x = 0$ requires that the fluxes to the left and right of $x = 0$ be equal:

$$\left( qf_w + k^- H \frac{\partial p}{\partial x} \right)_{x=0^-} = \left( qf_w + k^+ H \frac{\partial p}{\partial x} \right)_{x=0^+}$$

for all $t > 0$. A condition related to the pressure was obtained by a formal regularization procedure. Replacing in (1.1) $k(x)$ by $C^\infty$ approximations $k_n(x)$, according to

$$k_n(x) = \begin{cases} 
  k^- & \text{for } x \leq -\frac{1}{n}, \\
  \varphi(nx) & \text{for } -\frac{1}{n} < x < \frac{1}{n}, \\
  k^+ & \text{for } x \geq \frac{1}{n},
\end{cases}$$

with $\varphi$ smooth ($\varphi(-1) = k^-$, $\varphi(1) = k^+$, and $\varphi' \leq 0$), blowing up the transition region by $x \to nx$ and letting $n \to \infty$, we found the following. Let $S^*$ be defined by the relation

$$\frac{J(S^*)}{\sqrt{k^-}} = \frac{J(1)}{\sqrt{k^+}} > 0,$$

and let $S^-$ and $S^+$ denote, respectively, the left and right limits of $S$ at $x = 0$. Then for all $t > 0$ (see also Figure 1),

$$\left( \frac{J(S^-)}{\sqrt{k^-}} = \frac{J(S^+)}{\sqrt{k^+}} \right. \left. \text{if } S^- \leq S^* \text{ (pressure continuous)}, \right)$$

$$S^+ = 1 \text{ if } S^- > S^* \text{ (positive pressure jump)}.$$

Instead of analyzing (1.1) and conditions (M1-2) in the form presented above, we shall consider a further simplified model problem, without losing essential characteristic features. We take in (1.1)

$$f(S) = S, \quad H(S) = 1 - S, \quad \text{and} \quad J(S) = 2 - S.$$ 

After a trivial scaling, the following equations result for the oil saturation $u = 1 - S$:

$$\begin{align*}
  &u_t + f_x = 0 \quad (u \geq 0), \\
  &f = u - N_c k u p_x, \\
  &p = \frac{1 + u}{\sqrt{k(x)}},
\end{align*}$$

where $f$ denotes the flux and $N_c$ the dimensionless capillary number

$$N_c = \frac{\sigma \sqrt{K \phi}}{q \mu_w L}.$$ 

Here $K$ is a characteristic $k$-value, $L$ a characteristic length scale, and $\mu_w$ the water viscosity. By an additional scaling we may set $N_c = 1$. Further, $k$ is given by (1.2) and the subscripts $t$ and $x$ denote partial differentiation.
Fig. 2. Transformed capillary pressures.

We solve (1.6)–(1.8) in the subdomains

$$Q^\pm = \{(x, t) : x \in \mathbb{R}^\pm, \ t \in (0, \infty)\},$$

with transformed matching conditions at \( x = 0 \). These are

\[(M_1)\]

\[
\begin{cases}
1 + u^- = \frac{1 + u^+}{\sqrt{k^+}} & \text{if } u^- \geq u^* \\
u^+ = 0 & \text{if } u^- < u^*
\end{cases}
\]

in \((0, \infty)\),

and (see Figure 2)

\[(M_2)\]

\[u^+[p] = 0, \quad [p] \geq 0 \text{ in } (0, \infty).\]

Here \( u^* = \sqrt{\frac{k^+}{k^-}} - 1 \). As before, \( u^\pm = u^\pm(t) = u(0\pm, t) \), \([u] = u^+ - u^-\), and \( f \) and \( p \) have similar notation.

At \( t = 0 \) we prescribe

\[(1.9)\]

\[u(\cdot, 0) = u_0(\cdot) \text{ in } \mathbb{R},\]

with \( u_0 \) satisfying

\[(H)\]

\[
\begin{cases}
u_0 : \mathbb{R} \to [0, \infty), \quad \text{supp}(u_0) \subset \mathbb{R} \text{ is bounded}; \\
u_0 \text{ uniformly Lipschitz continuous in } \mathbb{R}\setminus\{0\}; \\
u_0^+[p_0] = 0, \quad f_0 := u_0 - \frac{\sqrt{k}}{2}(u_0^2)' \in BV(\mathbb{R}\setminus\{0\}).
\end{cases}
\]
The pressure condition at $t = 0$ is needed to construct an approximate sequence $\{u_{0n}\}$ for which the corresponding fluxes $f_{0n} := u_{0n} - k_n u_{0n} (p_{0n})'$ are uniformly bounded in $BV(\mathbb{R})$. This in turn will imply $f \in L^\infty((0, \infty); BV(\mathbb{R}))$, which is a crucial point in the existence proof. If the $k_n$ are taken as in (1.4), then $[p_0] \geq 0$ is needed as well. We will return to this in section 2 and in the appendix.

For steady state solutions, the role of $(M_2)$ can be seen explicitly. Assume $u = u(x)$ only, with $u(-\infty) = u(+\infty) = 0$. Then

$$f = u - k u p' = 0 \quad \text{in} \quad \mathbb{R}\{0\}.$$  

Using $u \geq 0$, we obtain

$$u(x) = 0 \quad \text{for} \quad x > 0.$$  

Hence the first condition in $(M_2)$ is always satisfied. Given any $u^- \geq 0$, we see that

$$u(x) = \left( u^- + \frac{1}{\sqrt{k^-}} x \right)_+$$  

satisfies (1.10) for $x < 0$. Here $(\cdot)_+ := \max\{\cdot, 0\}$. However, only for $u^- \in [0, u^*)$ we have $[p] \geq 0$. Thus we have a family of admissible steady state solutions, as shown in Figure 3.

Integrating the maximal steady state gives the maximal amount of oil that can be trapped to the left of the permeability discontinuity. It is given by

$$\bar{M} = \frac{1}{2} (u^*)^2 \sqrt{k^-}.$$  

Next we give the weak formulation of the trapping problem. Because the flux is expected to be continuous across $x = 0$, it will be defined globally in the formulation.
The saturation (and pressure) will be considered in the subdomains $Q^-$ and $Q^+$ separately. Let

$$Q^0 := Q^- \cup Q^+ \quad \text{and} \quad Q := \mathbb{R} \times (0, \infty).$$

Combining the saturation equations and the matching conditions gives the following.

**Problem P.** Find $u : Q^0 \rightarrow [0, \infty)$, $f : Q \rightarrow \mathbb{R}$ such that

(i) $u_x \in L^\infty(Q^0)$; $u$ is uniformly continuous in $Q^0$;

(ii) $f \in L^\infty((0, \infty); BV(\mathbb{R}))$;

(iii) $f = u - \frac{\sqrt{k^+}}{2} (u^2)_x$ a.e. in $Q^0$ and $\int_Q (u\zeta_x + f\zeta_x)dxdt + \int_\mathbb{R} u_0(x)\zeta(x,0)dx = 0$

for all $\zeta \in H^1(Q) \cap C(\bar{Q})$, with compact support in $Q$;

(iv) $u^+ [p] = 0$ and $[p] \geq 0$ in $(0, \infty)$, where $p := \frac{1 + u^+}{\sqrt{k^+}}$ in $Q^0$.

To prove existence we apply a $k$-regularization as in (1.4). This yields a sequence of approximating problems on $Q$ for which we derive the necessary estimates. This is done in section 2. In section 3 we consider the limit $n \rightarrow \infty$ giving existence for Problem P, with $u$ satisfying a porous media equation $(m = 2)$ with linear convection in $Q^0$. Clearly $(M_2')$ is satisfied. The weak equation in (iii) implies $f = 0$ a.e. in $(0, \infty)$. The comparison principle, with uniqueness as a consequence, is shown in section 4. In section 5 we give sufficient conditions for oil trapping; i.e., conditions that imply $u(x,t) = 0$ for $x > 0$ and for all $t > 0$. Finally, in section 6, we present some closing remarks about nonuniqueness, waiting times, and optimal regularity.

In a recent paper [DMP] considered oil transport in a multilayered porous medium. This work involves a discontinuous permeability which varies periodically in space. Using homogenization techniques they derived effective (upscaled) transport equations for the case where the periodicity length is small compared to the characteristic length $L$. In their analysis matching conditions ($\tilde{M}_1$) and ($\tilde{M}_2$) play a crucial role. They lead to a macroscopic irreducible oil saturation.

2. The approximate problem. In this section we study the approximate equation in which $k$ is replaced by the smooth function $k_n$, defined by (1.4). Together with $k$ we also need to approximate the initial value $u_0$. We construct approximations $u_{0n}$, so that the corresponding fluxes

$$f_{0n} := u_{0n} - k_n u_{0n} p_{0n}' , \quad p_{0n} := \frac{1 + u_{0n}}{\sqrt{k_n}} \tag{2.1}$$

have a uniformly bounded total variation. In addition we require that each $u_{0n}$ is strictly positive to eliminate the degeneracy of the equation at points where $u$ vanishes. The existence of such $u_{0n}$ is given in the following lemma. Since the proof is quite technical, it is given in the appendix.

**Lemma 2.1.** Let $n \in \mathbb{N}$ and let $k_n$ be defined by (1.4). Suppose $u_0$ satisfies hypothesis (H) and in addition

$$[p_0] = \frac{1 + u_0^+}{\sqrt{k^+}} - \frac{1 + u_0^-}{\sqrt{k^-}} \geq 0. \tag{2.2}$$

Then there exist $u_{0n} \in W^{1,\infty}(\mathbb{R})$ and $\varepsilon_n \in \mathbb{R}^+$ such that

(i) $u_{0n} \geq \varepsilon_n > 0$ in $\mathbb{R}$, and $u_{0n}(x) = \varepsilon_n$ for $|x|$ sufficiently large;

(ii) $u_{0n}$ is uniformly bounded in $\mathbb{R}$, and $f_{0n}$, defined by (2.1), is uniformly bounded in $BV(\mathbb{R})$;
As $n \to \infty$,

$$u_{on} \to u_0 \quad \text{uniformly in } \mathbb{R}\setminus\{0\}$$

and

$$u_{on} - \varepsilon_n \to u_0 \quad \text{in } L^1(\mathbb{R}).$$

For each $n \in \mathbb{N}$ we consider the approximate problem

$$(P_n) \begin{cases} u_t + u_x = (k_n u p_x)_x, & p = \frac{1 + u}{\sqrt{k_n}} \quad \text{in } Q, \\ u(x,0) = u_{on}(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

In the remainder of this section we prove the following results.

**Theorem 2.2.** Let $u_{on}$ be given by Lemma 2.1. Then problem $(P_n)$ has a solution $u_n \in C^\infty(Q) \cap C(\overline{Q})$ such that

(i) $0 < u_n \leq C$ in $Q$, where $C$ does not depend on $n$;

(ii) $f_n := u_{on} - k_n u_n \left(\frac{1 + u_n}{\sqrt{k_n}}\right)_x$ is uniformly bounded in $L^\infty([0,\infty); BV(\mathbb{R}))$;

(iii) $u_n$ is uniformly continuous in $\{(\mathbb{R}\setminus(-\varepsilon,\varepsilon)) \times [0,\infty)\}$ for all $\varepsilon > 0$.

**Proof.** Since $u_{on} \geq \varepsilon_n > 0$ in $\mathbb{R}$, problem $(P_n)$ is nondegenerate at $t = 0$. Hence it has a unique local (with respect to $t$) classical solution $u_n$; see, for instance, [LSU] and [F]. This solution can be continued as long as it remains bounded and bounded away from zero. Let $Q_{T_n} := \mathbb{R} \times (0, T_n)$ denote the maximal existence domain for $u_n$.

A positive lower bound follows from the maximum principle. Indeed, if we set $L_n := \max |(\sqrt{k_n})''|$ we observe that the solution of the initial value problem

$$(LB) \begin{cases} s' = -L_n s(1 + s) & \text{for } t > 0, \\ s(0) = \varepsilon_n \end{cases}$$

is a subsolution for problem $(P_n)$. Hence if $s_n$ denotes the solution of $(LB)$, we have

$$u_n(x,t) \geq s_n(t) > 0 \quad \text{for } (x,t) \in Q_{T_n}. \quad (2.3)$$

Before proving a uniform upper bound for $u_n$, we observe that the flux $f_n$ is uniformly bounded in $Q_{T_n}$. A straightforward calculation yields for $f_n$ the linear equation

$$f_t = a_n f_{xx} + b_n f_x, \quad (2.4)$$

where

$$a_n(x,t) := u_n \sqrt{k_n}, \quad b_n(x,t) := \frac{f_n}{u_n} - \frac{u_n k_n'}{2 \sqrt{k_n}}. \quad (2.5)$$

Hence, by the maximum principle

$$||f_n||_{L^\infty(Q_{T_n})} \leq ||f_{0n}||_{L^\infty(\mathbb{R})} \leq C \quad (2.6)$$

for all $n \in \mathbb{N}$.

We use this estimate to demonstrate a uniform upper bound for $u_n$ in $Q_{T_n}$. As a first observation we note that (2.6) implies the differential inequality

$$|u_n - \sqrt{k_n} u_n u_{nx}| \leq C \quad \text{in } \left(-\infty, -\frac{1}{n}\right) \times [0, T_n). \quad (2.7)$$
Then the upper bound for \( u_n \) in this set is immediate if we can control the decay of \( u_n \) as \( x \to -\infty \). This decay results from the following argument.

Let \( \bar u_n \) be a steady state solution satisfying

\[
\begin{align*}
    u - k_n u' &= \varepsilon_n, \\
    p &= \frac{1 + u}{\sqrt{k_n}} \quad \text{in } \mathbb{R}, \\
    u(\pm \infty) &= \varepsilon_n.
\end{align*}
\]

Clearly, \( \bar u_n(x) = \varepsilon_n \) for all \( x \geq \frac{1}{n} \). The corresponding pressure \( \bar p_n \) satisfies

\[
\begin{align*}
    k_n(p\sqrt{k_n} - 1)p' &= p\sqrt{k_n} - 1 - \varepsilon_n \quad \text{for } x < \frac{1}{n}, \\
    p(\frac{1}{n}) &= \frac{1 + \varepsilon_n}{\sqrt{k_n}}.
\end{align*}
\]

At points where \( \bar p_n' = 0 \), we must have \( \bar p_n > 0 \) and \( \bar p_n'' < 0 \). We use this to obtain \( \bar p_n' > 0 \) and \( \bar p_n > \frac{1 + \varepsilon_0}{\sqrt{k_n}} \) on \((-\infty, \frac{1}{n})\), and \( \bar p_n(x) \to \frac{1 + \varepsilon_n}{\sqrt{k_n}} \) as \( x \to -\infty \). In particular, \( \bar u_n(x) \to \varepsilon_n \) exponentially as \( x \to -\infty \) and \( \bar u_n - \varepsilon_n \in L^1(\mathbb{R}) \), uniformly in \( n \in \mathbb{N} \).

Now using Lemma 2.1(iii) and an argument as in the proof of Theorem 4.1, one finds for \( t > 0 \) the \( L^1 \)-contraction

\[
\int_{\mathbb{R}} |u_n(x, t) - \bar u_n(x)| \, dx \leq \int_{\mathbb{R}} |u_0(x) - \bar u_n(x)| \, dx.
\]

This inequality controls the behavior of \( u_n \) as \( |x| \to \infty \). Combined with (2.7) it gives the upper bound in \((-\infty, -\frac{1}{n}) \times (0, T_n)\). Arguing similarly for \( x > \frac{1}{n} \), we conclude that for all \( n \in \mathbb{Z}^+ \)

\[
(2.8) \quad u_n(x, t) \leq C \quad \text{for } |x| \geq \frac{1}{n}, \quad 0 \leq t < T_n.
\]

To obtain the upper bound in the remaining strip \([-\frac{1}{n}, \frac{1}{n}] \times (0, T_n)\) we express (2.6) in terms of the pressure \( p_n \):

\[
(2.9) \quad |p_n\sqrt{k_n} - 1 - k_n(p_n\sqrt{k_n} - 1)p_{nxx}| \leq C.
\]

By (2.8), \( p_n(\pm \frac{1}{n}, t) \) is uniformly bounded. Then (2.9) implies that \( p_n \), and thus \( u_n \), is uniformly bounded as well.

The uniform upper bound, together with lower bound (2.3), guarantees existence for all \( t > 0 \). Hence, \( T_n = \infty \) for each \( n \in \mathbb{N} \). This completes the proof of (i).

The proof of (ii) is a direct consequence of Lemma 2.1(ii) and the total variation estimate for the flux in Lemma 2.4 below.

We conclude by proving (iii). The boundedness of \( u_n \) and the flux estimate (2.7) imply that \( u_n \) is uniformly Hölder continuous (exponent \( \frac{1}{2} \)) with respect to \( x \) in \( \{(x, t) : x < -\frac{1}{n}, \ t > 0\} \). The same result holds in \( \{(x, t) : x > \frac{1}{n}, \ t > 0\} \). The smoothness and boundedness of the coefficients in the \( u_n \)-equation allow us to apply [G1], yielding that \( u_n \) is uniformly Hölder continuous (exponent \( \frac{1}{2} \)) with respect to \( t \) in \( \{(x, t) : |x| > \frac{1}{n}, \ t > 0\} \). Since, for fixed \( \varepsilon > 0 \), \( \frac{1}{n} < \varepsilon \) for \( n \) large enough, this proves (iii) and completes the proof of Theorem 2.2.

\[\Box\]

**Remark 2.3.** It is not difficult to show that the steady states \( \bar u_n \), corresponding to \( k = k_n \) and \( \bar u_n(\pm \infty) = \varepsilon_n \), approximate the maximal steady state in Figure 3. In essence this follows from \( \bar u_n(x) = \varepsilon_n \) for all \( x \geq \frac{1}{n} \) and, using the pressure equation,

\[
0 < \frac{1}{n} \frac{1}{k_n} \int_{-\frac{1}{n}}^{+\frac{1}{n}} \bar p_n(x)\sqrt{k_n} - 1 - \varepsilon_n \, dx \to 0
\]
as \( n \to \infty \).

It remains to prove the following lemma used in the proof of Theorem 2.2.

**Lemma 2.4.** Let \( u_{0n} \) be given by Lemma 2.1 and let \( u_n \) be the corresponding solution of problem \((P_n)\). Then

\[
TV_{\mathbb{R}}(f_n(t)) \leq TV_{\mathbb{R}}(f_{0n}) \quad \text{for all} \quad t > 0.
\]

**Proof.** Each flux \( f_n \) satisfies the linear problem

\[
\begin{aligned}
f_t &= a_n f_{xx} + b_n f_x \\
\{(f,x,0) &= f_{0n}(x) \quad \text{for} \quad x \in \mathbb{R},
\end{aligned}
\]

where \( a_n \) and \( b_n \), defined in (2.5), are bounded functions and where \( f_{0n} \) has uniformly bounded variation. First we proceed formally. Let us fix \( \varepsilon > 0 \) and calculate (dropping the subscript \( n \))

\[
\frac{d}{dt} \int_{\mathbb{R}} \left\{ \sqrt{f_x^2 + \varepsilon} - \sqrt{\varepsilon} \right\} = \int_{\mathbb{R}} \frac{f_x}{\sqrt{f_x^2 + \varepsilon}} (af_{xx} + bf_x)_x
\]

\[
= -\varepsilon \int_{\mathbb{R}} \frac{f_{xx}(af_{xx} + bf_x)}{(f_x^2 + \varepsilon)^{3/2}}.
\]

Integrating in time gives, for any \( t > 0 \),

\[
\int_{\mathbb{R}} \left\{ \sqrt{f_x^2(t) + \varepsilon} - \sqrt{\varepsilon} \right\} - \int_{\mathbb{R}} \left\{ \sqrt{f_{0n}^2 + \varepsilon} - \sqrt{\varepsilon} \right\} = -\varepsilon \int_{\mathbb{R} \times (0,t)} \frac{af_{xx}^2 + bf_x f_{xx}}{(f_x^2 + \varepsilon)^{3/2}}
\]

\[
- \varepsilon \int_{\mathbb{R} \times (0,t)} \frac{f_x f_{xx}}{(f_x^2 + \varepsilon)^{3/2}} b f_{xx}.
\]

Since

\[
\left| \frac{\varepsilon f_x}{(f_x^2 + \varepsilon)^{3/2}} \right| \leq 1
\]

and

\[
\frac{\varepsilon f_x}{(f_x^2 + \varepsilon)^{3/2}} \to 0, \quad \text{pointwise in} \quad Q \quad \text{as} \quad \varepsilon \to 0,
\]

the boundedness of \( b \) and Lebesgue’s dominated convergence theorem imply

\[
\int_{\mathbb{R}} |f_x(t)| \leq \int_{\mathbb{R}} |f'_{0n}|,
\]

provided \( f_{xx} \in L^1(\mathbb{R} \times (0,t)) \). To complete the proof of the lemma we need to make this argument rigorous.

It is enough to apply a mollifier to the initial function \( f_{0n} \) of the linear flux problem. This ensures the smoothness up to \( t = 0 \) necessary to carry out the above calculations. \( \square \)
3. Existence for Problem $P$. Let $u_n$ be the solution of problem $(P_n)$ as stated in Theorem 2.2. By a standard argument there exist a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, and $u \in L^\infty(Q) \cap C((\mathbb{R}^- \cup \mathbb{R}^+) \times [0, \infty))$ such that

$$u_n \rightarrow u \quad \text{in} \quad C_{\text{loc}}((\mathbb{R}^- \cup \mathbb{R}^+) \times [0, \infty))$$

as $n \rightarrow \infty$. We show the following theorem.

**Theorem 3.1.** $u$ is a solution of Problem $P$.

**Proof.** Clearly $u$ is a (weak) solution of the equation

$$u_t + u_x = \frac{1}{2} \sqrt{k^\pm (u^2)_{xx}} \quad \text{in} \quad Q^\pm$$

and

$$f = u - \frac{1}{2} \sqrt{k^\pm (u^2)} \in L^\infty([0, \infty); \ BV(\mathbb{R}^\pm)).$$

The boundedness of $u$ and $f$ implies that $u^2$ is uniformly Lipschitz continuous with respect to $x$ in $Q^0$. Hence the following quantities are well defined for each $t > 0$:

$$u^\pm(t), \quad f^\pm(t), \quad \text{and} \quad p^\pm(t) = \frac{1 + u^\pm(t)}{\sqrt{k^\pm}}.$$

Using the equation

$$u_t + f_x = 0 \quad \text{a.e. in} \quad Q^\pm$$

and again the boundedness of $f$, we obtain as in [DP] that the functions

$$t \rightarrow u^\pm(t)$$

are continuous in $[0, \infty)$.

Next we claim

$$f^+(t) = f^-(t) \quad \text{for almost all} \quad t > 0.$$  \hspace{1cm} (3.1)

Indeed, using the asymptotic behavior of $u_n(x, t)$ as $|x| \rightarrow \infty$, we find, for $n \rightarrow \infty$,

$$\int_R (u_n(x, t) - \varepsilon_n) dx = \int_R (u_0n(x) - \varepsilon_n) dx \rightarrow \int_R u_0(x) dx$$

and hence

$$\int_R u(x, t) dx = \int_R u_0(x) dx \quad \text{for all} \quad t > 0,$$

which expresses conservation of mass. This identity implies

$$0 = \lim_{\delta \rightarrow 0^+} \left( \int_{-\infty}^{-\delta} u(x, t) dx + \int_{\delta}^{\infty} u(x, t) dx - \int_{-\infty}^{-\delta} u_0(x) dx - \int_{\delta}^{\infty} u_0(x) dx \right)$$

$$= \int_0^t (f^+(s) - f^-(s)) ds \quad \text{for all} \quad t > 0.$$  \hspace{1cm} (3.1)

Together with the equations in $Q^\pm$, equality (3.1) implies the weak form (iii) of Problem $P$. 

It remains to prove

\begin{equation}
(3.2) \quad u^+[p] = 0 \quad \text{and} \quad [p] \geq 0 \quad \text{for all} \quad t > 0.
\end{equation}

For this purpose we study \( u_n \) and \( p_n \) in the interval \((-\frac{1}{n}, \frac{1}{n})\). Since \( k_n \) changes rapidly there, we make the blow-up

\[ y = nx \quad \text{for} \quad -\frac{1}{n} < x < \frac{1}{n}. \]

Knowing that the fluxes \( f_n \) are uniformly bounded, we obtain

\[ |u_n - nk_n u_n(p_n)y| \leq C. \]

Thus for appropriate \( C > 0 \) we have

\[ |u_n(p_n)y| \leq \frac{C}{n}, \]

or

\begin{equation}
(3.3) \quad \left|(u^2_n)_y - \frac{\varphi'}{\varphi} u_n(1 + u_n)\right| \leq \frac{C}{n}
\end{equation}

for all \(-1 < y < 1 \) and \( t > 0 \).

Hence \( u^2_n \) are Lipschitz continuous in \([-1, 1]\) uniformly with respect to \( n \) and \( t \). Up to a subsequence, \( u_n \to u \) uniformly in \([-1, 1]\), as \( n \to \infty \), for all \( t > 0 \); in particular \( u(-1,t) = u^-(t) \) and \( u(1,t) = u^+(t) \). In addition, it follows easily from (3.3) that \( u \) satisfies

\begin{equation}
(3.4) \quad (u^2)_y = \frac{\varphi'}{\varphi} u(1 + u)
\end{equation}

in \( \{(y,t) : -1 < y < 1, \ t > 0\} \). Since \( \varphi' \) is nonpositive, \( u \) is decreasing. Thus, if \( u^+(t) > 0 \), we have \( u(y,t) > 0 \) in \([-1, 1]\) and (3.4) reduces to

\begin{equation}
(3.5) \quad u_y = \frac{\varphi'}{2\varphi}(1 + u).
\end{equation}

A straightforward calculation gives \([p] = 0\).

Next suppose \( u^+(t) = 0 \). We have to show \([p] \geq 0\). If \( u^-(t) = 0 \), we get

\[ [p] = \frac{1}{\sqrt{k^+}} - \frac{1}{\sqrt{k^-}} > 0. \]

If \( u^-(t) > 0 \), define \( \bar{y} := \sup\{y \in [-1, 1] : u(y, t) > 0\} \) and solve (3.5) in \([-1, \bar{y}]\). This gives

\[ \frac{1 + u^-}{\sqrt{k^-}} = \frac{1}{\sqrt{\varphi(\bar{y})}} \leq \frac{1}{\sqrt{k^+}}, \]

which implies \([p] \geq 0\) and \( u^-(t) \leq u^* \).
4. The comparison principle. We start with some preliminary observations for solutions \((u, \rho)\) of Problem P. Choosing test functions with support in \(Q^\pm\), we obtain

\[
\int_{Q^\pm} u \zeta_t + \int_{Q^\pm} \left( u - \frac{\sqrt{\mathcal{F}}}{2} (u^2)_x \right) \zeta_x = 0.
\]

Thus away from \(x = 0\) we have two weak equations of “porous media” type \((m = 2)\) with linear convection, implying

\[
u_t + \left( u - \frac{\sqrt{\mathcal{F}}}{2} (u^2)_x \right)_x = 0 \quad \text{a.e. in } Q^\pm
\]

and

\[
supp(u(t)) \text{ is bounded in } \mathbb{R}
\]

for all \(t \in [0, \infty)\). Further, using hypothesis (H), we can apply the Bernstein argument of [A] in the truncated domain

\[
Q^\delta := \mathbb{R} \setminus (-\delta, \delta) \times (0, \infty) \quad (\text{for } \delta > 0, \text{ fixed})
\]

to obtain

\[
\|u_x\|_{L^\infty(Q^\delta)} \leq C(\delta).
\]

We use this to derive an estimate on \(u_t\) in \(Q^\delta\). Let \(u\) be a smooth solution of (4.1) in the sense of the usual “porous media” approximations, and let \(\xi : \mathbb{R} \to [0, 1]\) be an even \(C^1\) cut-off function satisfying

\[
\xi(x) = \begin{cases} 
0 & \text{for } 0 \leq x \leq \delta/2, \\
1 & \text{for } \delta \leq x \leq L, \\
0 & \text{for } x \geq L + 1 
\end{cases}
\]

for any \(L > \delta\). Multiplying (4.1) by \(\xi^2 u_t\) gives

\[
\int_{Q_\tau} \xi^2 u_t^2 = - \int_{Q_\tau} \xi^2 u_t u_x - \int_{Q_\tau} \xi \sqrt{\mathcal{F}} u_t (u^2)_x - \int_{Q_\tau} \frac{\sqrt{\mathcal{F}}}{2} u_{xt} (u^2)_x,
\]

where \(Q_\tau = \mathbb{R} \times (0, \tau)\) with \(\tau > 0\) arbitrarily chosen. Using \(u_{xt}(u^2)_x = u(u^2)_t\), the last integral becomes

\[
\int_{\mathbb{R}} \frac{\xi^2 \sqrt{\mathcal{F}}}{2} u u_x^2 \bigg|_0^\tau - \int_{Q_\tau} \frac{\xi^2 \sqrt{\mathcal{F}}}{2} u_t u_x^2.
\]

Then (i) of Problem P and (4.3) in \(Q^{\delta/2}\) give

\[
\int_{Q_\tau} \xi^2 u_t^2 \leq C(\delta, \tau),
\]

implying

\[
u_t \in L^2_{\text{loc}}(\tilde{Q}^{\delta}).
\]
We are now in a position to prove the following theorem.

**Theorem 4.1.** Let \((u_1, f_1)\) and \((u_2, f_2)\) be weak solutions of Problem P corresponding to initial values \(u_{01}\) and \(u_{02}\), respectively. Then \(u_{01} \leq u_{02}\) in \(\mathbb{R}\) implies \(u_1 \leq u_2\) in \(Q^0\).

**Proof.** Let \(\tau > 0\) be arbitrary. In the weak equation for the difference
\[
\int_Q \{(u_1 - u_2)\zeta_t + (f_1 - f_2)\zeta_x\} + \int_{\mathbb{R}} \zeta(u_{01} - u_{02}) = 0,
\]
we take the test function
\[
\zeta = \xi \psi S_\varepsilon(u_1^2 - u_2^2),
\]
where the following hold:

(i) \(\xi\) is an even \(C^1\) cut-off function near \(x = 0\),
\[
\xi(x) = \begin{cases} 
0 & \text{for } 0 \leq x \leq \delta/2, \\
1 & \text{for } x \geq \delta, 
\end{cases} \quad \xi'(x) \geq 0 \text{ for } \delta/2 < x < \delta.
\]

(ii) \(\psi\) is a \(C^1\) cut-off function near \(t = \tau\),
\[
\psi(t) = \begin{cases} 
1 & \text{for } 0 \leq t \leq \tau - \mu, \\
0 & \text{for } \tau \leq t,
\end{cases} \quad \psi'(t) \leq 0 \text{ for } \tau - \mu < t < \tau.
\]

(iii) \(S_\varepsilon : \mathbb{R} \to [0, 1]\) is given by
\[
S_\varepsilon(r) = \begin{cases} 
0, & r \leq 0, \\
\frac{r}{\sqrt{r^2 + \varepsilon^2}}, & r > 0.
\end{cases}
\]

Here \(\delta, \mu,\) and \(\varepsilon\) are small positive parameters. Note that for \(\varepsilon \searrow 0\)
\[
(4.5) \quad S_\varepsilon(r) \to \chi_{\{r > 0\}} := \begin{cases} 
1, & r > 0, \\
0, & r \leq 0,
\end{cases} \quad \text{pointwise in } \mathbb{R}.
\]

Integrating the first term by parts gives
\[
\int_{Q^\tau} (u_1 - u_2) \xi \psi S_\varepsilon(u_1^2 - u_2^2)
\]
\[
= \int_{Q^\tau} (f_1 - f_2) \psi \left\{\xi' S_\varepsilon(u_1^2 - u_2^2) + \xi S_\varepsilon'(u_1^2 - u_2^2)(u_1^2 - u_2^2)_x\right\}
\]
\[
\leq \int_{Q^\tau} (f_1 - f_2) \psi \xi' S_\varepsilon(u_1^2 - u_2^2) + \int_{Q^\tau} (u_1 - u_2) \psi \xi S_\varepsilon'(u_1^2 - u_2^2)(u_1^2 - u_2^2)_x.
\]

For fixed \(\mu, \delta > 0\), we first let \(\varepsilon \searrow 0\). Using (4.5), we have
\[
(u_1 - u_2) \psi \xi S_\varepsilon'(u_1^2 - u_2^2) \to 0 \quad \text{pointwise in } Q^\tau.
\]

Hence by (4.4) we obtain
\[
\int_{Q^\tau} \xi \psi((u_1 - u_2)_+)_{+} \leq \int_{Q^\tau} (f_1 - f_2) \psi \xi \chi_{\{u_1 > u_2\}}.
\]
Next we let $\mu \searrow 0$. This gives

$$\int_{\mathbb{R}} \xi(u_1 - u_2)_+(\tau) \leq \int_{0}^{\tau} \left\{ \int_{-\delta}^{-\delta/2} (f_1 - f_2)\xi'\chi_{u_1 > u_2} + \int_{\delta/2}^{\delta} (f_1 - f_2)\xi'\chi_{u_1 < u_2} \right\} =: \int_{0}^{\tau} \{ I_\delta^- + I_\delta^+ \}. \quad (4.6)$$

Let $t \in (0, \tau)$ be chosen such that $f^-, f^+$ exist. Consider the possibilities:

(i) $u_1^+ \neq u_2^+$, say $u_1^+ > u_2^+$. Then $u_1 > u_2$ in a right neighborhood of $x = 0$ and $\chi_{\{u_1 > u_2\}} = 1$ in $(\delta/2, \delta)$ for $\delta$ sufficiently small. The pressure conditions (M$_2$) give $u_1^+ > u_2^+$: if $u_2^+ > 0$, then $[p_1] = [p_2] = 0$ implies $u_1^+ > u_2^+$; if $u_2^+ = 0$, then $u_2^- \leq u^*$, while $u_1^- > u^*$. Therefore also $\chi_{\{u_1 > u_2\}} = 1$ in $(-\delta, -\delta/2)$. As a consequence

$$\lim_{\delta \searrow 0} (I_\delta^- + I_\delta^+) = [f_1] - [f_2] = 0.$$ 

(ii) $u_1^+ = u_2^+$. Now we need to compare the corresponding fluxes. Suppose that $f_1^+ = f_2^+$. Then

$$\sup_{(\delta/2, \delta)} (f_1 - f_2)\chi_{\{u_1 > u_2\}} \to 0 \quad \text{as} \quad \delta \searrow 0,$$

and the same applies in $(-\delta, -\delta/2)$. Thus again

$$\lim_{\delta \searrow 0} (I_\delta^- + I_\delta^+) = 0.$$ 

If $f_1^+ > f_2^+$, then $(u_1^+)_x < (u_2^+)_x$ and therefore $u_1 < u_2$ in $(\delta/2, \delta)$. Thus

$$I_\delta^- + I_\delta^+ = I_\delta^- \leq 0 \quad \text{for} \ \delta > 0 \ \text{sufficiently small.}$$

Finally, if $f_1^+ < f_2^+$, then $(u_1^+)_x > (u_2^+)_x$ and $u_1 > u_2$ in $(\delta/2, \delta)$. Thus $\lim_{\delta \searrow 0} I_\delta^+ = f_1^+ - f_2^+$. Furthermore, since

$$(f_1 - f_2)\xi'\chi_{\{u_1 > u_2\}} \leq (f_1 - f_2)\xi' \quad \text{in} \quad (-\delta, -\delta/2),$$

$$\limsup_{\delta \searrow 0} (I_\delta^- + I_\delta^+) \leq 0.$$ 

Combining these results, we obtain from (4.6)

$$u_1(\cdot, \tau) - u_2(\cdot, \tau) \leq 0 \quad \text{in} \quad \mathbb{R}\setminus\{0\},$$

which proves the theorem. $\Box$

As an immediate consequence we have the following.

**Corollary 4.2.** Problem P has at most one solution $(u, f)$. 
5. Oil trapping. The steady state solutions shown in Figure 3 suggest that oil may be trapped at the interface between coarse and fine material. Indeed, if $u_0(x) = 0$ for $x > 0$ and if for some $u^- \in (0, u^\ast]$
\[ u_0(x) \leq \left( u^- + \frac{1}{\sqrt{k^-}} x \right)_+ \text{ for } x < 0, \]
then the comparison principle guarantees
\[ u(x, t) \leq \left( u^- + \frac{1}{\sqrt{k^-}} x \right)_+ \text{ for all } (x, t) \in Q^- \]
and
\[ u = 0 \text{ in } Q^+. \]

The following theorem explains trapping in terms of the oil mass. For convenience, let
\[ \bar{u}(x) := \begin{cases} \left( u^\ast + \frac{1}{\sqrt{k^-}} x \right)_+ & \text{for } x < 0, \\ 0 & \text{for } x > 0 \end{cases} \]
denote the maximal admissible steady state having $\bar{M}$, given by (1.12), as corresponding mass.

**Theorem 5.1.** Let $u_0$ satisfy hypothesis (H) and let
\[ \int_{-\infty}^{x} u_0(s)ds \geq \int_{-\infty}^{x} \bar{u}(s)ds \text{ for } x < 0. \]
Then the solution of Problem P satisfies
\[ \int_{-\infty}^{0} u(s,t)ds \geq \bar{M} \text{ for all } t > 0. \]

**Proof.** Fix any $\delta > 0$ and set
\[ V_\delta(x, t) = \int_{-\infty}^{x} u(s,t)ds + \delta \text{ for } (x, t) \in \bar{Q}. \]
Then $V_\delta \in C(\bar{Q})$, $V(\cdot, t) \in C^1((\infty, 0)) \cup C^1([0, \infty))$ for all $t > 0$, and
\[ V_\delta = \delta \text{ to the left of the support of } u \text{ in } Q^-, \]
\[ V_\delta = \int_{\mathbb{R}} u_0(s)ds + \delta \text{ to the right of the support of } u \text{ in } Q^+. \]
As a consequence $V_\delta \geq \bar{M}$ in $Q^+$, and it satisfies
\[ V_t + V_x - \sqrt{k^-} V_x V_{xx} = 0 \text{ a.e. in } Q^- . \tag{5.1} \]
Setting
\[ \bar{v}(x) := \int_{-\infty}^{x} \bar{u}(s)ds \text{ for } x \in (-\infty, 0], \]
we have
\[ V_\delta > \bar{v} \quad \text{in} \quad Q^- := (-\infty, 0] \times (0, t) \]
for \( t \) sufficiently small. Let
\[ t_0 = \sup \{ t > 0 : V_\delta > \bar{v} \quad \text{in} \quad Q^- \}. \]
Below we show \( t_0 = \infty \). Suppose \( t_0 < \infty \). Then there exists \((x_0, t_0) \in Q^-\) such that
\[ (5.2) \quad V_\delta > \bar{v} \quad \text{in} \quad Q_{t_0}^- \]
and
\[ (5.3) \quad V_\delta(x, t_0) \geq \bar{v}(x) \quad \text{for all} \quad x \in (-\infty, 0] \quad \text{with} \quad V_\delta(x_0, t_0) = \bar{v}(x_0). \]
We first rule out \( x_0 = 0 \).
If \( x_0 = 0 \), we distinguish the three following cases:
(i) \( u(0^-, t_0) > u^* \). Then we have

\[ \frac{\partial V_\delta}{\partial x}(0^-, t_0) = u(0^-, t_0) > u^* = \frac{d\bar{v}}{dx}(0^-. \]
This contradicts (5.3).
(ii) \( u(0^-, t_0) < u^* \). By continuity there exists \( \varepsilon > 0 \) such that \( u(0^-, t) < u^* \) and \( u(0^+, t) = 0 \) for \( t_0 - \varepsilon < t < t_0 \). Since \( f^-(t) = f^+(t) \leq 0 \) for almost all \( t \in (t_0 - \varepsilon, t_0) \) (see also section 6), we find from integrating the \( u \)-equation in \((\infty, 0) \times (t_0 - \varepsilon, t_0)\)

\[ \int_{-\infty}^{0} u(s, t_0) ds - \int_{-\infty}^{0} u(s, t_0 - \varepsilon) ds = - \int_{t_0 - \varepsilon}^{t_0} f^-(t) dt \geq 0. \]
Hence
\[ V_\delta(0, t_0 - \varepsilon) \leq V_\delta(0, t_0) = \bar{v}(0), \]
which contradicts (5.2).
(iii) \( u(0^-, t_0) = u^* \). Then \( V_\delta(0^-, t_0) = \bar{v}(0) \) as well as

\[ \frac{\partial V_\delta}{\partial x}(0^-, t_0) = \frac{d\bar{v}}{dx}(0^-) = u^*. \]

Using (5.1) locally in \( Q^- \) and the strong maximum principle, we again obtain a contradiction.
Hence \( x_0 \neq 0 \) and \( V_0(0, \cdot) > \bar{v}(0) \) in \([0, t_0]\). We then apply the comparison principle to (5.1) in \( Q_{t_0}^- \) to find \( V_\delta > \bar{v} \) in \((\infty, 0] \times [0, t_0]\). This shows that \( t_0 = \infty \).
As a consequence \( V_\delta > \bar{v} \) in \( Q^- \) for any \( \delta > 0 \), which implies the assertion of the theorem.

Similarly we show the following.
THEOREM 5.2. Let \( u_0 \) satisfy hypothesis (H) and let

\[ \int_{-\infty}^{\infty} u_0(s) ds \leq \int_{-\infty}^{\infty} \bar{u}(s) ds \quad \text{for} \quad x \in \mathbb{R}. \]
Then
\[ u = 0 \quad \text{in} \quad \bar{Q}^+. \]
6. Closing remarks. In this section we briefly discuss some qualitative properties of solutions of Problem P.

6.1. Nonuniqueness. In the proof of the comparison principle, implying uniqueness, we have used the pressure condition

\[ [p] \geq 0. \]  

By means of a counterexample we show here that uniqueness fails if we drop condition (6.1). Let \( u_0 \) satisfy the structural properties

\[
(H) \quad \begin{cases}
  u_0(x) = 0 & \text{if } x > 0, \ u_0 \not\equiv \bar{u} \text{ in } \mathbb{R}, \\
  \bar{u}(x) \leq u_0(x) \leq (u^* + \delta x)_+ & \text{if } x < 0 \text{ for some } 0 < \delta < \frac{1}{\sqrt{k^-}}.
\end{cases}
\]

Based on the results of section 5, we expect that the corresponding solution \( u \) of Problem P will have a nontrivial component in \( Q^+ \); i.e., \( u \not\equiv 0 \) in \( Q^+ \). We will construct a second solution \( \tilde{u} \) which solves Problem P, except condition (6.1), and which satisfies \( \tilde{u} \equiv 0 \) in \( Q^+ \). This construction is based on a modification of \( k \).

Instead of (1.2) we consider

\[
(\tilde{H}) \quad \tilde{k}_n(x) = \begin{cases}
  k^- & \text{for } x < 0, \\
  \kappa & \text{for } 0 < x < \frac{1}{n}, \\
  k^+ & \text{for } x > \frac{1}{n},
\end{cases}
\]

where \( 0 < \kappa < k^+ < k^- \), and we let \( n \to \infty \).

**Theorem 6.1.** Let \( u_0 \) satisfy hypotheses (H) and (\( \tilde{H} \)) and let \( u \) denote the unique solution of Problem P. Then

(i) \( u \not\equiv 0 \) in \( Q^+ \);

(ii) there exists a second solution \( \tilde{u} \) of Problem P, except (6.1), which satisfies \( \tilde{u} \equiv 0 \) in \( Q^+ \).

**Proof.** We first show that \( u \not\equiv 0 \) in \( Q^+ \). Arguing by contradiction, we assume \( u(0^+, t) = 0 \) for all \( t > 0 \).

Using \([p] \geq 0\) and \( u \geq \bar{u} \) in \( Q \), we conclude

\[ u(0^-, t) = u^* \text{ for all } t > 0. \]

Hence \( u \) solves in \( Q^- \) the problem

\[
(P^-) \quad \begin{cases}
  u_t + (u - \sqrt{k^-} uu_x)_x = 0 & \text{in } Q^-, \\
  u(0, t) = u^* & \text{for } t > 0, \\
  u(x, 0) = u_0(x) & \text{for } x < 0.
\end{cases}
\]

Now observe that \( \tilde{z} := (u^* + \delta x)_+ \) is a supersolution for problem \((P^-)\). Hence the solution \( z(x, t) \) of problem \((P^-)\) with initial data \( z(\cdot, 0) = \tilde{z}(x) \) is decreasing with respect to time and converges to a steady state solution \( s(x) \). By comparison \( s \geq \bar{u} \) in \( \mathbb{R}^- \), but since \( \bar{u} \) is maximal we have

\[ s = \bar{u} \text{ in } \mathbb{R}^- \.]
Using
\[ \bar{u}(x) \leq u(x,t) \leq z(x,t) \quad \text{for all } (x,t) \in Q^-, \]
we obtain
\[ \lim_{t \to \infty} u(x,t) = \bar{u}(x) \quad \text{uniformly in } x < 0. \]
Combining this result with \( u \equiv 0 \) in \( Q^+ \), we find
\[ \lim_{t \to \infty} \int_{-\infty}^{+\infty} u(x,t) ds \to \int_{-\infty}^{+\infty} \bar{u}(x) dx < \int_{-\infty}^{+\infty} u_0(x) dx, \]
which contradicts mass conservation for \( u \).

Next we use (6.2) to explain the construction of \( \tilde{u} \). As a first observation we note that the class of steady states solutions of the equation
\[ \left( u - \hat{k}_n \left( \frac{1 + u}{\sqrt{k_n}} \right)' \right)' = 0 \quad \text{in } \mathbb{R}, \]
having compact support and satisfying \((M_1)\) and \((M_2)\), has the same structure as the one shown in Figure 3, but with \( u^* = \sqrt{\frac{2\epsilon}{\kappa}} - 1 \) replaced by \( \tilde{u}^* = \sqrt{\frac{2\epsilon}{\kappa}} - 1 \). In particular this class does not depend on \( n \). For \( \kappa \) sufficiently small we find, for \( \tilde{u} \), the maximal steady state
\[ u_0 \leq \tilde{u} \quad \text{in } \mathbb{R}. \]
As a consequence, the solution \( \tilde{u}_n \) of the problem
\[ \begin{cases} u_t + \left( u - \hat{k}_n \left( \frac{1 + u}{\sqrt{k_n}} \right)' \right)' \quad & \text{in } Q, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases} \]
satisfies
\[ \tilde{u}_n(x,t) \leq \tilde{u}(x) \quad \text{for all } (x,t) \in Q. \]
In particular
\[ \tilde{u}_n \equiv 0 \quad \text{in } Q^+ \]
for all \( n \in \mathbb{Z}^+ \). Finally, letting \( n \to \infty \), \( \tilde{u}_n \) converges along subsequences to a function \( \tilde{u} = \tilde{u}(x,t) \) which satisfies all properties required for Problem P except (6.1).

6.2. Waiting times and optimal regularity. Numerical simulations reported in [DMN] show that the right free boundary of \( u \) has a “waiting time” when it reaches the permeability discontinuity. The free boundary becomes stagnant there, while the oil saturation increases. It continues whenever the pressure exceeds the entry pressure of the low permeable region.

The following makes this precise.

**Theorem 6.2.** Let \( u_0 \) satisfy hypothesis \((H)\) and let \( \text{supp}(u_0) \subset \mathbb{R}^- \). Further, let the solution \( u \) of Problem P satisfy \( u \not\equiv 0 \) in \( Q^+ \). Set
\[ t_1 := \lim_{\varepsilon \to 0} \sup \{ \tau > 0 : u \equiv 0 \quad \text{in } (-\varepsilon, \infty) \times (0, \tau) \} \]
and
\[ t_2 := \sup \{ \tau > 0 : u \equiv 0 \text{ in } \mathbb{R}^+ \times (0, \tau) \}. \]

Then
\[ 0 < t_1 < t_2 < \infty \quad (t_2 - t_1 \text{ is the waiting time}) \]

and
\[ u(0^-, t_1) = 0, \quad u(0^-, t_2) = u^*. \]

Proof. Clearly \( t_1 \) and \( t_2 \) are well defined. Continuity of \( u^\pm(t) \) and (M2) imply directly \( t_2 > t_1 \) and \( u(0^-, t_1) = 0 \).

Suppose \( u(0^-, t_2) < u^* \). By continuity, there exists \( \delta > 0 \) such that \( u(0^-, t) < u^* \), and thus \( u(0^+, t) = 0 \), for \( t_2 \leq t < t_2 + \delta \). Thus \( u \equiv 0 \) in \( \mathbb{R}^+ \times (0, t_2 + \delta) \), contradicting the definition of \( t_2 \).

Next we consider the case where the oil initially is positioned in the fine material \((x > 0)\). If the initial position is sufficiently close to the interface at \( x = 0 \), diffusion may drive the oil towards \( x = 0 \), i.e., against the flow, where it will penetrate the coarse material. This follows from the transformation \( y = x - t, \ t = t \) and by considering an appropriate subsolution for the resulting porous media equation; see [G2].

Supposing the oil reaches \( x = 0 \), we have the following result.

**Theorem 6.3.** Let \( u_0 \) satisfy hypothesis (H) and let \( \text{supp}(u_0) \subset \mathbb{R}^+ \). Further, let the solution \( u \) of Problem P satisfy \( u \not\equiv 0 \) in \( Q^- \). Set
\[ t_1 := \sup \{ \tau > 0 : u \equiv 0 \text{ in } \mathbb{R}^- \times (0, \tau) \}. \]
\[ t_2 := \sup \{ \tau > 0 : u(0^+, t) = 0 \text{ for } 0 < t < \tau \}. \]

Then
\[ 0 < t_1 < t_2 \leq \infty. \]

In addition, there exists \( t \in (t_1, t_2) \) such that for some \( A > 0 \)
\[ u(x, t) = A\sqrt{x}(1 + o(1)) \quad \text{as } x \to 0^+. \]

Proof. By the finite speed of propagation we have \( t_1 > 0 \). Continuity of \( u(0^-, \cdot) \) implies \( u(0^-, t_1) = 0 \) and \( u(0^-, t) \leq u^* \) and hence \( u(0^+, t) = 0 \) for all \( t \) in an upper neighborhood of \( t_1 \). Hence \( t_2 > t_1 \). If \( u(0^-, t) \leq u^* \) for all \( t > 0 \), we have \( t_2 = \infty \). Since \( u \not\equiv 0 \) in \( \mathbb{R}^- \times (t_1, t_2) \) and \( u(0^+, \cdot) \equiv 0 \) in \((t_1, t_2)\), there exists \( t \in (t_1, t_2) \) such that
\[ f(t) = f^-(t) = f^+(t) < 0. \]

Hence, for this \( t \) fixed, setting \( f(t) = -\mathcal{C}(\mathcal{C} > 0) \),
\[ u - \sqrt{k^\pm}uu_x = -\mathcal{C}(1 + o(1)) \quad \text{as } x \to 0^+, \]
giving
\[ \frac{1}{2} u^2(x, t) = \frac{\mathcal{C}}{k^+} x(1 + o(1)) \quad \text{as } x \to 0^+. \]
Appendix. Proof of Lemma 2.1. Let $\varepsilon_n > 0$ be such that

$$\varepsilon_n = o\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty,$$

and set

$$u_{0n}(x) = \begin{cases} 
\sqrt{u_0^2(x - \frac{1}{n}) + \varepsilon_n^2} & \text{if} \quad x > \frac{1}{n}, \\
\sqrt{(u_0^+)^2 + \varepsilon_n^2} & \text{if} \quad x = \frac{1}{n},
\end{cases}$$

where $u_0^+ = \lim_{x \to 0} u_0(x)$. Since $|u_{0n}(x)| \leq |u_0'(x - \frac{1}{n})|$ for $x > \frac{1}{n}$, the uniform Lipschitz continuity of $u_0$ in $\mathbb{R}^+$ implies $u_{0n}$ is uniform Lipschitz continuous in $[\frac{1}{n}, \infty]$.

Since

$$f_0 = u_0 - \frac{1}{2} \sqrt{k^+}(u_0^+)' \quad \text{in} \quad \mathbb{R}^+,$$

$$f_{0n} = u_{0n} - \frac{1}{2} \sqrt{k^+}(u_{0n}^+) \quad \text{in} \quad \left[\frac{1}{n}, \infty\right),$$

the total variation of $(u_0^+)'$ in $\mathbb{R}^+$, $TV_{\mathbb{R}^+}((u_0^+))$, is bounded, and since $(u_{0n}^+)'(x) = (u_0^+)'(x - \frac{1}{n})$,

(A.1) \quad $TV_{\left[\frac{1}{n}, \infty\right)}(f_{0n}) \to TV_{\mathbb{R}^+}(f_0)$ as $n \to \infty$.

In order to extend $u_{0n}$ to the interval $[-\frac{1}{n}, \infty]$ we distinguish two different cases: $u_0^+ > 0$ and $u_0^+ = 0$. At this point we remind the reader that the constant $u^*$ is defined by

$$\frac{1 + u^*}{\sqrt{k^-}} = \frac{1}{\sqrt{k^+}}, \quad \text{i.e.,} \quad u^* = \sqrt{k^-} k^+ - 1.$$

(i) Case $u_0^+ > 0$. We define $u_{0n}$ in $[-\frac{1}{n}, \frac{1}{n}]$ by the relation $p_{0n} = p_0(n\frac{1}{n})$ in $[-\frac{1}{n}, \frac{1}{n}]$, i.e.,

$$u_{0n}(x) = -1 + \sqrt{\frac{k^-(x)}{k^+}} \left(1 + \sqrt{(u_0^+)^2 + \frac{\varepsilon_n^2}{k^-}} \right).$$

In particular, as $n \to \infty$,

$$u_{0n}(\frac{1}{n}) = -1 + \sqrt{\frac{k^-}{k^+}} \left(1 + \sqrt{(u_0^+)^2 + \frac{\varepsilon_n^2}{k^-}} \right)$$

(A.2)

$$\to -1 + \sqrt{\frac{k^-}{k^+}} (1 + u_0^+) = u_0^-,$$

where we have used, by hypothesis (H), $[p_0] = 0$ if $u_0^+ > 0$. Since $u_{0n}(\frac{1}{n}) > u_0^-$, there exist $\delta_n > 0$ such that

(A.3) \quad $u_{0n}(\frac{1}{n}) = \sqrt{(u_0^-)^2 + \delta_n^2}$. 


It follows directly from the construction of $u_{0n}$ that

\[ TV\left(\frac{1}{n} + \frac{1}{n}\right)(f_{0n}) = -u_{0n}\left(\frac{1}{n}\right) + u_{0n}\left(-\frac{1}{n}\right) \rightarrow -[u_0] \quad \text{as} \quad n \rightarrow \infty \tag{A.4} \]

and

\[ f_{0n}\left(\frac{1}{n} + \right) - f_{0n}\left(\frac{1}{n} - \right) = -\frac{1}{2}\sqrt{k^+}(u_{0n}^2)'\left(\frac{1}{n} + \right) = -\frac{1}{2}\sqrt{k^+}(u_0^2)'(0^+). \tag{A.5} \]

(ii) Case $u_0^+ = 0$. Since $|p_0| \geq 0$, $u_0^+ = 0$ implies that

\[ 0 \leq u_0^- \leq -1 + \sqrt{\frac{k^-}{k^+}} = u^* \]

Hence

\[ (1 + \varepsilon_n\sqrt{k^-}) > \sqrt{k^-} \geq \sqrt{k^+}(1 + u_0^-), \]

and there exist $\delta_n > 0$ such that

\[ \begin{cases} \delta_n \rightarrow 0 & \text{as} \quad n \rightarrow \infty, \\
(1 + \varepsilon_n)\sqrt{k^-} > \sqrt{k^+} \left(1 + \left(u_0^-\right)^2 + \delta_n^2\right), \\
\sqrt{(u_0^-)^2 + \delta_n^2} > \varepsilon_n. \end{cases} \]

These two inequalities imply that for some $\kappa_n \in (k^+, k^-)$

\[ (1 + \varepsilon_n)\sqrt{k^-} = \sqrt{\kappa_n\left(1 + \sqrt{(u_0^-)^2 + \delta_n^2}\right)}. \]

Then there exists $x_n \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ such that

\[ k_n(x_n) = \kappa_n, \]

and we define $u_{0n}$ in $[-\frac{1}{n}, \frac{1}{n})$ by the relations

\[ u_{0n}(x) \equiv u_{0n}\left(\frac{1}{n}\right) \quad (= \varepsilon_n) \quad \text{if} \quad x_n \leq x < \frac{1}{n} \]

and

\[ p_{0n}(x) \equiv p_{0n}(x_n) \quad \left(= \frac{1 + \varepsilon_n}{\sqrt{\kappa_n}}\right) \quad \text{if} \quad -\frac{1}{n} < x < x_n. \]

By the definition of $\kappa_n$ and $p_{0n}$, the latter relation can be written as

\[ u_{0n}(x) = -1 + \sqrt{k_n(x_n)/k^-} \left(1 + \sqrt{(u_0^-)^2 + \delta_n^2}\right) \quad \text{if} \quad -\frac{1}{n} \leq x < x_n. \]

In particular we have

\[ (A.6) \quad u_{0n}' \leq 0 \quad \text{in} \quad \left(-\frac{1}{n}, x_n\right) \quad \text{and} \quad u_{0n}\left(-\frac{1}{n}\right) = \sqrt{(u_0^-)^2 + \delta_n^2} \rightarrow u_0^- \quad \text{as} \quad n \rightarrow \infty, \]
and
\[ TV_{(-\frac{1}{n}, x_n)}(f_0^n) = TV_{(-\frac{1}{n}, x_n)}(u_0^n) \rightarrow -[u_0] \quad \text{as } n \rightarrow \infty. \]

Since \(|k_n^-| \leq \frac{C}{n}\) and \(\varepsilon_n = o\left(\frac{1}{n}\right)\) as \(n \rightarrow \infty\), and since
\[
f_0^n(x) = \varepsilon_n + \frac{1}{2}\varepsilon_n(1 + \varepsilon_n) \frac{k'_n(x)}{\sqrt{k_n(x)}} \quad \text{if } x_n < x < \frac{1}{n},
\]

it follows that
\[ TV_{(\varepsilon_n, \frac{1}{n})}(f_0^n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

In addition, as \(n \rightarrow \infty\),
\[ f_0^n\left(\frac{1}{n} + \right) - f_0^n\left(\frac{1}{n} - \right) \rightarrow -\frac{1}{2}\sqrt{k^+(u_0^n)'(0^+)} \]

and
\[ f_0^n(x_n^+) - f_0^n(x_n^-) = \frac{1}{2}\varepsilon_n(1 + \varepsilon_n) \frac{k'_n(x_n)}{\sqrt{k_n}} \rightarrow 0. \]

Combining (A.6)–(A.10) gives
\[ TV_{(-\frac{1}{n}, \frac{1}{n})}(f_0^n) \rightarrow -[u_0] \quad \text{as } n \rightarrow \infty. \]

Finally we have to define \(u_0^n(x)\) for \(x < -\frac{1}{n}\). In view of (A.3) and (A.6) it seems natural to set
\[ u_0^n(x) = \sqrt{u_0^2\left(x + \frac{1}{n}\right) + \delta_n^2} \quad \text{if } x < -\frac{1}{n}. \]

Arguing as in the interval \((\frac{1}{n}, \infty)\), we obtain as \(n \rightarrow \infty\)
\[ TV_{(-\infty, -\frac{1}{n})}(f_0^n) \rightarrow TV_{\mathbb{R}^-}(f_0) \]

and
\[ f_0^n\left(\left(-\frac{1}{n} + \right) - f_0^n\left(\left(-\frac{1}{n} - \right) \rightarrow -\frac{1}{2}\sqrt{k^-(u_0^n)'}(0^-). \]

Combining (A.1), (A.13), and (A.14) with, respectively, (A.4), (A.5) if \(u_0^+ > 0\) and (A.9), (A.11) if \(u_0^+ = 0\), we find
\[ TV_{\mathbb{R}^-}(f_0^n) \rightarrow TV_{\mathbb{R}}(f_0) \quad \text{as } n \rightarrow \infty. \]

Now, if \(\delta_n = \varepsilon_n\), \(u_0^n\) satisfies all properties of Lemma 2.1. In general, however, \(\delta_n \neq \varepsilon_n\) and we have to correct the construction of \(u_0^n\) in \((-\infty, -\frac{1}{n})\). Since \(u_0^n\left(-\frac{1}{n}\right) > u_0^n\left(\frac{1}{n}\right) \geq \varepsilon_n\), we can still use definition (A.12) in a neighborhood of \(x = -\frac{1}{n}\). Since \(k_n\) is constant in \((-\infty, -\frac{1}{n})\), the expression for the flux is simply
\[ f_0^n = u_0^n - \frac{1}{2}\sqrt{k^-(u_0^n)'} \quad \text{in } (-\infty, -\frac{1}{n}). \]

Therefore it is not difficult to change slightly the definition of \(u_0^n\) such that \(u_0^n \geq \varepsilon_n\) in \(\mathbb{R}\) and \(u_0^n(x) = \varepsilon_n\) for \(-x\) sufficiently large. We leave the details to the reader.
Acknowledgment. The authors gratefully acknowledge support of EU by the TMR program *Nonlinear Parabolic Partial Differential Equations: Methods and Applications*, FMRX-CT98-0201.

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